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On the distribution of k-free numbers and r-tuples of k-free numbers. A survey

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Abstract: This paper presents a brief survey of the current state of research the distribution of k-free numbers and r-tuples of k-free numbers. We state the main problems in the field, sketch their history and the basic machinery used to study them.
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1 Notations

Let k and n be integers and $k \geq 2$. We say that n is k-free if there is no prime p such that $p^k | n$. By convention, a 2-free integer is called square-free and a 3-free integer is called cube-free. We denote by μ_k the characteristic function of the k-free numbers, i.e.,

$$\mu_k(n) = \begin{cases} 1, & \text{if } n \text{ is a } k\text{-free number,} \\ 0, & \text{otherwise.} \end{cases}$$

As usual, $\varphi(n)$ is Euler's function, $\zeta(n)$ is Riemann's zeta function, $\mu(n)$ is the Möbius' function and $\tau(n)$ denotes the number of positive divisors of n. Let X be a sufficiently large positive number. By ε we denote an arbitrary small positive number, not necessarily the same in different occurrences. Moreover $[t]$ and $\{t\}$ denote the integer part and the fractional part of t, respectively. The letter p will always denote a prime number.

2 Introduction

The investigation of the k-freeness of the numbers is important and plays a significant role in the contemporary analytic number theory. Many Diophantine equations are solved with square-free numbers. For example, in 2014, Dudek [17] proved that every integer greater than two may be written as the sum of a prime and a square-free number. This paper presents a brief survey of the current state of the distribution of k-free numbers and r-tuples of k-free numbers. We state the main problems in the field, sketch their history and the basic machinery used to study them.

3 On the distribution of k-free numbers

3.1 k-free numbers of arbitrary type

It is well known that the density of k-free integers is $1/\zeta(k)$, and an elementary sieve shows

$$\sum_{n\leq X}\mu_k(n)=\frac{X}{\zeta(k)}+\mathcal{O}\left(X^{1/k}\right) .$$

No better exponent is known for the remainder term. In the case $k=2$ assuming RH the exponent $1/2$ has been refined several times [1,2,49] and currently to $17/54=0.31$ by [38]. It is expected that

$$\sum_{n\leq X}\mu_2(n)=\frac{6}{\pi^2}X+\mathcal{O}\left(X^{1/4+o(1)}\right) .$$

3.2 k-free numbers of the form $[n^c]$

3.2.1 Square-free numbers of the form $[n^c]$

In 1978, Rieger [57] showed that for any fixed $1<c<3/2$ the asymptotic formula

$$\sum_{n\leq X}\mu_2([n^c])=\frac{6}{\pi^2}X+\mathcal{O}\left(X^{\frac{2c+1}{4}+\varepsilon_0}\right)$$

holds.

From the above formula it follows that for any fixed $1<c<3/2$ there exist infinitely many square-free numbers of the form $[n^c]$.

Subsequently Cao and Zhai [9] using estimation of multiple exponential sums with monomials ([8, Theorem 7]), estimation of three-dimemsional exponential sums with monomials ([58, Theorem 3]) and Heath-Brown's identity [24], improved the result of Rieger by proving the following:

Theorem 1 ([9]). *For any fixed* $1<c<149/87$, $\gamma=c^{-1}$ *and* $0<\varepsilon<(149\gamma-87)/400$ *the asymptotic formulas*

$$\sum_{n\leq X}\mu_2([n^c])=\frac{6}{\pi^2}X+\mathcal{O}\left(X^{1-\varepsilon}\right) ,$$

$$\sum_{p\leq X}\mu_2([p^c])=\frac{6}{\pi^2}\int\limits_2^X\frac{dt}{\ln t}+\mathcal{O}\left(Xe^{-c_0\sqrt{\log X}}\right) ,$$

$$\sum_{\substack{n\le X\\ \mu_2(n)=1}} \mu_2([n^c]) = \frac{36}{\pi^4}X + \mathcal{O}\left(X^{1-\varepsilon}\right)$$

hold. Here $c_0 > 0$ *is an absolute constant.*

Their earlier result [7] covers the narrower range $1 < c < 61/36$.

3.2.2 Cube-free numbers of the form $[n^c]$

In 2017, Zhang and Li [67] using the method of Cao and Zhai [9] showed that for any fixed $1 < c < 11/6$ there exist infinitely many cube-free numbers of the form $[n^c]$. More precisely, they proved the following:

Theorem 2 ([67]). *For any fixed* $1 < c < 11/6$, $\gamma = c^{-1}$ *and* $0 < \varepsilon < 10^{-10}$ *the asymptotic formulas*

$$\sum_{n\le X} \mu_3([n^c]) = \frac{X}{\zeta(3)} + \mathcal{O}\left(X^{1-\varepsilon}\right) ,$$

$$\sum_{p\le X} \mu_3([p^c]) = \frac{1}{\zeta(3)}\int_2^X \frac{dt}{\ln t} + \mathcal{O}\left(Xe^{-c_0\sqrt{\log X}}\right) ,$$

$$\sum_{\substack{n\le X\\ \mu_3(n)=1}} \mu_3([n^c]) = \frac{X}{\zeta^2(3)} + \mathcal{O}\left(X^{1-\varepsilon}\right)$$

hold. Here $c_0 > 0$ *is an absolute constant.*

3.3 Square-free numbers of the form $[\alpha n]$

In 2008, Güloğlu and Nevans [22] showed that there exist infinitely many square-free numbers of the form $[\alpha n]$, where $\alpha > 1$ is an irrational number of finite type. More precisely, they proved that the asymptotic formula

$$\sum_{n\le N} \mu_2([\alpha n]) = \frac{6}{\pi^2}N + \mathcal{O}\left(\frac{N\log\log N}{\log N}\right)$$

holds.

Subsequently in 2013, Victorovich [64] showed that there exist infinitely many square-free numbers of the form $[\alpha n]$, where α is irrational number with bounded partial quotient or algebraic number. More precisely, he proved the following:

Theorem 3 ([64]). *For each* $A > 0$ *the asymptotic formula*

$$\sum_{n\le N} \mu_2([\alpha n]) = \frac{6}{\pi^2}N + \mathcal{O}\left(AN^{\frac{5}{6}}\log^5 N\right)$$

holds. Here $A = A(N) = \max\limits_{1\le m\le N^2} \tau(m)$.

3.4 Square-free numbers of the form $p-1$

Further in 2013, Victorovich [64] showed that there exist infinitely many square-free numbers of the form $p-1$, where p is prime. More precisely he proved the following:

Theorem 4 ([64]). *For each $A>0$ the asymptotic formula*

$$\sum_{p\le N}\mu_2(p-1)=\prod_p\left(1-\frac{1}{p(p-1)}\right)\int_2^N\frac{dt}{\ln t}+\mathcal{O}\left(\frac{N}{\log^A N}\right)$$

holds.

3.5 k-free values of polynomials

3.5.1 k-free values of polynomial of arbitrary type

Let k and n be integers and $k\ge 2$. Consider the irreducible polynomial $f(x)\in\mathbb{Z}[x]$ of degree d. Assume that for every prime p there is at least one integer n_p for which $p^k\nmid f(n_p)$. It is conjectured that the set $f(\mathbb{Z})=\{f(n),n\in\mathbb{Z}\}$ contains infinitely many k-free values. The first result in this direction belongs to Ricci [56], who proved the following:

Theorem 5 ([56]). *Let $f(x)\in\mathbb{Z}[x]$ be an irreducible polynomial of degree d. Then for $k\ge d$ the asymptotic formula*

$$N_{f,k}(x)\sim C(f,k)x\quad(x\to\infty)\tag{1}$$

holds. Here

$$N_{f,k}(x)=\#\{n\le x:\ f(n)\text{ is }k\text{-free}\},$$

$$C(f,k)=\prod_p\left(1-\frac{\rho_f(p^k)}{p^k}\right)$$

and

$$\rho_f(n)=\#\{a\,(\text{mod }n):n\,|\,f(a)\}.$$

Further progress was made by Erdős [18] who proved the conjecture in the case $k=d-1$ for $d\ge 3$. Later, Hooley [34] derived the asymptotic formula (1) for each such k. Using an alternative approach, Nair [50] established (1) for

$$k\ge\sqrt{2d^2+1}-\frac{d+1}{2}.$$

In 2006, Heath-Brown [27] showed how the determinant method could be applied to the problem, and demonstrated that the asymptotic formula (1) remained valid for

$$k\ge\frac{3d+2}{4}.$$

The idea behind Heath-Brown's approach is to translate the problem into one that involves counting suitably constrained integral points on a certain affine surface.

In 2011 Browning [6] proved that the asymptotic formula (1) holds for

$$k \geq \frac{3d+1}{4}$$

and $d \geq 3$.

There is a related question concerning k-free values of polynomial f at prime arguments. Such results can be found in [6, 30–33, 35, 39, 42, 50–52, 54, 63].

3.5.2 Square-free values of the form n^2+1

It was shown in 1931 by Estermann [19] that there exist infinitely many square-free numbers of the form n^2+1. More precisely, he proved the following:

Theorem 6 (**[19]**). *For $x \geq 2$ the asymptotic formula*

$$\mathcal{N}(x) = c_0 x + \mathcal{O}\left(X^{2/3}\log x\right)$$

holds. Here

$$\mathcal{N}(x) = \#\{n \leq x :\, n^2+1 \text{ is square-free}\}$$

and

$$c_0 = \prod_{p\equiv 1 \,(\mathrm{mod}\, 4)} \left(1 - \frac{2}{p^2}\right).$$

In 2012, Heath-Brown [29] improved the reminder term in the theorem of Estermann with $\mathcal{O}\left(X^{7/12+\varepsilon}\right)$. In order to obtain this result Heath-Brown used a variant of the determinant method, developed in his papers [26, 28].

3.5.3 k-free values of the form x^d+c

In 2013, Heath-Brown [30] investigated k-freeness of the polynomials of type x^d+c. He proved the following:

Theorem 7 (**[30]**). *Let $f(x) = x^d + c \in \mathbb{Z}[x]$ be an irreducible polynomial, and suppose that $k \geq (5d+3)/9$. Then, there is a constant $\delta(d)$ such that*

$$N_{f,k}(x) = C(f,k)x + \mathcal{O}\left(X^{1-\delta(d)}\right)$$

holds. Here

$$N_{f,k}(x) = \#\{n \leq x :\, f(n) \text{ is } k\text{-free}\},$$

$$C(f,k) = \prod_p \left(1 - \frac{\rho_f(p^k)}{p^k}\right)$$

and

$$\rho_f(n) = \#\{a \,(\mathrm{mod}\ n) : n \,|\, f(a)\}.$$

The implied constant may depend on f and k.

3.6 k-free values of multivariable polynomials

3.6.1 k-free values of multivariable polynomials of arbitrary type

Let $n \geq 1$, $d \geq 2$ be two integers. Consider the power-free values of the multivariable polynomial $F(x_1, ..., x_n)$ with integer coefficients and degree d. Denote

$$N_{F,k}(B) = \#\{(x_1, ..., x_n) \in \mathbb{Z}^n : |x_i| \leq B \text{ for } i = 1, ..., n,\ F(x_1, ..., x_n) \text{ is } k\text{-free}\}$$

Most of the work has been done for binary forms. The asymptotic formula for $N_{F,k}(B)$ for binary forms F was established for: $k \geq (d-1)/2$ by Greaves [21], $k > (2\sqrt{2}-1)d/4$ by Filaseta [20], $k > 7d/16$ by Browning [6] and $k > 7d/18$ by Xiao [65]. For other results concerning power-free values of polynomials in two variables we refer to [36, 37] and [60].

In 2018, Lapkova and Xiao [41] derived an asymptotic formula for $N_{F,k}(B)$.

Theorem 8 (**[41, Theorem 1]**). *Let $k \geq 2$ be a positive integer and let F be a polynomial with integer coefficients and degree $d \geq 2$, in n variables, such that for all primes p, there exists an integer n-tuple $(m_1, ..., m_n)$ such that $p^k \nmid F(m_1, ..., m_n)$. Then, there exists a positive number $C_{F,k}$ such that the asymptotic relation*

$$N_{F,k}(B) \sim C_{F,k} B^n$$

holds whenever $k \geq (3d+1)/4$.

Here the constant term is given by the limit of an absolutely convergent infinite product

$$C_{F,k} = \prod_p \left(1 - \frac{\rho_F(p^k)}{p^{kn}}\right)$$

and

$$\rho_F(m) = \#\{(m_1, ..., m_n) \in (\mathbb{Z}/m\mathbb{Z})^n : m \mid F(m_1, ..., m_n)\}.$$

Lapkova and Xiao [41] also proved a similar result when the inputs are restricted to be primes (see [41, Theorem 2]).

For another result concernings k-free values of multivariable polynomials we refer to [3,4,53] and [66].

3.6.2 Square-free values of the form $x^2 + y^2 + 1$

Using the properties of the Gauss sum and A. Weil's estimate for the Kloosterman sum in 2010 Tolev [61] showed that there exist infinitely many square-free numbers of the form $x^2 + y^2 + 1$. More precisely he proved the following:

Theorem 9 (**[61]**). *The asymptotic formula*

$$\sum_{1 \leq x,y \leq H} \mu_2(x^2 + y^2 + 1) = cH^2 + \mathcal{O}\left(H^{\frac{4}{3}+\varepsilon}\right),$$

holds. Here

$$c = \prod_p \left(1 - \frac{\lambda(p^2)}{p^4}\right)$$

and

$$\lambda(q) = \sum_{\substack{1 \le x, y \le q \\ x^2+y^2+1 \equiv 0 \ (q)}} 1\,.$$

3.6.3 k-free values of the form $t_1 \cdots t_r - 1$

Let $k, r \geq 2$ be two integers. Let $\mathcal{N}_{k,r}(x)$ denotes the number of the k-free values of the r variables polynomial $t_1 \cdots t_r - 1$ over $[1, x]^r \cap \mathbb{Z}^r$. In 2011 P. Le Boudec [43] proved an asymptotic formula for $\mathcal{N}_{k,r}(x)$.

Theorem 10 ([43]). *Let $\varepsilon > 0$ be fixed. As $x \to \infty$, if $\delta_{k,r} \le 1$ we have the estimate*

$$\mathcal{N}_{k,r}(x) = c_{k,r}x^r + \mathcal{O}\left(x^{r-\delta_{k,r}+\varepsilon}\right)\,,$$

where

$$c_{k,r} = \prod_p \left(1 - \frac{1}{p^k}\left(1 - \frac{1}{p}\right)^{r-1}\right)\,,$$

and if $1 < \delta_{k,r} \le 2$ we have the estimate

$$\mathcal{N}_{k,r}(x) = c_{k,r}x^r - \theta_{k,r}^{(1)}(x)x^{r-1} + \mathcal{O}\left(x^{r-\delta_{k,r}+\varepsilon}\right)\,,$$

where

$$\theta_{k,r}^{(1)}(x) = r\sum_{d=1}^{\infty} \frac{\mu(d)}{\varphi(d^k)}\left(\frac{\varphi(d)}{d}\right)^{r-1} \sum_{m|d} \mu(m)\left\{\frac{x}{m}\right\}\,,$$

and finally, if $\delta_{k,r} > 2$ we have the estimate

$$\mathcal{N}_{k,r}(x) = c_{k,r}x^r - \theta_{k,r}^{(1)}(x)x^{r-1} + \theta_{k,r}^{(2)}(x)x^{r-2} + \mathcal{O}\left(x^{r-\delta_{k,r}+\varepsilon}\right)\,,$$

where

$$\theta_{k,r}^{(2)}(x) = \frac{r(r-1)}{2}\sum_{d=1}^{\infty} \frac{\mu(d)}{\varphi(d^k)}\left(\frac{\varphi(d)}{d}\right)^{r-2} \left(\sum_{m|d} \mu(m)\left\{\frac{x}{m}\right\}\right)^2\,.$$

3.6.4 k-free values of the form $xy^k + C$

In 2012, Lapkova [40] considered the polynomial $f(x, y) = xy^k + C$ for $k \ge 2$ and any nonzero integer constant C. She derived an asymptotic formula for the k-free values of $f(xy)$ when $x, y \le H$.

Theorem 11 ([40, Theorem 1]). *Let $f(x, y) = xy^k + C \in \mathbb{Z}[x, y]$ for $k \ge 2$ and $C \neq 0$. Then, for some real $\delta = \delta(k, f) > 0$, we have*

$$S(H) = c_f H^2 + \mathcal{O}\left(H^{2-\delta}\right)$$

holds. Here

$$S(H) = \#\{1 \leq x, y \leq H : f(x,y) \text{ is } k\text{-free}\},$$
$$c_f = \prod_p \left(1 - \frac{\rho(p^k)}{p^{2k}}\right)$$

and

$$\rho(m) = \#\{(\mu,\nu) \in (\mathbb{Z}/m\mathbb{Z})^2 : m \mid f(\mu,\nu)\}.$$

Lapkova [40] also proved a similar result for the k-free values of $f(p,q)$ when $p, q \leq H$ are primes (see [40, Theorem 2]).

4 On the distribution of r-tuples of k-free numbers

4.1 Pairs of k-free numbers of arbitrary type

The problem for the pairs of k-free numbers arises in 1932 when Carlitz [10] proved the following:

Theorem 12 ([10]). *The asymptotic formula*

$$\sum_{n\leq X} \mu_k(n)\mu_k(n+1) = \prod_p \left(1 - \frac{2}{p^k}\right) X + \mathcal{O}\left(X^{\frac{2}{k+1}+\varepsilon}\right)$$

holds.

Further we find the result of Mirsky:

Theorem 13 ([47]). *The asymptotic formula*

$$\sum_{n\leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|h} \left(\frac{p^k-1}{p^k-2}\right) X + \mathcal{O}\left(X^{\frac{2}{k+1}+\varepsilon}\right)$$

holds.

Subsequently Mirsky [48] improved his result:

Theorem 14 ([48]). *The asymptotic formula*

$$\sum_{n\leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|h} \left(\frac{p^k-1}{p^k-2}\right) X + \mathcal{O}\left(X^{\frac{2}{k+1}}(\log X)^{\frac{k+2}{k+1}}\right)$$

holds.

Further Meng [46] improved the result of Mirsky as follows

Theorem 15 ([46]). *The asymptotic formula*

$$\sum_{n\leq X} \mu_k(n)\mu_k(n+h) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k|h} \left(\frac{p^k-1}{p^k-2}\right) X + \mathcal{O}\left(X^{\frac{2}{k+1}}\right)$$

holds.

In 1984, Heath-Brown [25] improved the result of Meng for $k = 2$, $h = 1$.

Theorem 16 (**[25]**). *The asymptotic formula*

$$\sum_{n \le X} \mu_2(n)\mu_2(n+1) = \prod_p \left(1 - \frac{2}{p^2}\right) X + \mathcal{O}\left(X^{\frac{7}{11}} (\log X)^7\right)$$

holds.

Finally, Reuss [55] using a generalization of the approximate determinant method proved the best result.

Theorem 17 (**[55]**). *The asymptotic formula*

$$\sum_{n \le X} \mu_k(n)\mu_k(n+h) = \prod_p \left(1 - \frac{2}{p^k}\right) \prod_{p^k | h} \left(\frac{p^k - 1}{p^k - 2}\right) X + \mathcal{O}\left(X^{\omega(k)+\varepsilon}\right)$$

holds. Here

$$\omega(k) = \begin{cases} \frac{26+\sqrt{433}}{81} & \text{if } k = 2, \\ \frac{169}{144k}, & \text{for } k \ge 3. \end{cases}$$

For intermediate results concerning the distribution of the pairs of k-free numbers of arbitrary type we refer to Brandes [5] and Dietmann and Marmon [11].

4.2 r-tuples of k-free numbers of arbitrary type

In 2014, Reuss [55], using a generalization of the approximate determinant method gave an asymptotic formula for r-tuples of k-free integers.

Theorem 18 (**[55]**). *Let $k \ge 2, r \ge 2$ and $l_i(x) = a_i x + b_i \in \mathbb{Z}[x]$ for $i = 1, ..., r$ such that $a_i b_j - a_j b_i \ne 0$ and $a_i \ne 0$ for all i, j with $1 \le i, j \le r$ and $i \ne j$. Then define*

$$\rho(p) = \#\{n (\text{mod } p^k) : p^k \,| l_i(n) \text{ for some } i\},$$

and let

$$c = \prod_p \left(1 - \frac{\rho(p)}{p^k}\right).$$

If $N(x)$ is the number of integers $n \le x$ such that $l_1(n), ..., l_r(n)$ are all k-free. Then for any $\varepsilon > 0$ and any sufficiently large x we have that

$$N(x) = cx + \mathcal{O}_\varepsilon\left(x^{\frac{3}{2k+1}+\varepsilon}\right).$$

It should be pointed that the implied constant in Theorem 18 depends on the choice of the l_i and that the best reminder term up to now for $k = 2$ was $\mathcal{O}(x^{7/11+\varepsilon})$ (See Tsang [62]). Tsang's proof uses a form of the Rosser–Iwaniec sieve and the version of Theorem 17 due to Heath-Brown. It should be noted that even though Tsang's error term is weaker than Reuss's, his implied constants are uniform in r and $\max_i ||l_i||$.

For other results on r-tuples of k-free numbers of arbitrary type we refer the reader to [**?**].

4.3 r-tuples of k-free numbers of the form $p+\alpha_1, \cdots, p+\alpha_s$

In 2016, Hablizel [23] using the circle method evaluated the behavior of limit-periodic functions on primes on average.

As an application he showed that for arbitrary $\alpha_i \in \mathbb{N}_0$, $r_i \in \mathbb{N}_{>1}$ and $s \in \mathbb{N}$ the asymptotic formula

$$\sum_{p\leq x} \mu_{r_1}(p+\alpha_1)\cdots\mu_{r_s}(p+\alpha_s) = \prod_p \left(1-\frac{D^*(p)}{\varphi(p^{r_s})}\right)\frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

holds. Here $D^*(p)$ is a computable function of the prime p, depending on the choice of the numbers α_i and r_i .

A weaker result related to this was obtained by Dimitrov in [15].

4.4 Consecutive k-free numbers of the form $[n^c], [n^c]+1$

4.4.1 Consecutive square-free numbers of the form $[n^c], [n^c]+1$

In 2018, Dimitrov [13] using the method of Cao and Zhai [8] showed that for any fixed $1 < c < 22/13$ there exist infinitely many consecutive square-free numbers of the form $[n^c], [n^c]+1$. More precisely he proved the following:

Theorem 19 (**[13]).** *Let* $1 < c < 22/13$, $\gamma = c^{-1}$ *and* $0 < \varepsilon < (22\gamma-13)/5(14-\gamma)$ *is a sufficiently small constant. Then*

$$\sum_{n\leq X} \mu_2([n^c])\mu_2([n^c]+1) = \prod_p \left(1-\frac{2}{p^2}\right) X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right),$$

$$\sum_{p\leq X} \mu_2([p^c])\mu_2([p^c]+1) = \prod_p \left(1-\frac{2}{p^2}\right)\int_2^X \frac{dt}{\ln t} + \mathcal{O}\left(Xe^{-c_0\sqrt{\log X}}\right),$$

$$\sum_{\substack{n\leq X\\ \mu_2(n)=1}} \mu_2([n^c])\mu_2([n^c]+1) = \frac{6}{\pi^2}\prod_p \left(1-\frac{2}{p^2}\right) X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right),$$

where $c_0 > 0$ *is an absolute constant.*

His earlier result [12] covers the narrower range $1 < c < 7/6$.

4.4.2 Consecutive cube-free numbers of the form $[n^c], [n^c]+1$

In 2018, Dimitrov [14] using the method of Zhang and Li [67] showed that for any fixed $1 < c < 31/17$ there exist infinitely many consecutive cube-free numbers of the form $[n^c], [n^c]+1$. More precisely, he proved:

Theorem 20 ([14]). *Let* $1 < c < 31/17$, $\gamma = c^{-1}$ *and* $0 < \varepsilon < \min\{(31\gamma - 17)/(9-9\gamma)\,, 10^{-10}\}$ *is a sufficiently small constant. Then*

$$\sum_{n\le X}\mu_3([n^c])\mu_3([n^c]+1) = \prod_p\left(1-\frac{2}{p^3}\right)X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right)\,,$$

$$\sum_{p\le X}\mu_3([p^c])\mu_3([p^c]+1) = \prod_p\left(1-\frac{2}{p^3}\right)\int\limits_2^X\frac{dt}{\ln t} + \mathcal{O}\left(Xe^{-c_0\sqrt{\log X}}\right)\,,$$

$$\sum_{\substack{n\le X\\ \mu_3(n)=1}}\mu_3([n^c])\mu_3([n^c]+1) = \frac{1}{\zeta(3)}\prod_p\left(1-\frac{2}{p^3}\right)X + \mathcal{O}\left(X^{1-\varepsilon^2/2}\right)\,,$$

where $c_0 > 0$ *is an absolute constant.*

4.5 Consecutive square-free numbers of the form $x^2 + y^2 + 1, x^2 + y^2 + 2$

Recently Dimitrov [16] using the method of Tolev [61] showed that there exist infinitely many consecutive square-free numbers of the form x^2+y^2+1, x^2+y^2+2. He also gave an asymptotic formula for the number of pairs of positive integers $x, y \le H$ such that x^2+y^2+1, x^2+y^2+2 are square-free.

Theorem 21 ([16]). *The asymptotic formula*

$$\sum_{1\le x,y\le H}\mu_2(x^2+y^2+1)\,\mu_2(x^2+y^2+2) = \sigma H^2 + \mathcal{O}\left(H^{\frac{8}{5}+\varepsilon}\right)$$

holds. Here

$$\sigma = \prod_p\left(1-\frac{\lambda(p^2,1)+\lambda(1,p^2)}{p^4}\right)$$

and

$$\lambda(q_1,q_2) = \sum_{\substack{1\le x,y\le q_1q_2\\ x^2+y^2+1\equiv 0\,(q_1)\\ x^2+y^2+2\equiv 0\,(q_2)}} 1\,.$$

4.6 On the distribution of consecutive square-free primitive roots modulo p

Let p be an odd prime. For any integer n with $(n,p)=1$, the smallest positive integer f such that $n^f \equiv 1 \pmod p$ is called the exponent of n modulo p. If the exponent of n modulo p is $p-1$, then n is called a primitive root mod p.

Let $A(n)$ be the characteristic function of the square-free primitive roots modulo p. In 2015 Liu and Dong [45] investigated the distribution of consecutive square-free primitive roots modulo p as follows.

Theorem 22 (**[45]**). *Let p be an odd prime, and let $A(n)$ be the characteristic function of the square-free primitive roots modulo p. Then we have*

$$\sum_{n\le x} A(n)A(n+1) = x\frac{\varphi^2(p-1)}{(p-1)^2}\frac{p(p-2)}{p^2-2}\prod_{p_1}\left(1-\frac{2}{p_1^2}\right) + \mathcal{O}\Big(4^{\omega(p-1)}p^{-1/2}(\log p)x + 4^{\omega(p-1)}p^{1/4}(\log p)^{1/2}x^{1/2}\log x\Big),$$

where the $\mathcal{O}$-constant is absolute and $\omega(q)$ denotes the number of the distinct prime factors of q.

For results concerning the distribution of positive square-free primitive roots modulo p not exceeding x we refer to Liu and Zhang [44] and Shapiro [59].

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