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A note on partially degenerate Legendre–Genocchi polynomials

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Abstract: In the past years, many researchers have worked on degenerate polynomials and a variety of its extentions and variants can be found in literature. Following up, in this article, we firstly introduce the partially degenerate Legendre–Genocchi polynomials, and further define a new generalization of degenerate Legendre–Genocchi polynomials. From our generalization, we establish some implicit summation formulae and symmetry identities by the generating function of partially degenerate Legendre–Genocchi polynomials. Eventually, it can be found that some recently demonstrated summations and identities stated in the article, are special cases of our results.

Keywords: Legendre polynomials, Partially degenerate Genocchi polynomials, Partially degenerate Legendre–Genocchi polynomials, Summation formula, Symmetric identities.
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1 Introduction

Throughout the paper, we make use of the following notations:

$$\mathbb{N} := \{1, 2, 3, ...\} \; and \; \mathbb{N}_o = \mathbb{N} \cup \{0\}.$$

Generalized and multivariable forms of the special functions of mathematical physics have witnessed a significant evolution during the recent years. In particular, the special polynomials of more than one variable provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization have been suggested by physical problems.

We recall that the 2-variable Legendre polynomials $S_n(x, y)$ and $R_n(x, y)$ are given by Dattoli et al. [2]

$$S_n(x, y) = n! \sum_{k=0}^{[\frac{n}{2}]} \frac{x^k y^{n-2k}}{[(n-2k)!(k!)^2]}, \tag{1}$$

and

$$R_n(x, y) = (n!)^2 \sum_{k=0}^{\infty} \frac{(-1)^{n-k} x^{n-k} y^k}{[(n-2k)!]^2 (k!)^2} \tag{2}$$

respectively, and are related with the ordinary Legendre polynomials $P_n(x)$ [24] as

$$P_n(x) = S_n\left(-\frac{1-x^2}{4}, x\right) = R_n\left(\frac{1-x}{2}, \frac{1+x}{2}\right). \tag{3}$$

From equation (1) and (2), we have

$$S_n(x, 0) = n! \frac{x^{[\frac{n}{2}]}}{[(\frac{n}{2})!]^2}, \qquad S_n(0, y) = y^n, \tag{4}$$

$$R_n(x, 0) = (-x)^n, \qquad R_n(0, y) = y^n. \tag{5}$$

The generating functions for two variable Legendre polynomials $S_n(x, y)$ and $R_n(x, y)$ are given by [2]

$$e^{yt} C_0(-xt^2) = \sum_{n=0}^{\infty} S_n(x, y) \frac{t^n}{n!} \tag{6}$$

$$C_0(xt) C_0(-yt) = \sum_{n=0}^{\infty} R_n(x, y) \frac{t^n}{(n!)^2} \tag{7}$$

where $C_0(x)$ is the 0-th order Tricomi function [24]

$$C_0(x) = \sum_{r=0}^{\infty} \frac{(-1)^r x^r}{(r!)^2}. \tag{8}$$

The classical Bernoulli polynomials $B_n(x)$, the classical Euler polynomials $E_n(x)$, and the classical Genocchi polynomials $G_n(x)$ each of degree n are defined respectively by the following generating functions (see [2]–[27]):

$$\frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \quad (|t| < 2\pi) \tag{9}$$

$$\frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad (|t| < \pi) \tag{10}$$

$$\frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}. \quad (|t| < \pi) \tag{11}$$

Note that

$$B_n(0) = B_n, \; E_n(0) = E_n \; and \; G_n(0) = G_n \quad (n \in \mathbb{N}).$$

The Daehee poynomials are defined by Kim and Kim [14], as follows

$$\frac{log(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_n(x)\frac{t^n}{n!}. \tag{12}$$

The case when $x = 0$ in (12), $D_n(0) := D_n$ are called Daehee numbers.

Jang et al. [3] considered the partially degenarate Genocchi polynomials which are given by means of the generating function

$$\frac{2log(1+t\lambda)^{\frac{1}{\lambda}}}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_{n,\lambda}(x)\frac{t^n}{n!}. \tag{13}$$

The case when $x = 0$, $G_{n,\lambda}(0) := G_n$ are called the partially degenerate Genocchi numbers.

Pathan and Khan [22] introduced the generalized Hermite–Bernoulli polynomials for two variables ${}_HB_n^{(\alpha)}(x,y)$ given by

$$\left(\frac{t}{e^t-1}\right)^{\alpha} e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(\alpha)}(x,y)\frac{t^n}{n!}. \tag{14}$$

On taking $\alpha = 1$, (14) reduces to known result of Dattoli et al. [1, p. 386 (1.6)], as follows

$$\left(\frac{t}{e^t-1}\right) e^{xt+yt^2} = \sum_{n=0}^{\infty} {}_HB_n(x,y)\frac{t^n}{n!}. \tag{15}$$

where, for the case $x = y = 0$ in (15), we have $B_n = {}_HB_n(0,0)$ are called the Bernoulli numbers.

For each $k \in \mathbb{N}_o$, $T_k(n)$ [19] defined by

$$T_k(n) = \sum_{j=0}^{n} (-1)^j j^k \tag{16}$$

is called the alternating sum. The exponential generating function for $T_k(n)$ is

$$\sum_{k=0}^{\infty} T_k(n)\frac{t^k}{k!} = \frac{1-(-e^t)^{(n+1)}}{e^t+1}. \tag{17}$$

The concept of degenerate numbers and polynomials was introduced with the study related to Bernoulli and Euler numbers and polynomials. Many researchers have studied the degenerate polynomials associated with special polynomials in various areas. (see [2]–[27] for a systematic work). Influenced by their works, we introduce partially degenerate Legendre–Genocchi polynomials and also a new generalization of partially degenerate Legendre–Genocchi polynomials and then give some of their applications. We also derive some implicit summation formula and general symmetry identities. For obtaining implicit summation formula and general symmetry identities, we use the proof techniques of Khan et al. ([4]–[13]), Dattoli et al. [1] and Pathan and Khan [22].

2 Partially degenerate Legendre–Genocchi polynomials

In this section, we assume that $\lambda, t \in \mathbb{C}$ with $|\lambda t| \leq 1$ and $\lambda t \neq 1$. Then we consider partially degenerate Legendre–Genocchi polynomials as follows:

$$\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt} C_0(-xt^2) = \sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x,y)\frac{t^n}{n!}, \tag{18}$$

so that

$${}_SG_{n,\lambda}(x,y) = \sum_{m=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) G_{m,\lambda}\, S_{n-m}(x,y).$$

The case when $x = y = 0$ in (18), we have ${}_SG_{n,\lambda}(0,0) := G_{n,\lambda}$ are called the partially degenerate Genocchi numbers introduced by Jang et al. [3].

Theorem 2.1. *For $n \in \mathbb{N}_0$, we have*

$${}_SG_{n,\lambda}(x,y) = \sum_{m=0}^{n} (n)_m\, m!(-\lambda)^m {}_SG_{n-m}(x,y). \tag{19}$$

Proof. It follows from (18) that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}\, e^{yt} C_0(-xt^2) \\ &= \left\{ \sum_{m=0}^{\infty} \frac{(-1)^m}{m+1}(\lambda t)^m \right\} \left\{ \sum_{n=0}^{\infty} {}_SG_n(x,y)\frac{t^n}{n!} \right\} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) \frac{(-\lambda)^m}{m+1} m!\; {}_SG_{n-m}(x,y) \right\} \frac{t^n}{n!}.\end{aligned}$$

Matching the coefficients $\frac{t^n}{n!}$ gives the desired result. $\square$

Remark 2.1. *In the case when $x = 0$ in Theorem 2.1, our result reduces to the result of Jang et al. [3, p. 3(13)].*

Theorem 2.2. *For $n \in \mathbb{N}_o$, we have*

$$
{}_SG_{n+1,\lambda}(x,y) = \sum_{m=0}^{n+1} \left(\begin{array}{c} n+1 \\ m \end{array} \right) (-\lambda)^m {}_SG_{n-m+1}(x,y) D_m. \tag{20}
$$

Proof. We first consider

$$
\begin{aligned}
I_1 = \frac{1}{t}\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt} C_0(-xt^2) &= \left\{\sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!}\right\} \left\{\sum_{n=0}^{\infty} {}_SG_n(x,y)\frac{t^n}{n!}\right\} \\
&= \sum_{n=1}^{\infty} \left\{\sum_{m=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) \frac{(\lambda)^m}{D_m}\ {}_SG_{n-m}(x,y)\right\} \frac{t^n}{n!} \\
&= t\sum_{n=0}^{\infty} \left\{\sum_{m=0}^{n+1} \left(\begin{array}{c} \mathrm{n+1} \\ \mathrm{m} \end{array} \right) \frac{(\lambda)^m}{D_m}\ \frac{{}_SG_{n-m}(x,y)}{n+1}\right\} \frac{t^n}{n!}.
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
I_2 = \frac{1}{t}\frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}\ e^{yt} C_o(-xt^2) &= \frac{1}{t}\sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x,y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \frac{{}_SG_{n+1,\lambda}(x,y)}{n+1}\frac{t^n}{n!}.
\end{aligned}
$$

Since $I_1 = I_2$, we conclude the proof of Theorem 2.2. □

Remark 2.2. *Taking $x = 0$ in Theorm 2.2 gives the result of Jang et al. [3, p.5(19)].*

Theorem 2.3. *For $n \in \mathbb{N}_0$, we have*

$$
{}_SG_{n,\lambda}(x,y) = n\sum_{m=0}^{n-1} \left(\begin{array}{c} n\text{-}1 \\ m \end{array} \right) (\lambda)^m {}_SE_{n-m-1}(x,y) D_m. \tag{21}
$$

Proof. From (18), we can write

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x,y)\frac{t^n}{n!} &= \frac{t\log(1+\lambda t)}{\lambda t}\frac{2}{e^t+1}\ e^{yt}C_0(-xt^2) \\
&= t\left\{\sum_{m=0}^{\infty} D_m\frac{(\lambda t)^m}{m!}\right\}\left\{\sum_{n=0}^{\infty} {}_SE_n(x,y)\frac{t^n}{n!}\right\} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{m=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) (\lambda)^m D_m\ {}_SE_{n-m}(x,y)\right\}\frac{t^{n+1}}{n!}.
\end{aligned}
$$

Thus, comparing the coefficients of t^n in both sides on the above, we conclude the proof. □

Remark 2.3. *On putting $x = 0$ in Theorem 2.3 yields the known result of Jang et al. [3, p.5(21)].*

Theorem 2.4. *For $n \in \mathbb{N}_o$, we obtain*

$$
{}_SG_{n,\lambda}(x, y+1) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) \left\{ {}_SG_{n-m,\lambda}(x, y) \right\}. \tag{22}
$$

Proof. By making use of (18), we see that

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \left\{ {}_SG_{n,\lambda}(x, y+1) - {}_SG_{n,\lambda}(x, y) \right\} \\
&= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{(y+1)t} C_0(-xt^2) - \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} e^{yt} C_0(-xt^2) \\
&= \sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x, y) \frac{t^n}{n!} \sum_{m=0}^{\infty} \frac{t^m}{m!} - \sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x, y) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) {}_SG_{n-m,\lambda}(x, y) - {}_SG_{n,\lambda}(x, y) \right\} \frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients $\frac{t^n}{n!}$ in both sides of the above equation, we get the result (22). □

Corollary 2.4.1. *In the case when $x = 0$ in Theorem 2.4, one can see*

$$
G_{n,\lambda}(y+1) = \sum_{m=0}^{n} \left(\begin{array}{c} n \\ m \end{array} \right) G_{n-m,\lambda}(y).
$$

Theorem 2.5. *For $n \in \mathbb{N}_o$, we have*

$$
{}_SG_{n,\lambda}(x, y) = \sum_{m=0}^{n} \sum_{k=0}^{m} \left(\begin{array}{c} n \\ m \end{array} \right) \left(\begin{array}{c} m \\ k \end{array} \right) G_{n-m} D_{m-k}\, \lambda^{m-k} S_k(x, y). \tag{23}
$$

Proof. Since,

$$
\begin{aligned}
\sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x, y) \frac{t^n}{n!} &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}\, e^{yt} C_0(-xt^2) \\
&= \left\{ \frac{2t}{e^t+1} \right\} \left\{ \frac{2\log(1+\lambda t)}{\lambda t} \right\} e^{yt} C_0(-xt^2) \\
&= \left\{ \sum_{n=0}^{\infty} G_n \frac{t^n}{n!} \right\} \left\{ \sum_{m=0}^{\infty} D_m \frac{(\lambda t)^m}{m!} \right\} \left\{ \sum_{k=0}^{\infty} S_k(x, y) \frac{t^k}{k!} \right\}
\end{aligned}
$$

we have

$$
= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \sum_{k=0}^{m} \left(\begin{array}{c} \mathrm{n} \\ \mathrm{m} \end{array} \right) \left(\begin{array}{c} \mathrm{m} \\ \mathrm{k} \end{array} \right) G_{n-m} D_{m-k} \lambda^{m-k} S_k(x, y) \right\} \frac{t^n}{n!}.
$$

We thus complete the proof of Theorem 2.5. □

We now give a multiplication formula for partially degenerate Laguerre–Genocchi polynomials.

Theorem 2.6. *For $n \in \mathbb{N}_o$, we have*

$$ {}_SG_{n,\lambda}(x,y) = d^{n-1}\sum_{a=0}^{d-1} {}_SG_{n,\frac{\lambda}{d}}\left(x, \frac{y+a}{d}\right) \tag{24}$$

Proof. From (18), we have

$$\sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}\, e^{yt}C_0(-xt^2)$$

$$= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}\, C_0(-xt^2)\sum_{a=0}^{d-1} e^{(a+y)t}$$

$$= \sum_{n=0}^{\infty}\left\{d^{n-1}\sum_{a=0}^{d-1} {}_SG_{n,\frac{\lambda}{d}}\left(x, \frac{y+a}{d}\right)\right\}\frac{t^n}{n!}$$

Equating the coefficients $\frac{t^n}{n!}$ of both the sides of above equation, we arrive at (24). □

Corollary 2.6.1. *The case when $x = 0$, we get*

$$G_{n,\lambda}(y) = d^{n-1}\sum_{a=0}^{d-1} G_{n,\frac{\lambda}{d}}\left(\frac{y+a}{d}\right).$$

3 Generalized partially degenerate Legendre–Genocchi polynomials

Let $d \in \mathbb{N}$ with $d \equiv 1 \pmod 2$ and χ be a Dirichlet character with conducter d. We consider the generalized partially degenerate Legendre–Genocchi polynomials attached to χ by means of the following generating function:

$$\sum_{n=0}^{\infty} {}_SG_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(y+a)t}C_0(-xt^2) \tag{25}$$

When $x = y = 0$ in (25), we have $G_{n,\chi,\lambda} = {}_SG_{n,\chi,\lambda}(0,0)$ that stands for the generalized partially degenerate Genocchi numbers attached to χ. Also we observe that

$$\lim_{\substack{\lambda \to 0 \\ y \to 0}} {}_SG_{n,\chi,\lambda}(x,y) = G_{n,\chi}(x)$$

is a generalized Genocchi polynomial (see [25]).

Theorem 3.1. *For $n \in \mathbb{N}_0$, we have*

$$ {}_SG_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{n} \binom{n}{m} \lambda^m D_m \, {}_SG_{n-m,\chi}(x,y). \tag{26}$$

Proof. It follows from (25) that

$$\begin{aligned}\sum_{n=0}^{\infty} {}_SG_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(y+a)t}C_0(-xt^2)\\ &= \left\{\frac{\log(1+\lambda t)}{\lambda t}\right\}\left\{\frac{2t}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{(y+a)t}C_0(-xt^2)\right\}\\ &= \left\{\sum_{m=0}^{\infty} D_m\frac{\lambda^m t^m}{m!}\right\}\left\{\sum_{n=0}^{\infty} {}_SG_{n,\chi}(x,y)\frac{t^n}{n!}\right\}\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ on both sides of the above equation, we compute the proof of Theorem 3.1. □

Theorem 3.2. *The following equality holds true:*

$$ {}_SG_{n,\chi,\lambda}(x,y) = d^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a)\, {}_SG_{n,\frac{\lambda}{d}}\left(x, \frac{a+y}{d}\right) \tag{27}$$

Proof. We consider

$$\begin{aligned}\sum_{n=0}^{\infty} {}_SG_{n,\chi,\lambda}(x,y)\frac{t^n}{n!} &= \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)\, e^{(y+a)t}\, C_0(-xt^2)\\ &= \frac{1}{d}\sum_{a=0}^{d-1}(-1)^a\chi(a)\frac{2\log(1+\lambda t)^{\frac{d}{\lambda}}}{e^{dt}+1}e^{\left(\frac{a+y}{d}\right)dt}\, C_0(-xt^2)\\ &= \sum_{n=0}^{\infty}\left\{d^{n-1}\sum_{a=0}^{d-1}(-1)^a\chi(a){}_SG_{n,\frac{\lambda}{d}}\left(x,\frac{a+y}{d}\right)\right\}\frac{t^n}{n!}\end{aligned}$$

Equating the coefficients $\frac{t^n}{n!}$ on both sides of the above equation, we compute the proof of Theorem 3.2. □

By using (25), the undermentioned theorems can be proved easily. So we omit the proofs.

Theorem 3.3. *The following equality holds true:*

$$ {}_SG_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{[\frac{n}{2}]}\frac{G_{n-m,\chi,\lambda}(y)\,(-x)^m\, n!}{(n-2m)!\,(m!)^2}. \tag{28}$$

Theorem 3.4. *The following equality holds true:*

$$ {}_SG_{n,\chi,\lambda}(x,y) = \sum_{m=0}^{n}\sum_{k=0}^{[\frac{n}{2}]}\frac{G_{n-m-2k,\chi,\lambda}\,(y)^m\,(-x)^k\, n!}{(n-m-2k)!\,(m)!\,(k!)^2}. \tag{29}$$

4 Implicit summation formulae invoving partially degenerate Legendre–Genocchi polynomials

Theorem 4.1. *The following implicit summation formulae for partially degenerate Legendre–Genocchi polynomials ${}_SG_{n,\lambda}(x,y)$ hold true:*

$$ {}_SG_{k+l,\lambda}(x,v) = \sum_{n,p=0}^{k,l} \left(\begin{array}{c} k \\ n \end{array} \right) \left(\begin{array}{c} l \\ p \end{array} \right) (v-y)^{n+p} {}_SG_{k+l-p-n,\lambda}(x,y) \tag{30} $$

Proof. We first replace t by $t+u$ and rewrite the generating function (18) as

$$ \frac{2\log(1+\lambda(t+u))^{\frac{1}{\lambda}}}{e^{t+u}+1} C_o(-x(t+u)^2) = e^{-y(t+u)} \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,y) \frac{t^k u^l}{k!l!} $$

Replacing y by v in the above equation and equating the resulting equation to the above equation, we have indeed

$$ e^{(v-y)(t+u)} \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,y) \frac{t^k u^l}{k!l!} = \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,v) \frac{t^k u^l}{k!l!} $$

and also

$$ \sum_{N=0}^{\infty} \frac{(v-y)^N (t+u)^N}{N!} \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,y) \frac{t^k u^l}{k!l!} = \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,v) \frac{t^k u^l}{k!l!}, \tag{31} $$

where on using the following formula taken in [26, p.52(2)]

$$ \sum_{N=0}^{\infty} f(N) \frac{(x+y)^N}{N!} = \sum_{n,m=0}^{\infty} f(n+m) \frac{x^n}{n!} \frac{y^m}{m!}, $$

in the left hand side of (31), it becomes

$$ \sum_{n,p=0}^{\infty} \frac{(v-y)^{n+p} t^n u^p}{n!p!} \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,y) \frac{t^k u^l}{k!l!} = \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,v) \frac{t^k u^l}{k!l!}. \tag{32} $$

Now replacing k by $k-n$, l by $l-p$, and using the lemma [26, p.100 (1)] in the left hand side of (32), we get

$$ \sum_{n,p=0}^{\infty} \sum_{k,l=0}^{\infty} \frac{(v-y)^{n+p}}{n!p!} {}_SG_{k+l-n-p,\lambda}(x,y) \frac{t^k u^l}{(k-n)!(l-p)!} = \sum_{k,l=0}^{\infty} {}_SG_{k+l,\lambda}(x,v) \frac{t^k u^l}{k!l!}. $$

Thus, on equating the coefficients of the like powers of t^k and u^l in the above equation, we arrive at the desired result. □

Corollary 4.1.1. *In the case $l=0$ in* (30)*, we have*

$$ {}_SG_{k,\lambda}(x,v) = \sum_{j=0}^{k} \left(\begin{array}{c} k \\ j \end{array} \right) {}_SG_{k-j,\lambda}(x,y). $$

Note that for special values of the parameters x and v in Theorem 4.1, one can obtain some identities of usual Genocchi polynomials. Now we give some theorems which, using (18), can easily be proved. So we choose to omit the proofs.

Theorem 4.2. *The following implicit summation formula for partially degenerate Legendre–Genocchi polynomials* ${}_SG_{n,\lambda}(x, y)$ *holds true:*

$$ {}_SG_{n,\lambda}(x, y+u) = \sum_{m=0}^{n} \binom{n}{m} u^m {}_SG_{n-m,\lambda}(x, y). $$

Theorem 4.3. *The following implicit summation formula for partially degenerate Legendre–Genocchi polynomials* ${}_SG_{n,\lambda}(x, y)$ *holds true:*

$$ \sum_{n=0}^{\infty} {}_SG_{n,\lambda}(x, y)\frac{t^n}{n!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1} \, e^{yt} C_o(-xt^2) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda} \, S_n(x, y), $$

$$ {}_SG_{n,\lambda}(x, y) = \sum_{m=0}^{n} \binom{n}{m} G_{n-m,\lambda}(x, y) \, S_n(x, y). $$

Theorem 4.4. *The following implicit summation formula for partially degenerate Legendre–Genocchi polynomials* ${}_SG_{n,\lambda}(x, y)$ *holds true:*

$$ {}_SG_{n,\lambda}(x, y+1) + {}_SG_{n,\lambda}(x, y) = 2n \sum_{m=0}^{n-1} \binom{n-1}{m} \frac{(-\lambda)^m m!}{m+1} \, S_{n-m-1}(x, y). $$

Theorem 4.5. *The following implicit summation formula for partially degenerate Legendre–Genocchi polynomials* ${}_SG_{n,\lambda}(x, y)$ *holds true:*

$$ {}_SG_{n,\lambda}(x, y+1) = \sum_{m=0}^{n} {}_SG_{n-m,\lambda}(x, y). $$

5 Symmetry identities for partially degenerate Legendre–Genocchi polynomials

In this section, we give general symmetry identities for the partially degenerate Legendre–Genocchi polynomials ${}_SG_{n,\lambda}(x, y)$ by making use of the generating functions (13) and (18).

Theorem 5.1. *For each pair of integers* a *and* b *with* $n \geq 0$*, the following symmetry identity holds true:*

$$ \sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \, {}_SG_{n-m,\lambda}(bx, by) \, {}_SG_{m,\lambda}(ax, ay) $$

$$ = \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \, {}_SG_{n-m,\lambda}(ax, ay) \, {}_SG_{m,\lambda}(bx, by) $$

Proof. We first consider

$$g(t) = \frac{\left\{2\log(1+\lambda)^{\frac{b}{\lambda}}\right\}\left\{2\log(1+\lambda)^{\frac{a}{\lambda}}\right\}}{(e^{at}+1)\,(e^{bt}+1)} e^{(a+b)yt} C_0(-axt^2) C_0(-bxt^2),$$

where $g(t)$ is symmetric in a and b, and can be expressed into series in two ways.

On the other hand,

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_S G_{n,\lambda}(bx, by) \frac{(at)^n}{n!} \sum_{m=0}^{\infty} {}_S G_{m,\lambda}(ax, ay) \frac{(bt)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{\mathrm{n}}{\mathrm{m}} b^m a^{n-m} \, {}_S G_{n-m,\lambda}(bx, by) \, {}_S G_{m,\lambda}(ax, ay) \right\} \frac{t^n}{n!} \end{aligned} \tag{33}$$

and on the other hand,

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} {}_S G_{n,\lambda}(ax, ay) \frac{(bt)^n}{n!} \sum_{m=0}^{\infty} {}_S G_{m,\lambda}(bx, by) \frac{(at)^m}{m!} \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{\mathrm{n}}{\mathrm{m}} a^m b^{n-m} {}_S G_{n-m,\lambda}(ax, ay) \, {}_S G_{m,\lambda}(bx, by) \right\} \frac{t^n}{n!} \end{aligned} \tag{34}$$

By comparing the coefficients t^n on the right hand sides of equations (33) and (34) we get the proof of the Theorem. □

Theorem 5.2. *For each pair of integers a and b with $n \geq 1$, the following symmetry identity holds true:*

$$\begin{aligned} &\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(x, by + \frac{b}{a} i + j \right) G_{m,\lambda}(az) \\ &= \sum_{m=0}^{n} \binom{n}{m} a^n b^{n-m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(x, ay + \frac{b}{a} i + j \right) G_{m,\lambda}(bz) \end{aligned}$$

Proof. Let

$$g(t) = \frac{\left\{2\log(1+\lambda)^{\frac{a}{\lambda}}\right\}\left\{2\log(1+\lambda)^{\frac{b}{\lambda}}\right\} e^{(abt+1)^2}}{(e^{at}+1)^2\,(e^{bt}+1)^2} e^{(ab)(y+z)t} \left\{C_0(-xt^2)\right\}.$$

We consider $g(t)$ with two ways. Firstly,

$$\begin{aligned} g(t) =& \frac{\left\{2\log(1+\lambda)^{\frac{a}{\lambda}}\right\}}{e^{at}+1} e^{abyt} C_o(-xt^2) \\ &\times \left(\frac{e^{abt}+1}{e^{bt}+1} \right) \frac{\left\{2\log(1+\lambda)^{\frac{b}{\lambda}}\right\}}{e^{bt}+1} e^{abzt} \left(\frac{e^{abt}+1}{e^{at}+1} \right) \\ =& \frac{\left\{2\log(1+\lambda)^{\frac{a}{\lambda}}\right\}}{e^{at}+1} e^{abyt} C_0(-xt^2) \left(\sum_{i=0}^{a-1} (-1)^i e^{bti} \right) \\ &\times \frac{\left\{2\log(1+\lambda)^{\frac{b}{\lambda}}\right\}}{e^{bt}+1} e^{abzt} C_0(-xt^2) \left(\sum_{j=0}^{b-1} (-1)^j e^{atj} \right). \end{aligned}$$

From where we have

$$g(t) = \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{\text{n}}{\text{m}} b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(ax, by + \frac{b}{a} i + j \right) G_{m,\lambda}(az) \right\} \frac{t^n}{n!}$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{n} \binom{\text{n}}{\text{m}} a^m b^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(x, ay + \frac{a}{b} i + j \right) G_{m,\lambda}(bz) \right\} \frac{t^n}{n!}.$$

Our assertion follows from comparing the coefficients of $\frac{t^n}{n!}$ on the right hand sides of last two equations, we arrive at desired result. □

We now give the following two theorems. We omit their proofs since the same technique is used as in the above theorems of the final section of this paper.

Theorem 5.3. *For each pair of integers a and b with $n \geq 0$, the following symmetry identity holds true:*

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} \sum_{i=0}^{a-1} \sum_{j=0}^{b-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(x, by + \frac{b}{a} i \right) G_{m,\lambda} \left(az + \frac{a}{b} j \right)$$

$$= \sum_{m=0}^{n} \binom{n}{m} a^m b^{n-m} \sum_{i=0}^{b-1} \sum_{j=0}^{a-1} (-1)^{i+j} {}_S G_{n-m,\lambda} \left(x, ay + \frac{a}{b} i + j \right) G_{m,\lambda} \left(bz + \frac{b}{a} j \right).$$

Theorem 5.4. *For each pair of integers a and b with $n \geq 0$, the following symmetry identity holds true:*

$$\sum_{m=0}^{n} \binom{n}{m} b^m a^{n-m} {}_S G_{n-m,\lambda}(bx, by) \sum_{i=0}^{m} \binom{m}{i} T_i(a-1)\, G_{m-i,\lambda}(ax)$$

$$= \sum_{m=0}^{n} \binom{n}{m} b^{n-m} a^m {}_S G_{n-m,\lambda}(ax, ay) \sum_{i=0}^{m} \binom{m}{i} T_i(b-1)\, G_{m-i,\lambda}(bx).$$

6 Concluding remarks

The paper aims at presenting the study of partially degenerate Legendre–Genocchi polynomials which play an important role in several various field of physics, applied mathematics and engineering. These special polynomials are important as they possess essential properties such as recurrence and explicit relations and functional and differential equations, summation formulae, symmetric and convolution identities, etc.

The results presented here are very useful and being very general, can be specialized to yield a large number of identities involving known or new simpler numbers and polynomials. For instance, just by letting $x = 0 = y$ in the generating function (18), the polynomials defined here reduce to partially degenerate Genocchi numbers and the corresponding results presented in [3] can directly be retrieved. Also the Genocchi polynomials in [25] can also be recovered from our results.

The technique used here could be used to establish further quite a wide variety of formulas for certain other special polynomials and can be extended to derive new relations for conventional and generalized polynomials. For example, to give an application, we introduce here the Laguerre-based partially degenerate Genocchi polynomials.

For $\lambda, t \in \mathbb{C}$ with $|t\lambda| \leq 1$ and $t\lambda \neq 1$. We introduce

$$\sum_{p=0}^{\infty} {}_L G_{p,\lambda}(x,y)\frac{t^p}{p!} = \frac{2\log(1+\lambda t)^{\frac{1}{\lambda}}}{e^t+1}e^{yt}C_0(xt). \tag{35}$$

They will have the closed form as

$${}_L G_{p,\lambda}(x,y) = \sum_{q=0}^{p} \begin{pmatrix} \text{p} \\ \text{q} \end{pmatrix} G_{q,\lambda}L_{p-q}(x,y).$$

We encourage other researchers to come up with other interesting properties of the polynomials defined in (35) and to generate more new polynomials from them.

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