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Extension factor: Definition, properties and problems. Part 1

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Abstract: A new arithmetic function, called "Extension Factor" is introduced and some of its properties are studied.
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1 Introduction

In a series of papers, published during the last 35 years, the authors introduced some new arithmetic functions. One of them was called "Restrictive Factor" (see, [2, 3]). For each natural number $n = \prod_{i=1}^{k} p_i^{\alpha_i}$, where $k, \alpha_1, \alpha_2, ..., \alpha_k \geq 1$ are natural numbers and $p_1, p_2, ..., p_k$ are different prime numbers,

$$RF(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1},$$

$$RF(1) = 1.$$

In the present paper, for each natural number n, having the above form, we will introduce a new arithmetic function, in some sense, opposite to the restrictive factor.

In the text, we will use also the definitions of the following three well-known arithmetic functions:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i - 1),\ \varphi(1) = 1 - \text{Euler's totient function},$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}.(p_i + 1),\ \psi(1) = 1 - \text{Dedekind's function},$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1},\ \sigma(1) = 1$$

(see [4, 7]).

We will use also the arithmetic functions

$$\underline{\text{mult}}(n) = \prod_{i=1}^{k} p_i,\ \underline{\text{mult}}(1) = 1,$$

$$B(n) = \sum_{i=1}^{k} \alpha_i.p_i,\ B(1) = 1,$$

(see [1, 7]), and

$$\delta(n) = \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1} ... p_{i-1}^{\alpha_{i-1}} p_i^{\alpha_i - 1} p_{i+1}^{\alpha_{i+1}} ... p_k^{\alpha_k},\ \delta(1) = 1.$$

(see [1]).

2 Main results

Here, we juxtapose to the natural number n the (natural) number

$$EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1},\ EF(1) = 1$$

that we call *Extension Factor*.

Hence,

$$EF(n) = n.\underline{\text{mult}}(n).$$

The first 40 values of EF are given in Table 1.

If $(m, n) = 1$, where for the natural numbers m, n, (m, n) is the Greatest Common Divisor (GCD), then

$$EF(m.n) = EF(m).EF(n),$$

i.e., EF is a multiplicative function,

$$EF(n) = \prod_{i=1}^{k} EF(p_i^{\alpha_i}),$$

n	$EF(n)$	n	$EF(n)$	n	$EF(n)$	n	$EF(n)$
1	1	11	121	21	441	31	961
2	4	12	72	22	484	32	64
3	9	13	169	23	529	33	1089
4	8	14	196	24	144	34	1156
5	25	15	225	25	125	35	1225
6	36	16	32	26	676	36	216
7	49	17	289	27	81	37	1369
8	16	18	108	28	392	38	1444
9	27	19	361	29	841	39	15321
10	100	20	200	30	900	40	400

Table 1

and

$$EF(n) = EF\left(\prod_{i=1}^{k} p_i^{\alpha_i}\right) = \prod_{i=1}^{k} p_i^{\alpha_i+1} \leq \prod_{i=1}^{k} p_i^{2\alpha_i} = n^2.$$

On the other hand, it can be seen that if for every i $(1 \leq i \leq k)$ $\alpha_i = 1$, then

$$EF(n) = n^2.$$

Therefore, for each prime number p:

$$EF(p) = p^2.$$

Moreover, for every natural number n:

$$\underline{\text{mult}}(n^2) \leq EF(n) \leq n^2.$$

From the definitions of functions RF and EF it follows the basic identity

$$EF(n).RF(n) = n^2. \tag{1}$$

Therefore, $EF(n) = n^2$ if and only if $RF(n) = 1$, i.e. when $n = \underline{\text{mult}}(n)$, so when n is a squarefree number.

Theorem 1. *For every two natural numbers m and n:*

$$EF(m).EF(n) = EF(m.n).\underline{\text{mult}}((m.n)).$$

Proof. Let $(m, n) = r \geq 1$ and let $m = s.r, n = t.r$. Then

$$EF(m).EF(n) = EF(s.r).EF(t.r) = (s.r.\underline{\text{mult}}(s.r)).(t.r.\underline{\text{mult}}(t.r))$$

$$= s.r^2.t.\underline{\text{mult}}(s).\underline{\text{mult}}(r)^2.\underline{\text{mult}}(t) = (s.r^2.t.\underline{\text{mult}}(s).\underline{\text{mult}}(r).\underline{\text{mult}}(t)).\underline{\text{mult}}(r)$$

$$= EF(m.n).\underline{\text{mult}}(r) = EF(m.n).\underline{\text{mult}}((m.n)).$$

□

Theorem 1 follows also from the definitions, and the following property of the function $\underline{\text{mult}}$:

$$\underline{\text{mult}}(n).\underline{\text{mult}}(m) = \underline{\text{mult}}(mn).\underline{\text{mult}}((m, n)).$$

Theorem 2. *For every natural number n:*

$$RF(EF(n)) = n \geq EF(RF(n)).$$

Proof. For $n = 1$, the statement is true. Let $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ and let for each real number x

$$\text{sg}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases}$$

Then

$$RF(EF(n)) = RF\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right)$$

$$= \prod_{i=1}^{k} p_i^{\alpha_i} = n \geq \prod_{i=1}^{k} p_i^{\alpha_i.\text{sg}(\alpha_i-1)}$$

(so, we eliminate the prime numbers with power 1)

$$= EF\left(\prod_{i=1}^{k} p_i^{\alpha_i-1}\right) = EF(RF(n)). \qquad \square$$

Another proof of Theorem 2 follows from:

$$\underline{\text{mult}}(n.\underline{\text{mult}}(n)) = n \tag{2}$$

and

$$\underline{\text{mult}}\left(\frac{n}{\underline{\text{mult}}(n)}\right) \leq n. \tag{3}$$

(2) follows from the fact that n and $\underline{\text{mult}}(n)$ have the same prime factors, while (3) from the fact that the prime factors of $\frac{n}{\underline{\text{mult}}(n)}$ are among the prime factors of n .

There is equality in (3) only when $n > 1$ is squarefull number (i.e. when from each prime power divisor p^a of n one has $a \geq 2$). Thus one has

$$\underline{\text{mult}}(RF(n)) \leq \underline{\text{mult}}(n)$$

and

$$\underline{\text{mult}}(EF(n)) = n$$

and the result follows.

It could be mentioned that there is equality in Theorem 2 only when n is squarefull.

Theorem 3. *For every natural number n:*

(a) $\varphi(EF(n)) = \varphi(n).\underline{\text{mult}}(n)$,

(b) $\psi(EF(n)) = \psi(n).\underline{\text{mult}}(n)$,

(c) $\sigma(EF(n)) \geq \sigma(n).\underline{\text{mult}}(n)$.

Proof. The statement is obviously true for $n = 1$. Let $n > 1$ be a natural number. Then

$$\varphi(EF(n)) = \varphi\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right) = \prod_{i=1}^{k} p_i^{\alpha_i}(p_i - 1) = \varphi(n).\underline{\text{mult}}(n),$$

i.e., (a) is valid. (b) is proved analogously, while the proof of (c) is the following.

$$\sigma(EF(n)) = \sigma\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+2} - 1}{p_i - 1} = \sigma(n).\prod_{i=1}^{k} \frac{p_i^{\alpha_i+2} - 1}{p_1^{\alpha_i+1} - 1} \geq \sigma(n).\underline{\text{mult}}(n). \qquad \square$$

Another proof of inequality (c) of Theorem 3 is based on the known inequality $\sigma(a.b) \geq a.\sigma(b)$, with equality only for $a = 1$. Let $a = \underline{\text{mult}}(n), b = n$, and the result follows.

When $a = n, b = \underline{\text{mult}}(n)$, one obtains another inequality:

$$\sigma(EF(n)) \geq n.\sigma(\underline{\text{mult}}(n)) = n.(p_1 + 1)...(p_k + 1),$$

where $p_1, ..., p_k$ are the distinct prime factors of n. Since

$$(p_1 + 1)...(p_k + 1) = \frac{\psi(n)}{RF(n)},$$

we get the inequality:

$$\sigma(EF(n)) \geq \frac{n\psi(n)}{RF(n)}.$$

Another result of this type is the following

Theorem 4. *For every natural number n:*

$$\sigma(EF(n)) \leq \frac{\sigma(n).\psi(n)}{RF(n)}.$$

Proof. For $n = 1$, the statement is obviously true. Applying the known inequality $\sigma(ab) \leq \sigma(a).\sigma(b)$ for $a = n$ and $b = \underline{\text{mult}}(n)$, and assuming the distinct prime factors of n to be $p_1, ..., p_k$, note that one has

$$\sigma(\underline{\text{mult}}(n)) = (p_1 + 1)...(p_r + 1) = \frac{\psi(n)}{RF(n)}.$$

The result follows by the definitions. $\square$

Theorem 5. *For every natural number $n > 1$:*

$$EF(n) > \sigma(n).$$

Proof. Let $n > 1$ be a natural number. Then

$$EF(n) = \prod_{i=1}^{k} p_i^{\alpha_i+1} > \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1} = \sigma(n). \qquad \square$$

The inequality of Theorem 5 can be improved when n is odd.

Theorem 6. *When $n > 1$ is odd number, then*

$$EF(n) > \sigma(n) + n.$$

Proof. Apply the known inequality $\sigma(n).\phi(n) < n^2$ (see e.g. [4, 7]). Thus, $\sigma(n) < \frac{n^2}{\phi(n)}$. Since $\frac{n}{\phi(n)} = \frac{p_1...p_k}{(p_1-1)...(p_k-1)}$, and $EF(n) - n = n(p_1...p_k - 1)$, it will be sufficient to prove that:

$$\frac{p_1...p_k}{(p_1-1)...(p_k-1)} \leq p_1...p_k - 1.$$

Put $p_i - 1 = x_i$. Since n is odd, one has $x_i \geq 2$ for all $i = 1, 2, ..., k$. We have to prove the inequality

$$x_1...x_k \leq (x_1+1)...(x_k+1)(x_1...x_k - 1),$$

or

$$x_1...x_k + (x_1+1)...(x_k+1) \leq x_1...x_k(x_1+1)...(x_k+1).$$

Put $x_1...x_k = a$, $(x_1+1)...(x_k+1) = b$. Then we have to prove that $a + b \leq a.b$, or, this can be written also as $(a-1)(b-1) \geq 1$. This is true, as $a - 1 \geq x_1 - 1 \geq 1$, and $b \geq x_1 + 1 \geq 3$. The inequality is strict. □

Now, we will formulate and prove the following common refinement of the last two theorems.

Theorem 7.

a) For any natural $n > 1$ one has

$$\sigma(n) < n(\omega(n) + 1) \leq EF(n) \tag{4}$$

b) For any odd $n > 1$ one has

$$\sigma(n) < n(\omega(n) + 1) \leq EF(n) - n, \tag{5}$$

where $\omega(n)$ denotes the number of distinct prime factors of n.

Proof. The first inequalities of both a) and b), namely

$$\sigma(n) < (\omega(n) + 1).n$$

appeared for the first time in paper [5] from 1989. A proof is included also in paper [6] from 2010.

Now, to prove the second inequality of (4), note that

$$\underline{\text{mult}}(n) = p_1...p_k \geq 2^k,$$

where $p_1, ..., p_k$ are the prime divisors of n, and $k = \omega(n)$. Now, $2^k \geq k + 1$ holds true for any $k \geq 1$. Thus (4) follows, as $EF(n) = n.\underline{\text{mult}}(n)$.

For the proof of second inequality of b), note that when $n > 1$ is odd, then $\underline{\text{mult}}(n) \geq 3^k$, as $p_1, ..., p_k \geq 3$. Now, the inequality $3^k \geq k + 2$ for $k \geq 1$ follows at once, e.g., by mathematical induction. This proves $\underline{\text{mult}}(n) \geq k + 2$, so (5) follows. □

Theorem 8. *For every natural number n:*

$$EF(n) + RF(n) \geq 2n,$$

with equality only for $n = 1$.

Proof. This follows from the classical inequality $x + y \geq= 2\sqrt{xy}$ applied for $x = EF(n), y = RF(n)$, and using the basic identity (1). □

Another simple related inequality is the following.

Theorem 9. *For every natural number $n > 1$:*

$$n^2 \geq \frac{EF(n)}{RF(n)} \geq 4^{\omega(n)},$$

where $\omega(n)$ is the number of distinct prime factors of n.

Proof. There is equality on the right only when n has a single prime factor, i.e., when $\omega(n) = 1$, and on the left, when n is squarefree number. This follows from

$$\frac{EF(n)}{RF(n)} = (\underline{\text{mult}}(n))^2.$$

Now, from $\underline{\text{mult}}(n) \leq n$, the left side inequality follows. For the right side, note that $\underline{\text{mult}}(n) \geq 2^k$ as any prime divisor is greater or equal to 2. □

Theorem 10. *For every natural number $n > 1$:*

$$B(EF(n)) = B(n) + B(\underline{\text{mult}}(n)).$$

Proof. Let $n > 1$ be a natural number. Then

$$\begin{aligned} B(EF(n)) = B(\prod_{i=1}^{k} p_i^{\alpha_i+1}) &= \sum_{i=1}^{k}(\alpha_i + 1).p_i = \sum_{i=1}^{k}(\alpha_i).p_i + \sum_{i=1}^{k} p_i \\ &= B(n) + B(\underline{\text{mult}}(n)). \end{aligned}$$

□

Theorem 11. *For every natural number $n > 1$:*

$$\delta(EF(n)) = \delta(n)\underline{\text{mult}}(n) + n\delta(\underline{\text{mult}}(n)).$$

Proof. Let $n > 1$ be a natural number. Then

$$\delta(EF(n)) = \delta\left(\prod_{i=1}^{k} p_i^{\alpha_i+1}\right) = \sum_{i=1}^{k}(\alpha_i + 1)p_1^{\alpha_1+1}...p_{i-1}^{\alpha_{i-1}+1}p_i^{\alpha_i}p_{i+1}^{\alpha_{i+1}+1}...p_k^{\alpha_k+1}$$

$$= \sum_{i=1}^{k} \alpha_i p_1^{\alpha_1+1}...p_{i-1}^{\alpha_{i-1}+1}p_i^{\alpha_i}p_{i+1}^{\alpha_{i+1}+1}...p_k^{\alpha_k+1} + \sum_{i=1}^{k} p_1^{\alpha_1+1}...p_{i-1}^{\alpha_{i-1}+1}p_i^{\alpha_i}p_{i+1}^{\alpha_{i+1}+1}...p_k^{\alpha_k+1}$$

$$= \delta(n)\underline{\text{mult}}(n) + n\delta(\underline{\text{mult}}(n)).$$

□

3 Conclusion

In conclusion, we will mention, that in the second part we will study the following problems.

Problem 1. To find other equalities and inequalities related to function EF.

Problem 2. To generalize the function EF to EF_s, so that for each natural number n : $EF_1(n) = EF(n)$.

Problem 3. To study the properties of EF_s.

References

[1] Atanassov, K. (1987). New integer functions, related to φ and σ functions, *Bulletin of Number Theory and Related Topics*, XI (1), 3-26.

[2] Atanassov, K. (2002). Restrictive factor: definition, properties and problems. *Notes on Number Theory and Discrete Mathematics*, 8(4), 117-119.

[3] Atanassov, K. (2016) On function "Restrictive factor", *Notes on Number Theory and Discrete Mathematics*,22(2), 17–22.

[4] Mitrinović, D. S. & Sándor, J., & Crstici, B. (1995). *Handbook of number theory*, Kluwer Acad. Publ.

[5] Sándor, J. (1989). On some Diophantine equations for particular arithmetic functions, *Seminarul de Teoria Structurilor, Univ. Timisoara, Romania*, 53, 1-10.

[6] Sándor, J. (2010). Two arithmetic inequalities, *Advanced Studies in Contemporary Mathematics*, 20(2), 197-202.

[7] Sándor, J. et al., *Handbook of number theory I*, Springer Verlag, 2005 (First printing 1995 by Kluwer Acad. Publ.)