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Generalized arithmetic subderivative

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Abstract: Let $\emptyset \neq S \subseteq \mathbb{P}$. The arithmetic subderivative of n with respect to S is defined as

$$D_S(n) = n \sum_{p \in S} \frac{\nu_p(n)}{p},$$

where $n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)} \in \mathbb{Z}_+$. In particular, $D_{\mathbb{P}}(n) = D(n)$ is the arithmetic derivative of n, and $D_{\{p\}}(n) = D_p(n)$ is the arithmetic partial derivative of n with respect to $p \in \mathbb{P}$.

For each $p \in S$, let f_p be an arithmetic function. We define generalized arithmetic subderivative of n with respect to S as

$$D_S^f(n) = n \sum_{p \in S} \frac{f_p(n)}{p},$$

where f stands for the collection $(f_p)_{p \in S}$ of arithmetic functions. In this paper, we examine for which kind of functions f_p the generalized arithmetic subderivative is obeys the Leibniz-rule, preserves addition, "usual multiplication" and "scalar multiplication".
Keywords: Arithmetic derivative, Arithmetic partial derivative, Arithmetic subderivative, Arithmetic function, Completely additive function, Completely multiplicative function, Leibniz rule.
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1 Introduction

For each $n \in \mathbb{Z}_+$ there is a unique sequence $(\nu_p(n))_{p \in \mathbb{P}}$ of nonnegative integers (with only finitely many nonzero terms) such that

$$n = \prod_{p \in \mathbb{P}} p^{\nu_p(n)}. \tag{1}$$

Let $\emptyset \neq S \subseteq \mathbb{P}$. The *arithmetic subderivative* of n with respect to S is defined [10] as

$$D_S(n) = n \sum_{p \in S} \frac{\nu_p(n)}{p}.$$

In particular, $D_{\mathbb{P}}(n) = D(n)$ is the *arithmetic derivative* of n, defined by Barbeau [3]. See also, e.g., [2, 13]. Another well-known special case is $D_{\{p\}}(n) = D_p(n)$, the *arithmetic partial derivative* of n with respect to $p \in \mathbb{P}$, defined by Kovič [8]. See also, e.g., [5, 6]. Various properties of the arithmetic derivative and its analogs have been investigated in the literature; for example, connections to classical arithmetic functions, inequalities and arithmetic differential equations and integrals, see e.g. [2, 3, 6, 8, 13].

We build a generalization of the arithmetic subderivative $D_S(n)$ replacing $\nu_p(n)$ with any arithmetic function. So, for each $p \in S$, let f_p be an arithmetic function. Assume that for all n, only a finite number of the values $f_p(n)$ with $p \in S$ is nonzero. This assures that the sum

$$\left(\sum_{p \in S} \frac{f_p}{p}\right)(n) = \sum_{p \in S} \frac{f_p(n)}{p}$$

is finite and thus defines an arithmetic function. For example, for $f_p = \nu_p$ this condition holds.

We define *generalized arithmetic subderivative* of n with respect to S as

$$D_S^f(n) = n \sum_{p \in S} \frac{f_p(n)}{p},$$

where f stands for the collection $(f_p)_{p \in S}$ of arithmetic functions. We refer to the function $D_{\mathbb{P}}^f$ as *generalized arithmetic derivative*, and to the function $D_{\{p\}}^f = D_p^f$ as *generalized arithmetic partial derivative*. For $f_p = \nu_p$, $D_S^f = D_S$ and, in particular, $D_{\mathbb{P}}^f = D$ and $D_{\{p\}}^f = D_p$. The concept of a generalized arithmetic subderivative is new in mathematical literature.

The motivation for this definition is to investigate for which kind of functions f_p the generalized arithmetic subderivative is Leibniz-additive (obeys the Leibniz-rule), linear (preserves addition and "scalar multiplication") and completely multiplicative (preserves "usual multiplication").

The fundamental property of arithmetic subderivative (including arithmetic derivative and arithmetic partial derivative) is that it obeys the Leibniz rule. The purpose of this note is to explain in terms of the function $f_p(n)$ why it is not linear and does not preserve multiplication in the usual sense. We carry out this with the aid of basic classes of arithmetic functions. It appears that the completely additivity of the functions $f_p(n)$ implies the Leibniz-additivity of generalized arithmetic subderivative D_S^f. (Note that the function $f_p = \nu_p$ really is completely additive.) If the functions $f_p(n)$ are constant functions, then generalized arithmetic subderivative D_S^f is linear. In the case of generalized arithmetic partial derivative D_p^f the above two conditions are even necessary and sufficient conditions. Generalized arithmetic subderivative D_S^f preserves multiplication if and only if the function $\sum_{p \in S} \frac{f_p}{p}$ is completely multiplicative. For generalized arithmetic partial derivative D_p^f this means that the function $f_p(n)$ is completely quasimultiplicative with $f_p(1) = p$.

2 Preliminaries

We here review the basic concepts on arithmetic functions applied in this paper.

An arithmetic function f is said to be additive if $f(mn) = f(m) + f(n)$, whenever $\gcd(m, n) = 1$, and an arithmetic function f is said to be multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$, whenever $\gcd(m, n) = 1$. An arithmetic function f is said to be completely additive if $f(mn) = f(m) + f(n)$ for all positive integers m and n, and an arithmetic function f is said to be completely multiplicative if $f(1) = 1$ and $f(mn) = f(m)f(n)$ for all positive integers m and n. See, e.g., [1, 9, 11]

An arithmetic function f is said to be quasimultiplicative if $f(1) \neq 0$ and there exists $c \neq 0$ such that $c\, f(mn) = f(m)f(n)$, whenever $\gcd(m, n) = 1$. Clearly, $c = f(1)$. We define that an arithmetic function f is completely quasimultiplicative if $f(1) \neq 0$ and there exists $c \neq 0$ such that $c\, f(mn) = f(m)f(n)$ for all m, n. It is easy to see that f is quasimultiplicative if and only if f/c is multiplicative for some $c \neq 0$, and f is completely quasimultiplicative if and only if f/c is completely multiplicative for some $c \neq 0$. See, e.g., [4, 12].

An arithmetic function f is said to be Leibniz-additive if there is a completely multiplicative function h_f such that

$$f(mn) = f(m)h_f(n) + f(n)h_f(m) \tag{2}$$

for all positive integers m and n. The arithmetic derivative D, the arithmetic subderivative D_S and the arithmetic partial derivative D_p are Leibniz-additive with $h_D = h_{D_S} = h_{D_p} = \mathrm{Id}$, where $\mathrm{Id}(n) = n$ for all n. See [7].

3 Properties of generalized arithmetic subderivative

Theorem 3.1. *The generalized arithmetic subderivative D_S^f is Leibniz-additive with $h_{D_S^f} = \mathrm{Id}$ if and only if $\sum_{p\in S} \frac{f_p}{p}$ is completely additive.*

Proof. The function D_S^f is Leibniz-additive with $h_{D_S^f} = \mathrm{Id}$ if and only if for all m and n,

$$D_S^f(mn) = D_S^f(m)n + D_S^f(n)m.$$

In other words,

$$mn \sum_{p\in S} \frac{f_p(mn)}{p} = m \sum_{p\in S} \frac{f_p(m)}{p} n + n \sum_{p\in S} \frac{f_p(n)}{p} m,$$

which is equivalent to

$$\sum_{p\in S} \frac{f_p(mn)}{p} = \sum_{p\in S} \frac{f_p(m)}{p} + \sum_{p\in S} \frac{f_p(n)}{p},$$

or,

$$\left(\sum_{p\in S} \frac{f_p}{p}\right)(mn) = \left(\sum_{p\in S} \frac{f_p}{p}\right)(m) + \left(\sum_{p\in S} \frac{f_p}{p}\right)(n).$$

which means that $\sum_{p\in S} \frac{f_p}{p}$ is completely additive. This completes the proof. □

Remark 3.1. Theorem 3.1 could also be proved using [7, Theorem 2.1].

Corollary 3.1. *Given $p \in \mathbb{P}$, the generalized arithmetic partial derivative D_p^f is Leibniz-additive with $h_{D_p^f} = \mathrm{Id}$ if and only if f_p is completely additive.*

Proof. According to Theorem 3.1 the function D_p^f is Leibniz-additive with $h_{D_p^f} = \mathrm{Id}$ if and only if $\frac{f_p}{p}$ is completely additive. But this holds if and only if f_p is completely additive. □

Corollary 3.2. *If f_p is completely additive for all $p \in S$, then the generalized arithmetic subderivative D_S^f is Leibniz-additive with $h_{D_S^f} = \mathrm{Id}$.*

Proof. Assume that f_p is completely additive for all $p \in S$. Then, for all m and n,

$$\begin{aligned}\left(\sum_{p\in S}\frac{f_p}{p}\right)(mn) &= \sum_{p\in S}\frac{f_p(mn)}{p} = \sum_{p\in S}\left(\frac{f_p(m)}{p}+\frac{f_p(n)}{p}\right)\\ &= \sum_{p\in S}\frac{f_p(m)}{p} + \sum_{p\in S}\frac{f_p(n)}{p} = \left(\sum_{p\in S}\frac{f_p}{p}\right)(m) + \left(\sum_{p\in S}\frac{f_p}{p}\right)(n).\end{aligned}$$

Thus $\sum_{p\in S}\frac{f_p}{p}$ is completely additive, and therefore, on the basis of Theorem 3.1, the function D_p^f is Leibniz-additive with $h_{D_p^f} = \mathrm{Id}$. □

Theorem 3.2. *The generalized arithmetic subderivative D_S^f preserves addition, i.e., for all m and n,*

$$D_S^f(m+n) = D_S^f(m) + D_S^f(n)$$

if and only if $\sum_{p\in S}\frac{f_p}{p}$ is a constant function.

Proof. Assume that $\sum_{p\in S}\frac{f_p}{p}$ is a constant function, say, $\sum_{p\in S}\frac{f_p(n)}{p} = c$ for all n. Then, for all m and n,

$$D_S^f(m+n) = (m+n)c = mc + nc = D_S^f(m) + D_S^f(n).$$

Conversely, assume that $D_S^f(m+n) = D_S^f(m) + D_S^f(n)$ for all m and n. Then

$$(m+n)\sum_{p\in S}\frac{f_p(m+n)}{p} = m\sum_{p\in S}\frac{f_p(m)}{p} + n\sum_{p\in S}\frac{f_p(n)}{p}.$$

Now, by induction, we can show that $\sum_{p\in S}\frac{f_p}{p}$ is a constant function. □

Corollary 3.3. *Given $p \in \mathbb{P}$, the generalized arithmetic partial derivative D_p^f preserves addition if and only if f_p is a constant function.*

Proof. According to Theorem 3.2 the function D_p^f preserves addition if and only if $\frac{f_p}{p}$ is a constant function, say c. This means that f_p is the constant function cp. □

Corollary 3.4. *If f_p is a constant function for all $p \in S$, then the generalized arithmetic subderivative D_S^f preserves addition.*

Proof. Assume that f_p is a constant function for all $p \in S$, say, $f_p(n) = c_p$ for all n. Then

$$\sum_{p\in S} \frac{f_p(n)}{p} = \sum_{p\in S} \frac{c_p}{p}$$

for all n. Thus $\sum_{p\in S} \frac{f_p(n)}{p}$ is a constant, and therefore, on the basis of Theorem 3.2, D_S^f preserves addition. □

Theorem 3.3. *The generalized arithmetic subderivative D_S^f preserves "scalar multiplication", i.e., for all a and n,*

$$D_S^f(an) = aD_S^f(n)$$

if and only if $\sum_{p\in S} \frac{f_p}{p}$ is a constant function.

Proof. Assume that $\sum_{p\in S} \frac{f_p}{p}$ is a constant function, say, $\sum_{p\in S} \frac{f_p(n)}{p} = c$ for all n. Then

$$D_S^f(an) = anc = aD_S^f(n).$$

Conversely, assume that $D_S^f(an) = aD_S^f(n)$ for all a and n. Then

$$an \sum_{p\in S} \frac{f_p(an)}{p} = an \sum_{p\in S} \frac{f_p(n)}{p},$$

or

$$\sum_{p\in S} \frac{f_p(an)}{p} = \sum_{p\in S} \frac{f_p(n)}{p}.$$

Taking $n = 1$ we see that $\sum_{p\in S} \frac{f_p}{p}$ is a constant function. □

Corollary 3.5. *Given $p \in \mathbb{P}$, for all a and n,*

$$D_p^f(an) = aD_p^f(n)$$

if and only if f_p is a constant function.

Proof of Corollary 3.5 is similar to that of Corollary 3.3.

Corollary 3.6. *If f_p is a constant function for all $p \in S$, then*

$$D_S^f(an) = aD_S^f(n)$$

for all a and n.

Proof of Corollary 3.6 is similar to that of Corollary 3.4.

Remark 3.2. The only arithmetic function $\sum_{p\in S} \frac{f_p}{p}$ that satisfies the conditions of Theorems 3.1, 3.2 and 3.3 simultaneously is the function that is identically zero.

Theorem 3.4. *The generalized arithmetic subderivative D_S^f is completely multiplicative (i.e., preserves "usual multiplication") if and only if $\sum_{p\in S} \frac{f_p}{p}$ is completely multiplicative.*

Proof. We first note that $D_S^f(1) = \sum_{p\in S} \frac{f_p(1)}{p}$. Assume now that this value equals 1. Then D_p^f is completely multiplicative if and only if for all m and n,

$$D_S^f(mn) = D_S^f(m)D_S^f(n).$$

In other words,

$$mn\sum_{p\in S}\frac{f_p(mn)}{p} = m\sum_{p\in S}\frac{f_p(m)}{p}n\sum_{p\in S}\frac{f_p(n)}{p},$$

which is equivalent to

$$\sum_{p\in S}\frac{f_p(mn)}{p} = \sum_{p\in S}\frac{f_p(m)}{p}\sum_{p\in S}\frac{f_p(n)}{p}.$$

This means that $\sum_{p\in S}\frac{f_p}{p}$ is completely multiplicative. This completes the proof. $\square$

Corollary 3.7. *Given $p \in \mathbb{P}$, the generalized arithmetic partial derivative D_p^f is completely multiplicative if and only if f_p is completely quasimultiplicative with $f(1) = p$.*

Proof. According to Theorem 3.4 the function D_p^f is completely multiplicative if and only if $\frac{f_p}{p}$ is completely multiplicative. This means that f_p is completely quasimultiplicative with $f_p(1) = p$. $\square$

Remark 3.3. There is no arithmetic function $\sum_{p\in S}\frac{f_p}{p}$ that satisfies the conditions of Theorems 3.1 and 3.4 simultaneously. In fact, if $\sum_{p\in S}\frac{f_p}{p}$ is completely additive, then $\sum_{p\in S}\frac{f_p(1)}{p} = 0$, and if it is completely multiplicative, then $\sum_{p\in S}\frac{f_p(1)}{p} = 1$. The only arithmetic function $\sum_{p\in S}\frac{f_p}{p}$ that satisfies the conditions of Theorems 3.2, 3.3 and 3.4 simultaneously is the function that is identically 1.

We next present "co-prime" analogs of Theorems 3.1 – 3.4. Proofs are similar to those of Theorems 3.1 – 3.4.

Theorem 3.5. *The generalized arithmetic subderivative D_S^f satisfies Leibniz-rule (2) with $h_{D_p^f} = \mathrm{Id}$ for all m and n with $\gcd(m,n) = 1$ if and only if $\sum_{p\in S}\frac{f_p}{p}$ is additive.*

Theorem 3.6. *We have*

$$D_S^f(m+n) = D_S^f(m) + D_S^f(n)$$

for all m and n with $\gcd(m,n) = 1$ if and only if $\sum_{p\in S}\frac{f_p}{p}$ is a constant function.

Theorem 3.7. *We have*

$$D_S^f(an) = aD_S^f(n)$$

for all a and n with $\gcd(a,n) = 1$ if and only if $\sum_{p\in S}\frac{f_p}{p}$ is a constant function.

Theorem 3.8. *The generalized arithmetic subderivative D_S^f is multiplicative if and only if $\sum_{p\in S}\frac{f_p}{p}$ is multiplicative.*

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