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Applications of Mollie Horadam's generalized integers to Fermatian and Fibonacci numbers

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Abstract: This paper extends some of the arithmetic functions which Mollie Horadam developed for sequences of generalized integers and apply them to some particular integer sequences, particularly the Fibonacci and Fermatian numbers.
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1 Introduction

In a series of papers [9–16], the late Mollie Horadam, wife of Alwyn F. Horadam and mother of Kathie Horadam (another Professor of Mathematics in Australia), extended the classic arithmetic functions of number theory to generalizations of the integers. We shall further extend some of these function and consider Fermatian and Fibonacci numbers as examples of divisibility sequences $\{u_n\}$ with, for instance, $\theta(u_n)$, $\varphi(u_n, u_m)$, $\mu(u_n)$, (u_n, u_m) as analogues of ordinary arithmetical functions.

The approach of Mollie Horadam [16] and others, particularly David Daykin [6], was to start with generalized primes as a foundation for generalized integers. Here, though the approach is to start with divisibility sequences, the elements of which we consider as generalized integers, and to define some of their elements as generalized primes. That is, up is a generalized prime (modulo a divisibility sequence) if its only divisors are itself and unity (within the system).

Our generalized integers cannot necessarily be represented as a product of distinct generalized primes, as we shall see. Thus new analogues of the classical arithmetical functions are needed in order to study these divisibility sequences as generalizations of integers. This is not to say that we cannot redefine our generalized primes to fit Mollie Horadam's theory but it leads to unnecessarily complicated questions about the distribution of these generalized primes within a divisibility sequence.

2 Fermatian numbers

In a series of papers [9–16], the late Mollie Horadam, wife of Alwyn F. Horadam and mother of Kathie Horadam (another Professor of Mathematics in Australia), extended the classic

$$\underline{z}_n = \begin{cases} -z^n \underline{z}_n & (n<0) \\ 1+z+z^2+\ldots z^{n-1} & (n>0) \\ 1 & (n=0) \end{cases} \tag{2.1}$$

so that

$$\underline{1}_n = n. \tag{2.2}$$

and

$$\underline{1}_n! = n!, \tag{2.3}$$

where

$$\underline{z}_n! = \underline{z}_n \underline{z}_{n-1} \cdots_z \underline{z}_1 \tag{2.4}$$

For example, if we consider the Fermatian numbers of index 2, we have $\underline{2}_2 = 3$, $\underline{2}_3 = 7$, $\underline{2}_4 = 15$, $\underline{2}_6 = 63$, $\underline{2}_8 = 255$, so that $\underline{2}_2$, $\underline{2}_3$ and $\underline{2}_4$ are generalized Fermatian primes, and $\underline{2}_6 = (\underline{2}_2)^2\underline{2}_3$, but $\underline{2}_8$ cannot be represented as a product of Fermatian numbers of index 2. Some properties of these numbers may be found in [18] and Carlitz and Moser [4]. Carlitz has also used Fermatian numbers in the development of q-Bernoulli numbers and polynomials [2]. The first ten Fermatian numbers of the first ten indices are displayed in Table 1.

Index ⇓	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	1	3	7	15	31	63	127	255	511	1023
3	1	4	13	40	121	364	1093	3280	9841	29524
4	1	5	21	85	341	1365	5461	21845	87381	349525
5	1	6	31	156	781	3906	19531	97656	488281	2441406
6	1	7	43	259	1555	9331	55987	335923	2015539	12093235
7	1	8	57	400	2801	19608	137257	960800	6725601	47079208
8	1	9	73	585	4681	37449	299593	2396745	19173961	153391689
9	1	10	91	820	7381	66430	597871	5380840	48427561	435848050
10	1	11	111	1111	11111	111111	1111111	11111111	111111111	1111111111

Table 1. First 10 Fermatian numbers of the first 10 indices

The corresponding row and column sequences, $\{\underline{z}_n\}_{n=1}^{\infty}$, $\{\underline{z}_n\}_{z=1}^{\infty}$, are obvious from their construction, but the sequence, $\left\{\sum_{n=1}^{z-1}\underline{(z-n)}_n\right\} \equiv \{1,3,6,11,21,45,105,315,1058,...\}$, formed from adding along the forward diagonals, does not seem to be well-known [20].

3 Some arithmetical functions

When dealing with ordinary integers we note that

$$\delta(m,s) = \begin{cases} 1, & m \mid s, \\ 0, & m \nmid s, \end{cases}, \tag{3.1}$$

has different interpretations in Hardy and Wright [7] and Horadam [15], namely

$$\begin{aligned} \delta(m,s) &= g(s,m)/m \quad [15] \\ &= \begin{cases} g(m)/m \\ E_s(m+1). \end{cases} \quad [7] \end{aligned}$$

With (3.1) we shall use

$$\rho(n,s) = \begin{cases} 0, & \mathit{if}\ \exists j : j \mid (n,s), 1 < j < n, \\ 1, & \mathit{otherwise}, \end{cases}, \tag{3.2}$$

and $\mu(n)$, the Möbius multiplicative function, defined for all positive integers n by

$$\mu(n) = \begin{cases} 1 \\ -1 \text{ if } n \text{ is} \\ 0 \end{cases} \begin{cases} \mathit{square-free\ \&\ positive\ with\ an\ even\ number\ of\ prime\ factors}, \\ \mathit{square-free\ \&\ positive\ with\ an\ odd\ number\ of\ prime\ factors}, \\ \mathit{not\ square-free}, \end{cases}$$

and $\mu(1) = 1$. Examples of $\rho(n, s)$ are set out in Table 2 which illustrates that it can detect primes for numbers in this range:

$n\rightarrow$ / $s\downarrow$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**
1	1	1	1	1	1	1	1	1
2	1	1	1	0	1	0	1	0
3	1	1	1	1	1	0	1	1
4	1	1	1	0	1	0	1	0
5	1	1	1	1	1	1	1	1
6	1	1	1	0	1	0	1	0
7	1	1	1	1	1	1	1	1
8	1	1	1	0	1	0	1	0

Table 2. $\rho(n, s)$ for n, s = 1, 2, …, 8

We shall also use

$$\zeta_m = \exp(2\pi i / m).$$

We note that

$$\sum_{n \le m} \zeta_m^{ns} = g(s,m)$$
$$= m\delta(m,s)$$

and

$$\sum_{\substack{n \le m \\ (n,m)=1}} \zeta_m^{ns} = c(s,m)$$

Ramanujan's sum, which are proved in Hardy and Wright. Of marginal relevance to some of Mollie Horadam's work is the neat definition of generalized (order r) Lucas numbers by Williams [21] and a modification of Carlitz [3]:

$$L_{s,n}^{(r)} = \frac{1}{d}\sum_{j=1}^{r} \alpha_{r,j}^{n} \zeta_r^{s(j-1)}, \; s = 0,1,2,\ldots,r-1,$$

in which d is some real number for $r > 2$. When $r = 2$, $d = \alpha_{2,1} - \alpha_{2,2}$, and

$$L_{0,n}^{(2)} = \alpha_{2,1}^{n} + \alpha_{2,2}^{n}$$

the primordial numbers of Lucas [17].

4 Some generalised integer divisors

For notational convenience with generalized integers $\{u_n\}$, we let

$$\delta(u_m, u_s) = \delta\{m,s\} = \begin{cases} 1 & u_m \mid u_s, \\ 0 & u_m \nmid u_s, \end{cases} \tag{4.1}$$

and

$$\rho(u_m, u_s) = \rho\{m,s\} = \begin{cases} 0 & \textit{if } \exists u_j : u_j \mid u_s, u_1 < u_j < u_m, \\ 1 & \textit{otherwise.} \end{cases} \tag{4.2}$$

For u_d, u_j, $u_k \in \{u_n\}$, we call $u_d = (u_j, u_k)$ the greatest common divisor of u_j and u_k, if

a) $u_d \mid u_j$ and $u_d \mid u_k$,

b) every $u_c \mid u_d$ if $u_c \mid u_j$ and $u_c \mid u_k$, $u_c \in \{u_n\}$.

When $u_1 = (u_j, u_k)$, shall say that u_j and u_k are generalized co-primes. A formula for the greatest common divisor is then given by

$$(u_s, u_t) = \max\{u_m \delta\{m,s\}, \delta\{m,t\}\}, \; 1 \le m \le \min\{s,t\}, \text{ for each } u_m \mid \min\{s,t\}. \tag{4.3}$$

For example, for the Fibonacci numbers: $(F_4, F_8) = (3, 21) = \max\{1, 2, 4\} = 3$, and also for the Fermatian numbers: $(\underline{3}_4, \underline{3}_8) = (40, 3280) = \max\{1, 2, 4\} = 40$.

Proof of (4.3). For each $\{(u_m \mid u_s) \wedge (u_m \mid u_t)\}$, $1 \le m \le \min\{s,t\}$, $u_m = u_m \delta(m,s)\delta(m,s)$, and so the maximum of all these will be the greatest common divisor of u_s and u_t. □

We now assert that $\rho\{m,s\}\delta\{m,s\} = 1$ *iff* u_m is unity or the smallest prime factor of u_s.

Proof. If u_m is the smallest prime factor of u_s, then $\delta\ \{m,s\} = 1$ because $u_m \mid u_s$, and $\rho\{m,s\} = 1$ because $\nexists u_j \mid u_s : 1 < u_j < u_m$. Also $\rho\{1,s\}\delta\{1,s\} \forall s$.

If $\rho\{m,s\}\delta\{m,s\} \ne 1$, then $\rho\{m,s\}\delta\{m,s\} = 0$, and either

(a) $\rho\{m,s\} = 0$, which implies that $\exists u_j : 1 < u_j < u_m$, $u_j \mid u_s$, or

(b) $\delta\{m,s\} = 0$, which implies that $u_m \nmid u_s$,

or both (a) and (b), all of which imply that u_m is not unity or is not the smallest prime divisor. □

For example, for the Fermatian numbers when $q = 2$,

$$\rho\{2,6\} = \rho\{3,63\} = 1,$$
$$\delta\{2,6\} = \delta\{3,63\} = 1,$$

so

$$\rho\{2,6\} = \delta\{2,6\} = 1;$$

but $\rho\{3,6\} = \rho\{7,63\} = 0,$ since $1 < 3 < 7$ and $3 \mid 63$, so

$$\rho\{3,6\}\delta\{3,6\} = 0.$$

Also

$$\rho\{7,7,\} = \rho\{127,127\} = 1,$$

since $\nexists \underline{2}_n : 1 < n < 7 : \underline{2}_n \mid 127$, and so

$$\rho\{7,7\}\delta\{7,7\} = 1,$$

but

$$\delta\{3,7\} = \delta\{7,127\} = 0,$$

so

$$\rho\{3,7\}\delta\{3,7\} = 0.$$

5 Sums of generalized divisors

Let

$$\sigma_k(u_n) = \sum_{u_d \mid u_n} u_d^k \tag{5.1}$$

so that $\sigma_0(u_n)$ represents the number of divisors u_d of u_n, where u_d, $u_n \in \{u_n\}$. We can also define a generalization of the greatest integer function, $[u_s / u_m]$, by the recurrence relation:

$$[u_s / u_m] - [u_{s-1} / u_m] = \delta\{m,s\},\ [u_0 / u_m] = 0. \tag{5.2}$$

Thus, when $\{u_n\} = \{n\}$, $[u_s / u_m] = [s / m]$, the ordinary greatest integer function.

Then, analogous to the well-known

$$\sum_{m=1}^{n} [n / m] = \sum_{s=1}^{n} \sigma_0(s),$$

we have

$$\sum_{m=1}^{n} [u_n / u_m] = \sum_{s=1}^{n} \sigma_0(u_s), \text{ if } [u_{j-1} / u_j] = 0 \, \forall j. \tag{5.3}$$

Proof.

$$\begin{aligned}
\sigma_0(u_s) &= \sum_{m=1}^{s} \delta\{m, s\} \\
&= \sum_{m=1}^{s} ([u_s / u_m] - [u_{s-1} / u_m]).
\end{aligned}$$

$$\begin{aligned}
\sum_{s=1}^{n} \sigma_0(u_s) &= \sum_{m=1}^{n} \left(\sum_{s=m}^{n} [u_s / u_m] - [u_{s-1} / u_m] \right) \\
&= \sum_{m=1}^{n} \left(\sum_{s=m}^{n} \left[\frac{u_s}{u_m} \right] - \sum_{s=m-1}^{n-1} \left[\frac{u_s}{u_m} \right] \right) \\
&= \sum_{m=1}^{n} \left(\left[\frac{u_n}{u_m} \right] - \left[\frac{u_{m-1}}{u_m} \right] \right) \\
&= \sum_{m=1}^{n} \left[\frac{u_n}{u_m} \right].
\end{aligned}$$

□

Note that

$$\left[\frac{u_s}{u_m} \right] = \sum_{j=1}^{s} \delta\{m, j\} \tag{5.4}$$

Proof.

$$\left[\frac{u_s}{u_m} \right] - \left[\frac{u_{s-1}}{u_m} \right] = \delta\{m, s\}$$

$$\left[\frac{u_{s-1}}{u_m} \right] - \left[\frac{u_{s-2}}{u_m} \right] = \delta\{m, s-1\}$$

$$\cdots$$

$$\left[\frac{u_1}{u_m} \right] - \left[\frac{u_0}{u_m} \right] = \delta\{m, 1\}.$$

On adding the corresponding sides of these equations we get the result we seek, since $\left[{}^{u_0}\!/_{u_m} \right] = 0.$ □

A corollary of this is that $[u_s]=\sum_{j=1}^{s}\delta\{1,j\}$, which is the number of the elements of the sequence which are less than or equal to u_s. We can then obtain that

$$\left\lfloor\frac{u_s}{u_m}\right\rfloor-\left\lfloor\frac{u_{s-1}}{u_m}\right\rfloor=\int_m^{m+1}\delta\{\lfloor y\rfloor,s\}dy. \tag{5.5}$$

Proof.

$$\begin{aligned}\int_m^{m+1}\delta\{\lfloor y\rfloor,s\}dy&=\begin{cases}\int_m^{m+1}dy, & u_m \mid u_s,\\ 0, & u_m \nmid u_s,\end{cases}\\&=\begin{cases}1, & u_m \mid u_s,\\ 0, & u_m \nmid u_s,\end{cases}\\&=\delta\{m,s\}\\&=\left\lfloor\frac{u_s}{u_m}\right\rfloor-\left\lfloor\frac{u_{s-1}}{u_m}\right\rfloor.\end{aligned}$$

□

For ordinary integers this becomes

$$\left\lfloor\frac{s}{m}\right\rfloor-\left\lfloor\frac{s-1}{m}\right\rfloor=\int_m^{m+1}\delta\{\lfloor y\rfloor,s\}dy.$$

Thus,

$$\sigma_0(u_n)=\int_n^{n+1}\int_1^{n+1}\delta\{\lfloor y\rfloor,\lfloor x\rfloor\}dydx. \tag{5.6}$$

Proof.

$$\begin{aligned}\sigma_0(u_n)&=\sum_{u_d\mid u_n}1\\&=\sum_{i=1}^{n}\delta\{i,n\}\\&=\sum_{i=1}^{n}\int_i^{i+1}\delta\{\lfloor y\rfloor,n\}\\&=\int_n^{n+1}\int_1^{n+1}\delta\{\lfloor y\rfloor,\lfloor x\rfloor\}dydx.\end{aligned}$$

□

For example, for the sequence of Fibonacci numbers, since the divisors of $F_4 = 3$, are F_1, F_2 and F_4, $\sigma_0(F_4)=3$, and

$$\begin{aligned}\int_4^5\int_1^5\delta\{\lfloor y\rfloor,\lfloor x\rfloor\}dydx&=\int_4^5\left(\int_1^3 dy+\int_4^5 dy\right)dx\\&=3\int_4^5 dx\\&=3.\end{aligned}$$

For ordinary integers, the result (5.6) becomes similar to a result of Graeme Cohen, formerly Editor of the *Bulletin of the Australian Mathematical Society*, namely

$$\sigma_0(n) = \int_n^{n+1}\int_1^{n+1} \delta(\lfloor y \rfloor, \lfloor x \rfloor) dydx.$$

Cohen [5] has also considered aspects of Gaussian integers as generalized integers.

6 Prime divisors

Let $w\{u_n\}$ denote the number of prime divisors of u_n as in [9]. For notational convenience, let $w\{1\} = 1$. Then

$$w\{n\} = \sum_{1<u_i<u_n} \rho\{i,i\}\delta\{i,n\}. \tag{6.1}$$

Proof. Let

$$u_n = \prod_{j=1}^{s} u_{p_j}^{a_g N}$$

in which $(N = 1) \vee (N \notin \{u_n\})$. Then

$$\begin{aligned}\sum_{1<u_i<u_n} \rho\{i,i\}\delta\{i,n\} &= \sum_{j=1}^{n-1} \rho(p,p)\delta\{j,n\} \\ &= w\{n\}\end{aligned}$$

where $p \equiv u_{pj}^{a_j}$, because $\rho\{p,p\} = 0$ except when $a_j = 1$, since $u_{p_j} \mid u_{p_j}^{a_j}$, $u_1 < u_{p_j} < u_{p_j}^{a_j}$ for $a_j > 1$. □

Examples follow:

(i) For the ordinary integers when $n = 12$:

$$\begin{aligned} w(12) &= \sum_{i=2}^{11} \rho\{i,i\}\delta\{i,12\} \\ &= \rho(2,2)\delta(2,12) + \rho(3,3)\delta(3,12) + \\ &\quad \rho(4,4)\delta(4,12) + \rho(6,6)\delta(6,12) \\ &= 2 \end{aligned}$$

Since in Table 1, $\rho(2,2) = \rho(3,3) = 1, \rho(4,4) = \rho(6,6) = 0.$

(ii) For the Fibonacci numbers, where $F_8 = 21 = 7F_4$, and

$$\begin{aligned} w\{8\} &= \sum_{1<F_1<F_8} \rho\{i,i\}\delta\{i,8\} \\ &= \rho\{4,4\}\delta\{4,8\} \\ &= 1. \end{aligned}$$

We note that we can express $w\{n\}$ similarly to (5.6) as

$$w\{n\} = \int_n^{n+1}\int_2^{n+1} \rho\{j,j\}\delta\{\lfloor y \rfloor, \lfloor x \rfloor\} dydx. \tag{6.2}$$

We now define an analogue of the Möbius function

$$\mu(u_n)=\begin{cases}0, & u_p^2 \mid u_n, \\ (-1)^{w\{n\}}, & otherwise.\end{cases} \quad (6.3)$$

For example, for the Fibonacci numbers:

$$\mu(F_3)=\mu(2)=(-1)^1=1,\ \mu(F_1)=\mu(1)=(-1)^0=1.$$

Then

$$\mu(u_n)=(-1)^{w\{n\}}\left(1-\delta\left(u_p^2,u_n\right)\right) \quad (6.4)$$

Proof: When n = 1, u_n = 1, and

$$\mu(u_1)=(-1)^{w\{1\}}\left(1-\delta\left(u_p^2,u_1\right)\right)$$

When $n = p$> 1, $u_n = u_p$, and

$$\begin{aligned}\mu(u_p)&=(-1)^{w(u_p)}\left(1-\delta\left(u_p^2,u_p\right)\right)\\&=-1.\end{aligned}$$

$$u_n=u_p^aN, a>1, \delta\left(u_p^2,u_n\right)=1$$

and

$$\mu(u_n)=0. \quad \square$$

If (u_m, u_n) = 1, then $\mu(u_n)$ is multiplicative, that is,

$$\mu(u_n)\mu(u_m)=\mu(u_nu_m); \quad (6.5)$$

Proof: If u_m and u_n have no common divisors in the sequence $\{u_n\}$, then

$$w\{mn\}=w\{m\}+w\{n\}.$$

Analogous to the Mőbius function theorem we have

$$\sum_{\substack{u_du_c=u_n\\(u_d,u_c)=1}}\mu(u_d)=\left.\begin{cases}1, & n=1,\\0, & n>1,\end{cases}\right\}=e(n). \quad (6.6)$$

Proof: If n = 1, then

$$\begin{aligned}\mu(u_d)&=\mu(u_1)\\&=(-1)^0\\&=1.\end{aligned}$$

If n > 1, then

$$\begin{aligned}\mu(u_n)&=\mu\left(\prod_{i=1}^{s}u_{p_i}^{a_i}\right)\\&=\prod_{i=1}^{s}\mu\left(u_{p_i}^{a_i}\right)\end{aligned}$$

$$\begin{aligned}\sum_{\substack{u_du_c=u_{p_a}^{a_i}\\(u_d,u_c)=1}}\mu(u_d)&=\mu(u_1)+\sum\mu\left(u_{p_i}^{a_i}\right)\\&=1+(-1)^1\\&=0.\end{aligned} \quad \square$$

In the same spirit we now define $\varphi(u_n)$ to be the number of elements of the set $\{u_1, u_2, ..., u_{n-1}\}$ which are co-prime with u_n. For instance, if $\{u_n\} = \{1, 2, 3, 1, 2, 3, ...\}$, then the co-primes of u_5 are $u_1 = 1$, $u_3 = 3$, $u_3 = 3$, $u_4 = 1$, and so $\varphi(u_5) = 3$. We are then able to assert the following theorem:

$$\varphi(u_n) = n - 1 - \psi(u_n)$$

in which

$$\psi(u_n) = \sum_{\substack{m=2 \\ u_m \neq 1}}^{n-1} \sum_{s=2}^{n-1} \delta\{m,s\}\delta\{m,n\}\rho\{m,s\}. \qquad (6.7)$$

Proof. Let $A = \{u_1, u_2, ..., u_n\}$. Then for each m

- $\sum_{m=1}^{n-1} \sum_{s=1}^{n-1} \delta\{m,s\}\delta\{m,n\}$ represents the number of elements $u_s,\ u_1 \leq u_s < u_n : (u_s, u_n) = u_m \geq 1$;
- $\sum_{\substack{m=2 \\ u_m \neq 1}}^{n-1} \sum_{s=2}^{n-1} \delta\{m,s\}\delta\{m,n\}$ represents the number of elements $u_s,\ u_1 \leq u_s < u_n : (u_s, u_n) = u_m > 1$;
- $\sum_{\substack{m=2 \\ u_m \neq 1}}^{n-1} \sum_{s=2}^{n-1} \delta\{m,s\}\delta\{m,n\}\rho\{m,s\}$: number of distinct elements $u_s,\ u_1 \leq u_s < u_n$: $(u_s, u_n) = u_m > 1$.

Thus $n - 1$ represents the number of distinct elements of A which are co-prime with u_n, and this is $\varphi(u_n)$; that is,

$$\varphi(u_n) = n - 1 - \psi(u_n). \qquad \square$$

Observe that $\psi(u_p) = 0$ only when u_p is a generalized prime. More generally, we note that the problem of Lehmer, discussed by Alter [1], of whether there exists a composite integer n and integer $k > 1$ so that the equation

$$k\varphi(n) = n - 1$$

has a solution can be made equivalent to finding whether there is a solution to

$$j\varphi(n) = \psi(n)$$

for a composite integer n and a positive integer j.

Examples of the use of (6.7) now follow in (i), (ii) and (iii) below.

(i) For $\{u_n\} = \{1, 2, 3, 1, 2, 3, \ldots\}$, $u_1 = 1$, $u_2 = 2$, $u_3 = 3$, $u_4 = 1$, $u_5 = 2$, and 1, 1, 3, 1 are co-prime with 2, so that

$$\varphi(u_5) = 3$$

$$\psi(u_5) = \sum_{\substack{m=2 \\ u_m \neq 1}}^{4} \sum_{s=2}^{4} \rho\{m,s\}\delta\{m,s\}\delta\{m,5\}$$

$$= \sum_{s=2}^{4} \rho\{2,s\}\delta\{2,s\}$$

$$= \rho\{2,2\}$$

$$= 1;$$

that is,

$$\varphi(u_5) = n - 1 - \psi(u_5)$$

$$= 5 - 1 - 1.$$

(ii) For $\{F_n\}$, $F_1 = 1$, $F_2 = 1$, $F_4 = 3$, $F_5 = 5$ are co-prime with $F_6 = 8$, so that

$$\varphi(F_6) = 4$$

$$\psi(F_6) = \sum_{\substack{m=2 \\ F_m \neq 1}}^{5} \sum_{s=2}^{5} \rho\{m,s\}\delta\{m,s\}\delta\{m,6\}$$

$$= \sum_{s=2}^{5} \rho\{3,s\}\delta\{3,s\}$$

$$= \rho\{3,3\}$$

$$= 1;$$

so that

$$\varphi(F_6) = n - 1 - \psi(F_6)$$

$$= 6 - 1 - 1.$$

(iii) For Z, 1,5,7,11 are co-prime with 12, so that

$$\varphi(12) = 4$$

$$\psi(12) = \sum_{m=2}^{11} \sum_{s=2}^{11} \rho(m,s)\delta(m,s)\delta(m,12)$$

$$= \rho(2,2)\delta(2,2) + \rho(2,4)\delta(2,4) + \rho(2,6)\delta(2,6) + \rho(2,8)\delta(2,8) + \rho(2,10)\delta(2,10) + \rho(3,3)\delta(3,3) + \rho(3,9)\delta(3,9)$$

$$= 7,$$

$$\varphi(12) = 12 - 1 - \psi(12)$$

$$= 4.$$

A consequence of this is that for $u_p \neq 1$ if $p > 2$:

$$\varphi(u_p) = p - 1,$$

Since

$$\delta\{m, p\} = 0,\ m > 1.$$

Furthermore, if $u_p \neq 1$ if $p > 2$, then u_p is a generalized prime iff p is an ordinary prime.

Proof: $u_n \mid u_p$ iff $n \mid p$ iff $n = 1, p$ ($u_p \neq 1$ if $p > 2$), and thus $u_n \mid u_p$ iff $u_n = u_1 = 1$ or $u_n = u_p$, which is a result used in a proof of a generalized Staudt–Clausen theorem [8]. □

7 Concluding comments

Analogues of other functions can be similarly defined; for example, the Nagell totient function for elements of a divisibility sequence. We define $\theta(u_n, u_t)$ as the number of elements u_j of the set $\{u_1, u_2, \ldots, u_t\}$, $t \leq n$, such that

$$(u_{n-j}, u_t) = (u_j, u_t) = 1.$$

We then have

$$\theta(u_n, u_t) = \sum_{j=1}^{t} \delta((u_{n-j}, u_t), u_1)\delta((u_j, u_t), u_1). \tag{7.1}$$

Proof. The two delta functions in the summation are both unity only when $(u_{n-j}, u_t) \mid u_1$ and when $(u_j, u_t) \mid u_1$; that is, only when

$$(u_{n-j}, u_t) = (u_j, u_t) = 1.$$

When we sum over all j up to and including t we have the number of elements which satisfy the conditions of Nagell's function for divisibility sequences. □

As a corollary we have

$$\varphi(u_n) = \theta(u_n, u_n) \tag{7.2}$$

$$= \sum_{j=1}^{n} \delta((u_{n-j}, u_n), u_1)\delta((u_j, u_n, u_1))$$

since this yields the number of elements of the set $\{u_1, u_2, \ldots, u_{n-1}\}$ which are coprime with u_n.

We can also define $\pi(u_n)$ to represent the number of generalized primes $\leq u_n$. Then

$$\pi(u_n) = \sum_{m=2}^{n} \rho\{m, m\}. \tag{7.3}$$

Proof. For $m \geq 2$,

$$\rho\{m, m\} = \begin{cases} 0, & \textit{if } u_m \textit{ has a factor between } u_1 \textit{ and } u_m; \\ 1, & \textit{if } u_m \textit{ is a generalized prime.} \end{cases}$$

For example,

$$\pi(F_8) = \sum_{m=2}^{8} \rho\{m, m\}$$
$$= 5$$
$$\neq \pi(8).$$

□

Of course there are still unsolved problems with some of the classical functions in terms of the ordinary integers [22]. A further development of a calculus of convolutions is unnecessary as this has been achieved by Dr Mollie Horadam; what has been done here is to outline some properties of the ρ,δ functions which are pertinent to the study of divisibility sequences.

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