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Indispensable digits for digit sums

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Abstract: Let b be an integer greater than 1 and $g = b - 1$. For any nonnegative integer n, we define *indispensable digits* in the base-b representation of n so that we can calculate the digit sum of the base-b representation of $g \cdot n$: Instead of adding every digit in it, we multiply g by the number of the indispensable digits in the base-b representation of n. Then, we find the formula to calculate the digit sum of $g \cdot n + r$ using the number of indispensable digits in n, for any nonnegative integers n and r with $0 < r < g$.
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1 Introduction

Throughout this paper, let b be an integer greater than 1 and $g = b - 1$.

Definition 1.1. For any nonnegative integer m, the *digit sum of the base-b representation of* m, denoted by $s_b(m)$, is defined as

$$s_b(m) = \sum_{i=0}^{k} a_i,$$

where a_i's $\in \{0, 1, 2, \dots, g\}$ such that $m = \sum_{i=0}^{k} a_i b^i$.

It is well known that for any nonnegative integer m, $m \equiv s_b(m) \pmod{g}$ [5]. That is, for any nonnegative integers n and r with $0 \leq r < g$,

$$s_b(g \cdot n + r) = g \cdot k + r, \tag{1.1}$$

for some integer k.

In this paper, we define *indispensable digits* in the base-b representation of any nonnegative integer as demonstrated in Definition 3.1. Then, we find k satisfying (1.1) using the number of indispensable digits in the base-b representation of n, instead of adding every digit in the base-b representation of $g \cdot n + r$.

In Section 2, we clarify notation for this paper. In Section 3, we define indispensable digits in the base-b representation of a nonnegative integer. In Section 4, we calculate the digit sum of the base-b representation of a multiple of g. Finally, in Section 5, we calculate the digit sum of the base b representation of an integer with a nonzero residue modulo g.

2 Notation

The set containing every finite string consisting of digits in $\{0, 1, 2, \ldots, g\}$, including the empty string ϵ, is denoted by $\{0, 1, 2, \ldots, g\}^*$. For any strings x and y in $\{0, 1, 2, \ldots, g\}^*$, the *concatenation* of x and y, denoted by xy, is the string obtained by joining x and y end-to-end. The concatenation of n x's is denoted by x^n, for any positive integer n [4]. That is, if $x = a_k \cdots a_1 a_0$ and $y = c_l \cdots c_1 c_0$ for $a_i, c_i \in \{0, 1, 2, \ldots, g\}$,

$$xy = a_k \cdots a_1 a_0 c_l \cdots c_1 c_0 \text{ and } x^n = xxx \cdots x (n \text{ times}).$$

For example, the concatenation of 123 and 10 is $123\ 10 = 12310$. The concatenation of n digit 1's is the string $1^n = \overbrace{11 \cdots 1}^{n \text{ times}}$.

Every nonempty string in $\{0, 1, 2, \ldots, g\}^*$ represents a nonnegative integer. That is, for any $a_i \in \{0, 1, 2, \ldots, g\}$, $a_k a_{k-1} \cdots a_1 a_0$ represents $\sum_{i=0}^{k} a_i b^i$, and we write

$$[a_k \cdots a_1 a_0]_b = \sum_{i=0}^{k} a_i b^i.$$

For example, $[011]_3 = 4 = [0011]_3$. If $a_k \neq 0$, we have a unique base-b representation of a nonnegative integer: for any nonnegative integer n, there exist unique a_i's in $\{0, 1, 2, \ldots, g\}$ such that $n = \sum_{i=0}^{\lfloor \log_b n \rfloor} a_i b^i$. We write

$$(n)_b = a_{\lfloor \log_b n \rfloor} a_{\lfloor \log_b n \rfloor - 1} \cdots a_2 a_1 a_0,$$

and we call it the b-ary string of n. For example, $(4)_3 = 11$. Note that $[(n)_b]_b = n$ for any nonnegative integer n, but it is not always true that $([x])_b = x$ for its b-ary string x [4].

For any string x, $|x|$ denotes the number of digits in x, and for any nonnegative integer n, $l_b(n)$ denotes the number of digits in $(n)_b$: if $(n)_b = x = a_k a_{k-1} \cdots a_1 a_0$ for some digit $a_i \in \{0, 1, 2, \ldots, g\}$,

$$|x| = k + 1 = \lfloor \log_b n \rfloor + 1 = l_b(n).$$

For example, $l_3(4) = 2$.

3 Indispensable digits

Definition 3.1. For any digits a_i's and a string $x = a_m a_{m-1} \cdots a_1 a_0$, a_i is called *indispensable* in x, if $a_i = a_{i-1} = a_{i-2} = \cdots = a_{i-k+1} > a_{i-k}$ for some positive integer $k \leq i+1$, considering $a_{-1} = 0$; otherwise, *dispensable* in x. The number of indispensable digits in x is denoted by $\iota(x)$.

For example, consider $x = 288943771$. Then, digits 9, 4, 7, 7, 1 are indispensable in x, because $9 > 4 > 3$ and $7 = 7 > 1 > 0 = a_{-1}$, and digits 2, 8, 8, 3, are dispensable in x, because $2 < 8 = 8 < 9$ and $3 < 7$. Hence, $\iota(288\dot{9}\dot{4}3\dot{7}\dot{7}\dot{1}) = 5$

Note 3.2. *Digit 0 is always dispensable in any string.*

For a nonzero digit to be dispensable, we have the following:

Lemma 3.3. *For any digit a_i's and any string $x = a_m a_{m-1} \cdots a_1 a_0$, if $a_i \neq 0$ and a_i is dispensable in x, $a_i = a_{i-1} = a_{i-2} = \cdots = a_{i-k+1} < a_{i-k}$ for some positive integer $k \leq i$.*

Proof. Suppose $a_i = a_{i-1} = \cdots = a_{i-k+1} \geq a_{i-k}$ for all $k \leq i$. If $a_i = a_{i-1} = \cdots = a_{i-k+1} > a_{i-k}$ for some $k \leq i$, a_i is indispensable. Otherwise, $a_i = a_{i-1} = \cdots = a_1 = a_0$. Since a_i is nonzero, $a_i = a_0 > 0 = a_{-1}$. Hence, a_i is indispensable. □

Now we consider the number of indispensable digits in the b-ary string of a nonnegative integer:

Definition 3.4. For any nonnegative integer n, we denote the number of indispensable digits in $(n)_b$ by $\iota_b(n)$.

For example, $\iota_3(50) = \iota(1\dot{2}1\dot{2}) = 2$, since $(50)_3 = 1212$. Then, the following is obvious:

Note 3.5. *For any nonnegative integer n,*

1. *If $n = [a_m \cdots a_1 a_0]_b$, $\iota_b(n) = \iota(a_m \cdots a_1 a_0)$;*
2. *$\iota_b(n) = 0$ if and only if $n = 0$;*
3. *$\iota_b(n) \leq l_b(n)$.*

In general, $n < m$ does not imply $\iota_b(n) \leq \iota_b(m)$. However, $[1^k]_b = [\overbrace{111\cdots1}^{k}]_b$ is the least positive integer with k indispensable digits in the base-b system.

Lemma 3.6. *For any nonnegative integer n and a positive integer k,*

$$\text{if } n < [1^k]_b,\ \ \iota_b(n) < k.$$

Proof. Since $n < [1^k]_b$, $l_b(n) \leq |1^k| = k$. If $l_b(n) < k$, $\iota_b(n) \leq l_b(n) < k$. If $l_b(n) = k$, $(n)_b = a_{k-1} \cdots a_1 a_0$ for some digits a_i's in $\{0, 1, 2, \ldots, g\}$. Since $n < [1^k]_b$, $a_i < 1$ for some i, i.e., $a_i = 0$ for some i. Since digit 0 is dispensable, $\iota_b(n) \leq k - 1$. □

For example, for any positive integer n,

$$\text{if } n < [11]_b = [1^2]_b,\ \iota_b(n) < 2 \text{ so } \iota_b(n) = 1. \tag{3.1}$$

To simplify the further discussion in the following sections, we define a sequence as follows:

Definition 3.7. For any positive integer k, u_k is defined by $u_1 = [11]_b$; $u_k = [1^{u_{k-1}}]_b$.

That is,

$$\begin{aligned} u_1 &= [11]_b; \\ u_2 &= [1^{[11]_b}]_b = [\overbrace{11\cdots 1}^{[11]_b}]_b; \\ &\vdots \\ u_k &= [1^{u_{k-1}}]_b = [\overbrace{11111111111111111111\cdots 1}^{u_{k-1}}]_b. \end{aligned}$$

By Definition 3.7 and Lemma 3.6, for any nonnegative integer n and a positive integer k,

$$\text{if } n < u_k,\ \iota_b(n) < u_{k-1}. \tag{3.2}$$

Theorem 3.8. *For any positive integers k and n,*

$$\textit{if } n < u_k,\ \iota_b^k(n) = 1 \tag{3.3}$$

Proof. The proof is done by mathematical induction on k. The base case is shown in (3.1). Induction hypothesis: assume $\iota_b^{k-1}(n) = 1$ for any positive integer $n < u_{k-1}$. Suppose $n < u_k$. Then, $\iota_b(n) < u_{k-1}$ by (3.2). Hence, by the induction hypothesis, $\iota_b^k(n) = \iota_b^{k-1}(\iota_b(n)) = 1$. □

4 Digit sums for multiples of g

In the base-b system, the product of g by a single digit is as follows:

Lemma 4.1. *For any digit $a \in \{0, 1, 2, \ldots, g\}$,*

$$g \cdot a = [a - \delta, b \cdot \delta - a]_b, \textit{ where } \delta = \begin{cases} 0, & \textit{if } a = 0; \\ 1, & \textit{if } a \neq 0. \end{cases}$$

Proof. If $a = 0$, $g \cdot 0 = 0 = [00]_b$. Otherwise, $1 \le a \le g$, so $a - 1$ and $b - a$ are digits in $\{0, 1, 2, \ldots, g\}$, and $[a - 1, b - a]_b = (a - 1)b + (b - a) = g \cdot a$. □

Hence, the digit sum, $s_b(g \cdot n)$, for any nonnegative integer n with $l_b(n) = 1$, is as follows:

Corollary 4.2. *For any integer n in $\{0, 1, 2, \ldots, g\}$,*

$$s_b(g \cdot n) = \begin{cases} g \cdot 0, & \textit{if } n = 0; \\ g \cdot 1, & \textit{if } n > 0. \end{cases}$$

Proof. It is obtained by Lemma 4.1. □

To help the further discussion on the digit sum, we define the following:

Definition 4.3. For any nonnegative integer n and any digit a_i in $\{0, 1, 2, \ldots, g\}$, if $(n)_b = a_m a_{m-1} \ldots a_1 a_0$, we define the *$b^i$-th place sum in $g \cdot n$*, denoted by σ_i, as follows:

$$\sigma_0 = b \cdot \delta_0 - a_0;\ \sigma_i = b \cdot \delta_{i+1} - a_{i+1} + a_i - \delta_i (k = 1, 2, \ldots, m);\ \sigma_{m+1} = a_m - \delta_m,$$

where $\delta_i = 0$ if $a_i = 0$; 1 otherwise, for all i.

If $0 < a_k = a_{k-1} = \cdots = a_{k-j+1} \neq a_{k-j}$, $\sigma_i = b - a_k + a_k - 1 = g$ for all $i = k, k-1, \ldots, k-j+2$. That is,

$$\begin{array}{ccccccccc}
 & & & & & & & & g \\
\times) & \cdots & a_k & a_{k-1} & \cdots & a_{k-j+2} & a_{k-j+1} & a_{k-j} & \cdots \\
\hline
 & & & & & & a_{k-j} - \delta_{k-j} & * & \cdots \\
 & & & & & a_k - 1 & b - a_k & & \\
 & & & & & b - a_k & & & \\
 & & & & \iddots & & & & \\
 & & & a_k - 1 & & & & & \\
 & & a_k - 1 & b - a_k & & & & & \\
 & a_k - 1 & b - a_k & & & & & & \\
\cdots & * & & & & & & & \\
\hline
\cdots & a_k - 1 + * & g & g & \cdots & g & b - a_k & \cdots & \cdots \\
 & & & & & & + a_{k-j} - \delta_{k-j} & &
\end{array}$$

If $a_k > a_{k-j} > 0$, $\sigma_{k-j+1} < g - 1$. Even if the b^{k-j}-th place sum exceeds b so there is an increase in σ_{k-j+1} by 1, $\sigma_{k-j+1} + 1 \leq g$. If $a_k > a_{k-j} = 0$, $\sigma_{k-j+1} \leq g$ but there is an increase in σ_{k-j+1}, so $\sigma_{k-j+1} + 1 \leq g$. Hence, the b^i-th place digit in $(g \cdot n)_b$ becomes g for all $i = k, k-1, \ldots, k-j+2$. Since we gain another g by canceling a_k's in the b^{k+1}-th place digit and the b^{k-j+1}-th place digit, every digit a_i for $i = k, k-1, \ldots, k-j+1$ affects the digit sum $s_b(g \cdot n)$.

If $a_k < a_{k-j}$, $\sigma_{k-j+1} > g$, so the b^{k-j+1}-th place sum decreases by b and the b^{k-j+2}-th place sum increases by 1. Since $g + 1 = b$, so the b^i-th place digit $(g \cdot n)_b$ becomes 0 for all $i = k, k-1, \ldots, k-j+2$. Since we cancel a_k by adding tbe b^{k+1}-th place digit and the b^{k-j+1}-th place digit, every digit a_i for $i = k, k-1, \ldots, k-j+1$ does not affect the digit sum $s_b(g \cdot n)$.

Therefore, the number of digits to determine the digit sum $s_b(g \cdot n)$ is the number of indispensable digits in the b-ary string of n.

Theorem 4.4. *For any nonnegative integer n, $s_b(g \cdot n) = g \cdot \iota_b(n)$.*

Proof. The proof is done by mathematical induction on $l_b(n)$. The base case, when $l_b(n) = 1$, is covered by Corollary 4.2. Induction Hypothesis: assume if $l_b(n) \leq k$, $s_b(g \cdot n) = g \cdot \iota_b(n)$.

Consider n with $l_b(n) = k + 1$. Then, $(n)_b = a_k a_{k-1} \cdots a_1 a_0$ for some $a_i \in \{0, 1, 2, \ldots, g\}$ and $a_k \neq 0$. Then, by Definition 3.1 and Lemma 3.3,

$$\text{if } a_k \text{ is indispensable, } a_k = a_{k-1} = \cdots = a_{k-j+1} > a_{k-j}$$

for some positive integer $j \le k+1$ and

$$\text{if } a_k \text{ is dispensable, } a_k = a_{k-1} = \cdots = a_{k-j+1} < a_{k-j}$$

for some positive integer $j \le k$. Since $a_k \ne 0$, by Lemma 4.1,

$$\begin{array}{ll} g \cdot a_i = [a_k - 1, b - a_k]_b & \text{for all } i = k, , k-1, \ldots, k-j+1; \\ g \cdot a_{k-j} = [a_{k-j} - \delta_{k-j},\ *\ ]_b, & \text{where } \delta_{k-j} = 0 \text{ if } a_{k-j} = 0; 1 \text{ else.} \end{array}$$

Then, the b^i-th place sum in $g \cdot n$ for $i = k-j+1, \ldots, k+1$ is

$$\sigma_{k+1} = a_k - 1;\ \sigma_i = g\ (i = k, k-1, \ldots, k-j+2);\ \sigma_{k-j+1} = b - a_k + a_{k-j} - \delta_{k-j}.$$

Let c_i be a digit satisfying $g \cdot n = [c_{k+1} c_k \cdots c_1 c_0]_b$. Consider δ' as a possible increase 1 in the b^{k-j+1}-th place sum in $g \cdot n$. Then,

$$g \cdot [a_{k-j} \cdots a_1 a_0]_b = [a_{k-j} - \delta_{k-j} + \delta', c_{k-j} \cdots c_1 c_0]_b,$$

where

$$\delta' = \begin{cases} 1, & \text{if the } b^{k-j}\text{-th place sum in } g \cdot n \text{ excceeds } g; \\ 0, & \text{otherwise.} \end{cases}$$

Let $\delta = \delta_{k-j} - \delta'$. Since there is no increase in b^{k-j+1}-th place sum in $g \cdot n$ when $a_{k-j} = 0$,

$$\delta = \begin{cases} 0, & \text{if } a_{k-j} = 0; \\ 0, & \text{if } a_{k-j} \ne 0 \text{ and the } b^{k-j}\text{-th place sum in } g \cdot n \text{ excceeds } g; \\ 1, & \text{if } a_{k-j} \ne 0 \text{ and the } b^{k-j}\text{-th place sum in } g \cdot n \text{ does not excceed } g. \end{cases}$$

Hence,

$$s_b(g \cdot [a_{k-j} \cdots a_1 a_0]_b) = (a_{k-j} - \delta) + \sum_{i=0}^{k-j} c_i \text{ for some } \delta = 0 \text{ or } 1.$$

If a_k is indispensable, $-g \le a_{k-j} - a_k \le -1$. Then,

$$0 \le \sigma_{k-j+1} + \delta' = a_{k-j} - a_k + b - \delta \le g.$$

Thus, there is no increase in each b^i-th place sum for all $i = k+1, k, \ldots, k-j+1$, so

$$c_i = \begin{cases} \sigma_{k-j+1} + \delta' = b - a_k + a_{k-j} - \delta, & \text{if } i = k-j+1; \\ \sigma_i = g, & \text{if } i = k, k-1, \ldots, k-j+2; \\ \sigma_{k+1} = a_k - 1, & \text{if } i = k+1. \end{cases}$$

Hence,

$$\begin{aligned} s_b(g \cdot n) &= a_k - 1 + (j-1) \cdot g + (b - a_k) + (a_{k-j} - \delta) + \textstyle\sum_{i=0}^{k-j} c_i \\ &= g \cdot j + s_b(g \cdot [a_{k-j} \ldots a_1 a_0]_b). \end{aligned}$$

Since a_i is indispensable for all $i = k, k-1, \ldots, k-j+1$, $\iota_b([a_{k-j} \cdots a_1 a_0]_b) = \iota_b(n) - j$. By the induction hypothesis, $s_b(g \cdot [a_{k-j} \ldots a_1 a_0]_b) = g \cdot (\iota_b(n) - j)$. Therefore,

$$s_b(g \cdot n) = g \cdot j + g \cdot (\iota_b(n) - j) = g \cdot \iota_b(n).$$

If a_k is dispensable, $1 \le a_{k-j} - a_k < g$, since $a_k \neq 0$. Then,

$$b \le \sigma_{k-j+1} + \delta' = a_{k-j} - a_k + b - \delta < b + g.$$

To find c_i's, the b^{k-j+1}-th place sum decreases by b and the b^{k-j+2}-th place sum increases by 1. Since $\sigma_i + 1 = b$ for all $i = k-j+2, \ldots, k$, every $\sigma_i + 1$ decreases by b again, and the b^{k+1}-th place sum increases by 1. Thus,

$$c_i = \begin{cases} \sigma_{k-j+1} + \delta' - b = -a_k + a_{k-j} - \delta, & \text{if } i = k-j+1; \\ \sigma_i + 1 - b = 0, & \text{if } i = k, k-1, \ldots, k-j+2; \\ \sigma_{k+1} + 1 = a_k, & \text{if } i = k+1, \end{cases}$$

Hence,

$$s_b(g \cdot n) = a_k + (j-1) \cdot 0 + (-a_k) + (a_{k-j} - \delta) + \sum_{i=0}^{k-j} c_i = s_b(g \cdot [a_{k-j} \cdots a_1 a_0]_b)$$

By the induction hypothesis, $s_b(g \cdot [a_{k-j} \cdots a_1 a_0]_b) = g \cdot \iota(a_{k-j} \cdots a_1 a_0)$. Since a_i is dispensable for all $i = k, k-1, \ldots, k-j+1$, $\iota_b([a_{k-j} \cdots a_1 a_0]_b) = \iota_b(n)$. Therefore, $s_b(g \cdot n) = g \cdot \iota_b(n)$. □

Example 4.5.

$$s_{10}(9 \cdot 11123455567000) = 9 \cdot \iota_{10}(1112345556\dot{7}000) = 9 \cdot 1 = 9;$$
$$s_{10}(9 \cdot 4355722256611) = 9 \cdot \iota_{10}(\dot{4}355\dot{7}22256\dot{6}\dot{1}\dot{1} = 9 \cdot 6 = 54.$$

Corollary 4.6. *For any positive integer n, if $n < u_k$, $s_b^k(g \cdot n) = g$.*

Proof. It is obtained by Theorem 4.4 and Theorem 3.8. □

5 Digit sums of integers with a nonzero residue modulo g

Now we consider integers with a nonzero remainder when divided by g: consider $g \cdot n + r$ for any nonnegative integer n and r with $0 < r \le g$. Let a_i, c_i, and c_i' be digits in $\{0, 1, 2, \ldots, g\}$ such that

$$n = [a_m a_{m-1} \cdots a_1 a_0]_b; \; g \cdot n = [c_{m+1} c_m \ldots c_1 c_0]_b; \; g \cdot n + r = [c'_{m+1} c'_m \ldots c'_1 c'_0]_b.$$

By Lemma 4.1, $c_0 = 0$ if $a_0 = 0$; $b - a_0$ if $a_0 > r$. Hence, $c_0 + r = r < b$ if $a_0 = 0$; $b - a_0 + r < b$ if $a_0 > r$, so $c'_0 = c_0 + r$ and $c'_i = c_i$ for all $i > 0$. Therefore,

$$\text{if } a_0 = 0 \text{ or } a_0 > r, \; s_b(g \cdot n + r) = \sum_{i=0}^{m+1} c_i + r = s_b(g \cdot n) + r. \tag{5.1}$$

Assume $0 < a_0 \le r$. If $a_p \neq a_{p-1} = a_{p-2} \cdots = a_1 = a_0$, the b^i-th place sum in $g \cdot n$ for $i = p, p-1, \ldots, 1, 0$ is

$$\sigma_p = b \cdot \delta_p - a_p + a_0 - 1; \; \sigma_i = b - a_0 + a_0 - 1 = g \; (i = 1, 2, \ldots, p-1); \; \sigma_0 = b - a_0,$$

where $\delta_p = 0$ if $a_p = 0$; 1 else. That is,

$$\begin{array}{ccccccccc}
 & & & & & & & & g \\
\times) & \cdots & a_p & a_{p-1} & a_{p-2} & \cdots & a_2 & a_1 & a_0 \\
\hline
 & & & & & & & a_0-1 & b-a_0 \\
 & & & & & & a_0-1 & b-a_0 & \\
 & & & & & & b-a_0 & & \\
 & & & & & \iddots & & & \\
 & & & & a_0-1 & & & & \\
 & & & a_0-1 & b-a_0 & & & & \\
 & & a_0-1 & b-a_0 & & & & & \\
 & * & b\cdot\delta_p - a_p & & & & & & \\
\hline
 & \cdots & a_0-1 & g & g & \cdots & g & g & b-a_0 \\
 & & +b\cdot\delta_p - a_p & & & & & &
\end{array}$$

If $a_p = 0$, $\sigma_p = a_0 - 1 < g$, so $c_p = \sigma_p < g$. If $a_p \neq 0$, $\sigma_p = g - (a_p - a_0)$, Since $a_p \neq a_0$, $1 \leq |a_p - a_0| \leq g - 1$, so $1 \leq \sigma_p < g$ or $b \leq \sigma_p < 2g$. Thus, if $1 \leq \sigma_p < g$, $c_p = \sigma_p < g$, and if $b \leq \sigma_p < 2g$, $c_p = \sigma_p - b < 2g - b < g$. Therefore,

$$c_p < g;\ c_i = g\ (i = p-1, p-2, \ldots, 1);\ c_0 = b - a_0.$$

Since $c_0 + r = b - a_0 + r \geq b$ and $c_p < g$,

$$c'_p = c_p + 1;\ c'_i = 0\ (i = p-1, p-2, \ldots, 1);\ c'_0 = c_0 + r - b.$$

Hence,

$$s_b(g\cdot n + r) = (c_0 + r - b) + (c_p + 1) + \sum_{i=p+1}^{m+1} c_i = \sum_{i=0}^{m+1} c_i - \sum_{i=1}^{p-1} c_i - g + r = s_b(g\cdot n) - g\cdot p + r.$$

That is,

$$\text{if } 0 < a_0 \leq r,\ s_b(g\cdot n + r) = s_b(g\cdot n) - g\cdot p + r. \tag{5.2}$$

Theorem 5.1. *For any nonnegative integers n and r with $0 < r < g$, let a_i be a digit in $\{0, 1, 2, \ldots, g\}$ such that $(n)_b = a_m a_{m-1} \cdots a_1 a_0$ and $a_p \neq a_{p-1} = a_{p-2} = \cdots = a_1 = a_0$. Then,*

$$s_b(g\cdot n + r) = \begin{cases} g\cdot \iota_b(n) + r, & \text{if } a_0 = 0 \text{ or } a_0 > r; \\ g\cdot(\iota_b(n) - p) + r, & \text{if } 1 \leq a_0 \leq r. \end{cases} \tag{5.3}$$

Proof. It is obtained by Theorem 4.4, (5.1), and (5.2). □

Example 5.2.

$$\begin{aligned}
&s_{10}(9\cdot \dot{7}\dot{6}23\dot{4}\dot{4}000 + 3) = 9\cdot 4 + 3 = 39;\\
&s_{10}(9\cdot \dot{7}\dot{6}2344\dot{5}\dot{5}\dot{5} + 3) = 9\cdot 5 + 3 = 48;\\
&s_{10}(9\cdot \dot{7}\dot{6}2344\dot{5}\dot{5}\dot{5} + 5) = 9\cdot(5-3) + 5 = 23;\\
&s_{10}(9\cdot \dot{7}\dot{6}\dot{6}23445\dot{5}\dot{5} + 6) = 9\cdot(5-2) + 6 = 33.
\end{aligned}$$

Combining Theorem 4.4 and Theorem 5.1, we have the following:

Remark 5.3. *For any nonnegative integers n and r with $0 \leq r < g$,*

$$s_b(g \cdot n + r) = g \cdot (\iota_b(n) - p) + r$$

for some nonnegative integer $p \leq \iota_b(n)$.

Corollary 5.4. *For any nonnegative integers n and r with $0 < r < g$,*

$$\text{if } n < u_k, \;\; s_b^{k+1}(g \cdot n + r) = r.$$

Proof. The proof is done by mathematical induction on k. When $k = 1$, $n < u_1 = [1^2]_b$. Then, by Lemma 3.6, $\iota_b(n) = 0$ or 1, and thus, by Theorem 5.1,

$$s_b(g \cdot n + r) = \text{ either } \; g \cdot 0 + r = [r]_b \; \text{ or } \; g \cdot 1 + r = b + r - 1 = [1, r-1]_b.$$

Hence, $s_b(s_b(g \cdot n + r)) = r$.

Induction hypothesis: assume $s_b^k(g \cdot n + r) = r$ for any nonnegative integer $n < u_{k-1}$. If $n < u_k$, $\iota_b(n) < u_{k-1}$ by (3.2). By Remark 5.3,

$$s_b(g \cdot n + r) = g \cdot (\iota_b(n) - p) + r \text{ for some nonnegative integer } p \leq \iota_b(n).$$

Since $\iota_b(n) < u_{k-1}$, $\iota_b(n) - p < u_{k-1}$. Hence, by the induction hypothesis,

$$s_b^k(s_b(g \cdot n + r)) = s_b^k(g \cdot (\iota_b(n) - p) + r) = r.$$

□

References

[1] Atanassov, K. (1985). An arithmetic function and some of its applications, *Bull. of Number Theory and Related Topics*, 9, 18–27.

[2] Atanassov, K. (2015). A digital arithmetical function and some of its applications, *Proceedings of the Jangjeon Mathematical Society*, 18 (4), 511–528.

[3] Datta, D. & Singh, A. N. (1962). *History of Hindu Mathematics: A Source Book*, Asia Publishing House, Bombay, Available online at: `https://ia801600.us.archive.org/20/items/HinduMathematics/HinduMathematics.pdf`

[4] Shallit, J. (2009). *A Second Course in Formal Languages and Automata Theory*, Cambridge Universisty Press, New York.

[5] Weisstein, E. W. *MathWorld-A Wolfram Web Resource*, Available online at: `http://mathworld.wolfram.com/DigitSum.html`.