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Representation of higher even-dimensional rhotrix

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Abstract: The multiplication of higher even-dimensional rhotrices is presented and generalized. The concept of empty rhotrix, and the necessary and sufficient conditions for an even-dimensional rhotrix to be represented over a linear map, are investigated and presented.
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1 Introduction

A rhotrix is an arrangement of numbers in a rhomboid shape. This is similar to a matrix, which is an arrangement of numbers in a rectangular form. Rhotrix was first introduced by Ajibade [1], as an extension of the idea suggested by Atanassov and Shannon [4] in their work titled "matrix -tertions and matrix-noitrets". A formal definition of a real rhotrix as presented in the maiden paper is given below:

Definition 1.1. *[1] A real rhotrix set of dimension three, denoted as $\hat{R}_3(\Re)$ is defined as:*

$$\hat{R}_3(\Re) = \{ \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \ \in \Re \}$$

where $c = h(R)$ is called the heart of any rhotrix R belonging to $\hat{R}_3(\Re)$ (a set of all real rhotrices of dimension 3) and $\Re$ is the set of real numbers.

Examples showing extension of this set and analysis are copious in literature. A few are presented in these references [2, 3, 5–7, 10–14, 18, 19]. It has been noted that these heart-oriented rhotrices are always of odd dimension. Thus, a rhotrix with even dimension is recently being introduced by Isere [8, 9]. The algebra and analysis establishing this new structure as mathematically tractable were all presented in [9]. The heart-based rhotrices are classified as classical rhotrices, while even-dimensional rhotrices are classified as non-classical rhotrices [8].

Mean while, the addition and multiplication of heart-based rhotrices (h-rhotrices) were first presented in [1]. Thus, addition and multiplication of two heart-based rhotrices are defined as:

$$R + Q = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle + \left\langle \begin{array}{ccc} & f & \\ g & h(Q) & j \\ & k & \end{array} \right\rangle = \left\langle \begin{array}{ccc} & a+f & \\ b+g & h(R)+h(Q) & d+j \\ & e+k & \end{array} \right\rangle$$

and

$$R \circ Q = \left\langle \begin{array}{ccc} & ah(Q) + fh(R) & \\ bh(Q) + gh(R) & h(R)h(Q) & dh(Q) + jh(R) \\ & eh(Q) + kh(R) & \end{array} \right\rangle,$$

respectively. A generalization of this hearty multiplication is given in [14] and in [6]. A row-column multiplication of heart-based rhotrices was proposed by Sani [15] as:

$$R \circ Q = \left\langle \begin{array}{ccc} & af + dg & \\ bf + eg & h(R)h(Q) & aj + dk \\ & bj + ek & \end{array} \right\rangle.$$

A generalization of this row-column multiplication was also later given by Sani [16] as:

$$R_n \circ Q_n = \langle a_{ij}, c_{ij} \rangle \circ \langle b_{ij}, d_{lk} \rangle = \left\langle \sum_{i,j=1}^{t} (a_{ij} b_{ij}), \sum_{l,k=1}^{t-1} (c_{lk} d_{lk}) \right\rangle, t = (n+1)/2,$$

where R_n and Q_n are n-dimensional rhotrices (with n rows and n columns). These two methods of multiplication of rhotrices are very popular in literature. In both methods, the heart plays a significant role as shown above. A lot of work has been done on h-rhotrices. These works are also well known in literature, such as the conversion of a rhotrix into a coupled matrix by Sani [17]. A generalization of rhotrix was introduced as paraletrix by Aminu and Michael [3]. This concept shows more flexibility in mathematical arrays of numbers, where the number of rows and columns need not be the same. It was noted that not every paraletrix has a heart. Consequently, a rhotrix without a heart was introduced in [8, 9] as heartless rhotrices (*hl-rhotrices*). Such rhotrices were found to be even-dimensional. The simplest non-trivial even-dimensional rhotrix is of dimension two, and it is stated below:

Definition 1.2. *A real rhotrix of dimension two is given as*

$$A = \{ \left\langle \begin{array}{ccc} & a & \\ b & & d \\ & e & \end{array} \right\rangle : a, b, d, e \in \mathbb{R} \}.$$

It is to be noted that an n-dimensional rhotrix with n being even has its cardinality as $|R_n| = \frac{1}{2}(n^2 + 2n) \quad \forall \ n \in 2N$. The multiplication of h-rhotrices, as remarked in [1], can be done in many ways. This is also true with even-dimensional rhotrices. In this work, we define multiplication of two even-dimensional rhotrices elementwise as follows:

$$A \circ B = \left\langle \begin{matrix} & a_{11} & \\ a_{21} & & a_{12} \\ & a_{22} & \end{matrix} \right\rangle \circ \left\langle \begin{matrix} & b_{11} & \\ b_{21} & & b_{12} \\ & b_{22} & \end{matrix} \right\rangle = \left\langle \begin{matrix} & a_{11}b_{11} & \\ a_{21}b_{21} & & a_{12}b_{12} \\ & a_{22}b_{22} & \end{matrix} \right\rangle .$$

Moreover, we shall be looking at multiplication of higher even-dimensional rhotrices, the concept of empty rhotrix and the representation of an even-dimensional rhotrix over a linear map. The concept of rhotrix linear transformation was first investigated by Mohammed *et al* [13]. The necessary and sufficient conditions for a rhotrix to be represented by a linear map were given in [13]. It is to be noted that the rhotrix investigated was an h-rhotrix. These conditions will be stated in the next section. However, an extension of these conditions will be considered in this work, and the necessary and sufficient conditions for an even-dimensional rhotrix to be represented by a linear map will be presented.

2 Preliminaries

Some definitions will be considered in this section that will be useful in achieving the results anticipated in this work.

Definition 2.1. *[13] A rhotrix R of dimension n is given as:*

$$R_n = \left\langle \begin{matrix} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & - & - & - & - & - & \\ a_{t1} & - & - & - & - & - & a_{1t} \\ & - & - & - & - & - & \\ & & a_{tt-1} & c_{t-1t-1} & a_{t-1t} & & \\ & & & a_{tt} & & & \end{matrix} \right\rangle .$$

The element $a_{ij}(i,j = 1,2,...,t)$ and $c_{kl}(k,l = 1,2,...t-1)$ are called the major and minor entries of R, respectively. This is usually denoted as $R_n = \langle a_{ij}, c_{kl} \rangle$.

Definition 2.2. *[13] Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an n dimensional rhotrix. Then, a_{ij} is the (i,j)-entries called the major entries of R_n and c_{kl} is the (k,l)-entries called the minor entries of R_n.*

Definition 2.3. *[16] A rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ of n dimension is a couple of two matrices (a_{ij}) and (c_{kl}) consisting of its major and minor matrices of R_n.*

Definition 2.4. *[13] Let $R_n = \langle a_{ij}, c_{kl} \rangle$ be an n dimensional rhotrix. Then, rows and columns of a_{ij}(c_{kl}) will be called the major (minor) rows and columns of R_n, respectively.*

Definition 2.5. *[13] For any odd integer n, a $n \times n$ matrix (a_{ij}) is called a filled coupled matrix if $a_{ij} = 0$ for all i, j whose sum $i + j$ is odd. We shall refer to these entries as the null entries of the filled coupled matrix.*

Remark 2.1. (i) $R_n = \langle a_{ij}, c_{kl} \rangle$ *is a representation of any rhotrix.* (ii) *Moreover, an even-dimensional rhotrix can also be represented as $R_n = \langle a_{ij}, c_{kl} \rangle$ or simply as $R_n = \langle a_i, \rangle$.* (iii) *a $(n \times n)$ filled coupled matrix has n^2 entries.*

Definition 2.6. *For any odd integer n, a $(n \times n)$ matrix (a_{ij}) is called a completely filled coupled matrix if $a_{ij} = 0$ for all i, j whose sum $i + j$ is odd and for all $i = j = \frac{n+1}{2}$. The entry corresponding to $a_{ij} = 0, i = j = \frac{n+1}{2}$ is a special null-entry called the null entry of the completely filled coupled matrix.*

Definition 2.7. *The entries a_{ij} whose sum $i+j$ is even, except when $i = j = \frac{n+1}{2} \quad \forall\, n \in 2Z^+ + 1$, are called the real entries of the completely filled coupled matrix.*

Theorem 2.1. *[13] Let $n \in 2Z^+ + 1$ and F be a field. Then, a linear map $T : F^n \mapsto F^n$ can be represented by a rhotrix with respect to the standard basis if and only if T is defined as:*

$$\begin{aligned} T(x_1, y_1, x_2, y_2, ..., y_{t-1}, x_t) = (&\alpha_1(x_1, x_2, ..., x_t), \beta_1(y_1, y_2, ..., y_{t-1}),\\ &\alpha_2(x_1, x_2, ..., x_t), \beta_2(y_1, y_2, ..., y_{t-1}), ...,\\ &\beta_{t-1}(y_1, y_2, ..., y_{t-1}), \alpha_t(x_1, x_2, ..., x_t)) \end{aligned}$$

where $t = \frac{n+1}{2}$, $\alpha_1, \alpha_2, ..., \alpha_t$ and $\beta_1, \beta_2, ..., \beta_{t-1}$ are any linear maps on F^t and F^{t-1}, respectively.

Lemma 2.2. *Let $[a_{ij}]_n$ be a $(n \times n)$ filled coupled matrix, then:*

(a) *The number of all the real entries is given as*

$$\Pi_n = \frac{1}{2}(n^2 + 1) \quad \forall\, n \in 2Z^+ + 1$$

(b) *The number of all the null entries is given as*

$$\emptyset_n = \frac{1}{2}(n^2 - 1) \quad \forall\, n \in 2Z^+ + 1$$

Proof. Since a $(n \times n)$ filled coupled matrix has n^2 entries, then $(a) + (b) = n^{2\cdot}$. Consider:

$$\frac{1}{2}(n^2 + 1) + \frac{1}{2}(n^2 - 1) = n^2$$

Then, (a) and (b) are true. □

Remark 2.2. $\Pi_n + \emptyset_n$ *as in Lemma 2.2 is an odd-dimensional rhotrix, i.e., the real entries are odd.*

Lemma 2.3. *Let $[a_{ij}]_n$ be a completely filled coupled matrix, then:*

(a) *The number of all the real entries is given as*

$$\Pi_n = \frac{1}{2}(n^2 - 1) \quad \forall n \in 2Z^+ + 1$$

(b) *The number of all the null entries is given as*

$$\emptyset_n = \frac{1}{2}(n^2 + 1) \quad \forall n \in 2Z^+ + 1.$$

Proof. The proof is similar to the proof of Lemma 2.2 above. □

Remark 2.3. $\Pi_n + \emptyset_n$ *as in Lemma 2.3 is an even-dimensional rhotrix, i.e., the real entries are even.*

Theorem 2.4. *There is a one-to-one correspondence between the set of all n-dimensional rhotrices over a field F and the set of all $n \times n$ completely filled coupled matrices over F.*

Proof. The proof follows from Lemma 2.3 and the fact that any n-dimensional rhotrix is n^2 entries. □

Remark 2.4. (i) *The set of all real entries* (Π_n) *of the completely filled coupled matrix corresponds to the entries of an even-dimensional rhotrix $R_n = \langle a_{ij}, c_{kl} \rangle$ or simply as $R_n = \langle a_i \rangle$.*

(ii) *A filled coupled matrix and a completely filled coupled matrix comprise of both real and null entries.*

(iii) *All heart-based rhotrices result in a filled coupled matrix while all even-dimensional rhotrices result in a completely filled coupled matrix.*

3 Main Results

This section presents the main results starting with the concept of empty rhotrix, then some examples of filled and completely filled coupled matrices and multiplication of higher even-dimensional rhotrices.

3.1 The concept of empty rhotrix

Definition 3.1. *A rhotrix that has no entry is an empty rhotrix, e.g., $A = \langle\ \rangle$.*

Lemma 3.1. *An empty rhotrix A of n-th dimension contains null-entry of a completely- filled matrix as its only entry.*

Proof. Recall that for an even-dimensional rhotrix $|R_n| = \frac{1}{2}(n^2 + 2n) \quad \forall\, n \in 2\mathbb{N}$. Since $n \in 2\mathbb{N}$ implies that $0 \in 2\mathbb{N}$ and $R_0 = \langle\ \rangle$. The proof follows. □

Corollary 3.1.1. *An empty real rhotrix is even-dimensional.*

Proof. We prove by contradiction. Let R_n be any n-dimensional real rhotrix. Suppose, n is odd, then, its cardinality can be represented as

$$|R_n| = \frac{1}{2}(n^2 + 1) \ \ n \in 2\mathbb{Z}^+ + 1.$$

Since, an empty rhotrix has no entry, its cardinality is zero. That is

$$0 = \frac{1}{2}(n^2 + 1)$$

implies that $n = \pm i$. Then, we have a contradiction. Now, suppose that n is even, then

$$|R_n| = \frac{1}{2}(n^2 + 2n) \ \ n \in 2\mathbb{N}$$

implies that $n = 0 \in 2\mathbb{N}$. Thus, an empty rhotrix is even-dimensional. □

Remark 3.1. $\mathbb{N}$ *is a set of non-negative integers*

3.2 Some examples of filled and completely filled coupled matrices

Example 3.1. *A rhotrix of dimension five* (R_5) *is given by:*

$$R_5 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & a_{22} & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle$$

Then its corresponding filled coupled matrix is presented below:

$$M(R_5) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & c_{11} & 0 & c_{12} & 0 \\ a_{21} & 0 & a_{22} & 0 & a_{23} \\ 0 & c_{21} & 0 & c_{22} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{bmatrix}$$

Example 3.2. *A rhotrix of dimension seven* (R_7) *is given by:*

$$R_7 = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ a_{41} & c_{31} & a_{32} & c_{22} & a_{23} & c_{13} & a_{14} \\ & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & \\ & & a_{43} & c_{33} & a_{34} & & \\ & & & a_{44} & & & \end{array} \right\rangle$$

Then its corresponding filled coupled matrix will be presented below:

$$M(R_7) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} & 0 & a_{14} \\ 0 & c_{11} & 0 & c_{12} & 0 & c_{13} & 0 \\ a_{21} & 0 & a_{22} & 0 & a_{23} & 0 & a_{24} \\ 0 & c_{21} & 0 & c_{22} & 0 & c_{23} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} & 0 & a_{34} \\ 0 & c_{31} & 0 & c_{32} & 0 & c_{33} & 0 \\ a_{41} & 0 & a_{42} & 0 & a_{43} & 0 & a_{44} \end{bmatrix}$$

Example 3.3. *A rhotrix of dimension four* (R_4) *is given by:*

$$R_4 = \left\langle \begin{array}{ccccc} & & a_{11} & & \\ & a_{21} & c_{11} & a_{12} & \\ a_{31} & c_{21} & & c_{12} & a_{13} \\ & a_{32} & c_{22} & a_{23} & \\ & & a_{33} & & \end{array} \right\rangle$$

Then its corresponding completely filled coupled matrix is presented below:

$$C(R_4) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} \\ 0 & c_{11} & 0 & c_{12} & 0 \\ a_{21} & 0 & 0^* & 0 & a_{23} \\ 0 & c_{21} & 0 & c_{22} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} \end{bmatrix}$$

Example 3.4. *A rhotrix of dimension six* (R_6) *is given by:*

$$R_6 = \left\langle \begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{21} & c_{11} & a_{12} & & \\ & a_{31} & c_{21} & a_{22} & c_{12} & a_{13} & \\ a_{41} & c_{31} & a_{32} & & a_{23} & c_{13} & a_{14} \\ & a_{42} & c_{32} & a_{33} & c_{23} & a_{24} & \\ & & a_{43} & c_{33} & a_{34} & & \\ & & & a_{44} & & & \end{array} \right\rangle$$

Then its corresponding completely filled coupled matrix is:

$$C(R_6) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 & a_{13} & 0 & a_{14} \\ 0 & c_{11} & 0 & c_{12} & 0 & c_{13} & 0 \\ a_{21} & 0 & a_{22} & 0 & a_{23} & 0 & a_{24} \\ 0 & c_{21} & 0 & 0^* & 0 & c_{23} & 0 \\ a_{31} & 0 & a_{32} & 0 & a_{33} & 0 & a_{34} \\ 0 & c_{31} & 0 & c_{32} & 0 & c_{33} & 0 \\ a_{41} & 0 & a_{42} & 0 & a_{43} & 0 & a_{44} \end{bmatrix}$$

Remark 3.2. *A completely filled coupled matrix is obtained from even-dimensional rhotrix, and contains the null-entry of the completely filled coupled matrix denoted as* 0^**, while a filled coupled matrix is obtained from odd-dimensional rhotrices.*

3.3 Multiplication of higher even-dimensional rhotrices

Multiplication of higher even-dimensional rhotrices whether even or odd dimensional can be defined in many ways. In this work, elementwise multiplication method is presented for higher even-dimensional rhotrices. Examples of rhotrices of dimension four are presented for the purpose of demonstration. Let

$$A = \left\langle \begin{array}{ccccc} & & a_1 & & \\ & a_2 & a_3 & a_4 & \\ a_5 & a_6 & & a_7 & a_8 \\ & a_9 & a_{10} & a_{11} & \\ & & a_{12} & & \end{array} \right\rangle, \quad B = \left\langle \begin{array}{ccccc} & & b_1 & & \\ & b_2 & b_3 & b_4 & \\ b_5 & b_6 & & b_7 & b_8 \\ & b_9 & b_{10} & b_{11} & \\ & & b_{12} & & \end{array} \right\rangle$$

then

$$A \odot B = \left\langle \begin{array}{ccccc} & & a_1b_1 & & \\ & a_2b_2 & a_3b_3 & a_4b_4 & \\ a_5b_5 & a_6b_6 & & a_7b_7 & a_8b_8 \\ & a_9b_9 & a_{10}b_{10} & a_{11}b_{11} & \\ & & a_{12}b_{12} & & \end{array} \right\rangle$$

Example 3.5. *Let*

$$A = \left\langle \begin{array}{ccccc} & & 2 & & \\ & 3 & 1 & 4 & \\ 5 & 6 & & 7 & 8 \\ & 9 & 10 & 5 & \\ & & 3 & & \end{array} \right\rangle, \quad B = \left\langle \begin{array}{ccccc} & & 3 & & \\ & 2 & 4 & 1 & \\ 7 & 8 & & 9 & 5 \\ & 6 & 8 & 3 & \\ & & 10 & & \end{array} \right\rangle$$

then

$$A \odot B = \left\langle \begin{array}{ccccc} & & 2 \odot 3 & & \\ & 3 \odot 2 & 1 \odot 4 & 4 \odot 1 & \\ 5 \odot 7 & 6 \odot 8 & & 7 \odot 9 & 8 \odot 5 \\ & 9 \odot 6 & 10 \odot 8 & 5 \odot 3 & \\ & & 3 \odot 10 & & \end{array} \right\rangle = \left\langle \begin{array}{ccccc} & & 6 & & \\ & 6 & 4 & 4 & \\ 35 & 48 & & 63 & 40 \\ & 54 & 80 & 15 & \\ & & 30 & & \end{array} \right\rangle$$

Generally, a rhotrix R of dimension n (n being even) can be written as:

$$R_n = \left\langle \begin{array}{ccccccc} & & & r_1 & & & \\ & & r_2 & r_3 & r_4 & & \\ & - & - & - & - & - & \\ - & - & - & & - & - & - \\ & - & - & - & - & - & \\ & & r_{\frac{n^2+2n-6}{2}} & r_{\frac{n^2+2n-4}{2}} & r_{\frac{n^2+2n-2}{2}} & & \\ & & & r_{\frac{n^2+2n}{2}} & & & \end{array} \right\rangle.$$

A generalization of the elementwise multiplication of even-dimensional rhotrices is as follows. Let $R_n = \langle a_i \rangle$ and $Q_n = \langle b_j \rangle$, be two even-dimensional rhotrices, then their multiplication is as follows

$$R_n \odot Q_n = \langle a_i \rangle \odot \langle b_j \rangle = \left\langle \sum_{i=1}^{t} a_i \right\rangle \odot \left\langle \sum_{j=1}^{t} b_j \right\rangle = \left\langle \sum_{k=1}^{t} (a_k b_k) \right\rangle, \ \ t = (n^2 + 2n)/2, \ \ n \in 2\mathbb{N},$$

where the product $(a_{ij} b_{ij})$ is empty whenever $i = j = \frac{t+1}{2} \ \forall \ t \in 2Z^+ + 1$.

4 Linear maps on an even-dimensional rhotrix

The concept of representation by a linear map helps to establish the existence of a linear structure. In this section, we investigate the representation of an even-dimensional rhotrix over a linear map.

Theorem 4.1. *Let $n \in 2Z^+ + 1$ and F be a field. Then, a linear map $\tau : F^n \mapsto F^n$ can be represented by an even-dimensional rhotrix with respect to the standard basis if and only if τ is defined as:*

$$\begin{aligned} \tau(x_1, y_1, x_2, y_2, ..., y_{t-1}, x_t) = (&\alpha_1(x_1, x_2, ..., x_t), \beta_1(y_1, y_2, ..., y_{t-1}), \\ &\alpha_2(x_1, x_2, ..., x_t), \beta_2(y_1, y_2, ..., y_{t-1}), ..., \\ &\beta_{\frac{t}{2}}(y_1, y_2, ..., 0(y_{\frac{t}{2}}), ..., y_{t-1}) \ \forall \ t-1 \in 2Z^+ + 1, \\ &\alpha_{\frac{t+1}{2}}(x_1, x_2, ..., 0(x_{\frac{t+1}{2}}), ..., x_t) \ \forall \ t \in 2Z^+ + 1, ..., \\ &\beta_{t-1}(y_1, y_2, ..., y_{t-1}), \alpha_t(x_1, x_2, ..., x_t)), \end{aligned}$$

where $t = \frac{n+2}{2}$, $\alpha_1, \alpha_2, ..., \alpha_{\frac{t+1}{2}}, ..., \alpha_t$ and $\beta_1, \beta_2, ..., \beta_{\frac{t}{2}}, ..., \beta_{t-1}$ are any linear maps on F^t and F^{t-1}, respectively.

Proof. Case 1 (when $t \in 2Z^+ + 1$).
Given that

$$\begin{aligned} \tau(x_1, y_1, x_2, y_2, ..., y_{t-1}, x_t) = (&\alpha_1(x_1, x_2, ..., x_t), \beta_1(y_1, y_2, ..., y_{t-1}), \\ &\alpha_2(x_1, x_2, ..., x_t), \beta_2(y_1, y_2, ..., y_{t-1}), ..., \\ &\beta_{\frac{t}{2}}(y_1, y_2, ..., 0(y_{\frac{t}{2}}), ..., y_{t-1}) \ \forall \ t-1 \in 2Z^+ + 1, \\ &\alpha_{\frac{t+1}{2}}(x_1, x_2, ..., 0(x_{\frac{t+1}{2}}), ..., x_t) \ \forall \ t \in 2Z^+ + 1, ..., \\ &\beta_{t-1}(y_1, y_2, ..., y_{t-1}), \alpha_t(x_1, x_2, ..., x_t)) \end{aligned}$$

where $t = \frac{n+2}{2}$, $\alpha_1, \alpha_2, ..., \alpha_{\frac{t+1}{2}}, ..., \alpha_t$ and $\beta_1, \beta_2, ..., \beta_{\frac{t}{2}}, ..., \beta_{t-1}$ are any linear maps on F^t and F^{t-1}, respectively.

Now let us consider the standard basis:

$$\begin{aligned}
\tau(e_1) = &\quad [\alpha_1(1,0,\ldots,0),0,\ldots,\alpha_t(1,0.\ldots,0)]\\
\tau(e_1) = &\quad [0,\beta_1(1,0,\ldots,0),0,\ldots,\beta_{t-1}(1,0.\ldots,0)]\\
&\vdots\\
\tau(e_1) = &\quad [\alpha_1(1,0,\ldots,0),0,\ldots,\alpha_{\frac{t+1}{2}}(0.\ldots,0(x_{\frac{t+1}{2}}),\ldots 0),\ldots,\alpha_t(1,0.\ldots,0)]\\
&\vdots\\
\tau(e_t) = &\quad [0,\beta_1(0,\ldots,t),0,\ldots,\beta_{t-1}(0.\ldots,0),1]\\
\tau(e_t) = &\quad [\alpha_1(0,\ldots,t),0,\ldots,\alpha_t(0.\ldots,1)]
\end{aligned}$$

Putting the above linear equations into a matrix, we have

$$\begin{bmatrix}
\alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1t-1} & 0 & \alpha_{1t}\\
0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1t-1} & o\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{\frac{t+1}{2}1} & 0 & \alpha_{\frac{t+1}{2}2} & \ldots & 0 & \ldots & \alpha_{\frac{t+1}{2}t}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & \beta_{t-11} & 0 & \ldots & 0 & \beta_{t-1t-1} & 0\\
\alpha_{t1} & 0 & \alpha_{t2} & \ldots & \alpha_{tt-1} & 0 & \alpha_{tt}
\end{bmatrix}$$

The transpose of the above matrix is the matrix of transformation denoted as

$$m(\tau) = \begin{bmatrix}
\alpha_{11} & 0 & \alpha_{12} & \ldots & \alpha_{1t-1} & 0 & \alpha_{1t}\\
0 & \beta_{11} & 0 & \ldots & 0 & \beta_{1t-1} & o\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
\alpha_{\frac{t+1}{2}1} & 0 & \alpha_{\frac{t+1}{2}2} & \ldots & 0 & \ldots & \alpha_{\frac{t+1}{2}t}\\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\
0 & \beta_{t-11} & 0 & \ldots & 0 & \beta_{t-1t-1} & 0\\
\alpha_{t1} & 0 & \alpha_{t2} & \ldots & \alpha_{tt-1} & 0 & \alpha_{tt}
\end{bmatrix}^T$$

The result is a completely filled coupled matrix from which we have the even-dimensional rhotrix representation.

Conversely, suppose that $\tau : F^n \mapsto F^n$ has an even-dimensional rhotrix representation $\langle \alpha_{ij}, \beta_{kl} \rangle$ in the standard basis. Then, the corresponding matrix representation of τ is the completely filled coupled matrix given above. From this, we obtain the linear system below:

$$\begin{aligned}
\tau(e_1) = &\quad [\alpha_1(1,0,\ldots,0),0,\ldots,\alpha_t(1,0.\ldots,0)]\\
\tau(e_1) = &\quad [0,\beta_1(1,0,\ldots,0),0,\ldots,\beta_{t-1}(1,0.\ldots,0)]\\
&\vdots\\
\tau(e_1) = &\quad [\alpha_1(1,0,\ldots,0),0,\ldots,\alpha_{\frac{t+1}{2}}(0.\ldots,0(x_{\frac{t+1}{2}}),\ldots 0),\ldots,\alpha_t(1,0.\ldots,0)]\\
&\vdots\\
\tau(e_t) = &\quad [0,\beta_1(0,\ldots,t),0,\ldots,\beta_{t-1}(0.\ldots,0),1]\\
\tau(e_t) = &\quad [\alpha_1(0,\ldots,t),0,\ldots,\alpha_t(0.\ldots,1)]
\end{aligned}$$

Case 1 (when $t-1 \in 2Z^+ + 1$).
The proof follows similarly. □

Remark 4.1. *The above theorem is seeing our even-dimensional rhotrix as a completely filled couple matrix.*

Example 4.1. *Consider the linear mapping $\tau : \mathfrak{R} \mapsto \mathfrak{R}$ defined by $\tau(x,y,z) = (ax+dz, 0, bx+ez)$. Find the hl-rhotrix represented by the linear transformation(linear map) τ with respect to the standard basis.*

Solution:

$$\begin{aligned}
\tau(1,0,0) =&(a,0,b)\\
\tau(0,1,0) =&(0,0,0)\\
\tau(0,0,1) =&(d,0,e)
\end{aligned}$$

Then, putting this into matrix gives

$$\begin{pmatrix} a & 0 & b\\ 0 & 0 & 0\\ d & 0 & e \end{pmatrix}$$

Thus, the matrix of representation is the transpose of the above matrix

$$m(\tau) = \begin{pmatrix} a & 0 & b\\ 0 & 0 & 0\\ d & 0 & e \end{pmatrix}^T = \begin{bmatrix} a & 0 & d\\ 0 & 0 & 0\\ b & 0 & e \end{bmatrix}$$

which is a completely filled coupled matrix. Then the even-dimensional rhotrix by τ is

$$R(\tau) = \left\langle \begin{matrix} & a & \\ b & & d\\ & e & \end{matrix} \right\rangle$$

Example 4.2. *Consider the linear mapping $\tau : \mathfrak{R} \mapsto \mathfrak{R}$ defined by $\tau(a,b,c,d,e) = (a+2c-5e, 3b+6d, 4a+10e, 8b-11d, 9a+12c+13e)$. Find the even-dimensional rhotrix represented by the linear transformation(linear map) τ with respect to the standard basis.*

Solution:

$$\begin{aligned}\tau(1,0,0,0,0) =&(1,0,4,0,9)\\ \tau(0,1,0,0,0) =&(0,3,0,8,0)\\ \tau(0,0,1,0,0) =&(2,0,0,0,12)\\ \tau(0,0,0,1,0) =&(0,6,0,-11,0)\\ \tau(0,0,0,0,1) =&(-5,0,10,0,13)\end{aligned}$$

Thus, the matrix of representation is given below:

$$m(\tau) = \begin{bmatrix} 1 & 0 & 4 & 0 & 9 \\ 0 & 3 & 0 & 8 & 0 \\ 2 & 0 & 0 & 0 & 12 \\ 0 & 6 & 0 & -11 & 0 \\ -5 & 0 & 10 & 0 & 13 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 & 2 & 0 & -5 \\ 0 & 3 & 0 & 6 & 0 \\ 4 & 0 & 0 & 0 & 10 \\ 0 & 8 & 0 & -11 & 0 \\ 9 & 0 & 12 & 0 & 13 \end{bmatrix}$$

which is a completely filled coupled matrix. Then the even-dimensional rhotrix by τ is

$$R(\tau) = \left\langle \begin{array}{ccccc} & & 1 & & \\ & 4 & 3 & 2 & \\ 9 & 8 & & 6 & -5 \\ & 12 & -11 & 10 & \\ & & 13 & & \end{array} \right\rangle$$

This is an even-dimensional rhotrix of dimension 4.

Example 4.3. *Consider the linear mapping $\tau : \mathfrak{R} \mapsto \mathfrak{R}$ defined by $\tau(a,b,c,d,e,f,g) = (3a + 2c-4g-2e, 5b+4d+3f, 5a-7c+3e-g, 8b-5f, 7a+12c-3e+5g, -4b+2d+f, a+14c-7e+10g)$. Find the even-dimensional rhotrix represented by the linear transformation(linear map) τ with respect to the standard basis.*

Solution:

$$\begin{aligned}\tau(1,0,0,0,0,0,0) =&(3,0,5,0,7,0,1)\\ \tau(0,1,0,0,0,0,0) =&(0,5,0,8,0,-4,0)\\ \tau(0,0,1,0,0,0,0) =&(2,0,-7,0,12,0,14)\\ \tau(0,0,0,1,0,0,0) =&(0,4,0,0,0,2,0)\\ \tau(0,0,0,0,1,0,0) =&(-2,0,3,0,-3,0,-7)\\ \tau(0,0,0,0,1,0,0) =&(0,3,0,-5,0,1,0)\\ \tau(0,0,0,0,1,0,0) =&(-4,0,-1,0,5,0,10)\end{aligned}$$

Thus, the matrix of representation is given below:

$$m(\tau) = \begin{bmatrix} 3 & 0 & 5 & 0 & 7 & 0 & 1 \\ 0 & 5 & 0 & 8 & 0 & -4 & 0 \\ 2 & 0 & -7 & 0 & 12 & 0 & 14 \\ 0 & 4 & 0 & 0 & 0 & 2 & 0 \\ -2 & 0 & 3 & 0 & -3 & 0 & -7 \\ 0 & 3 & 0 & -5 & 0 & 1 & 0 \\ -4 & 0 & -1 & 0 & 5 & 0 & 10 \end{bmatrix}^T = \begin{bmatrix} 3 & 0 & 2 & 0 & -2 & 0 & -4 \\ 0 & 5 & 0 & 4 & 0 & 3 & 0 \\ 5 & 0 & -7 & 0 & 3 & 0 & -1 \\ 0 & 8 & 0 & 0 & 0 & -5 & 0 \\ 7 & 0 & 12 & 0 & -3 & 0 & 5 \\ 0 & -4 & 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 14 & 0 & -7 & 0 & 10 \end{bmatrix}$$

which is a completely filled coupled matrix. Then the even-dimensional rhotrix by τ is

$$R(\tau) = \left\langle \begin{array}{ccccccc} & & & 3 & & & \\ & & 5 & 5 & 2 & & \\ & 7 & 8 & -7 & 4 & -2 & \\ 1 & -4 & 12 & & 3 & 3 & -4 \\ & 14 & 2 & -3 & -5 & -1 & \\ & & -7 & 1 & 5 & & \\ & & & 10 & & & \end{array} \right\rangle$$

This is an even-dimensional rhotrix of dimension 6.

5 Conclusion

A strenuous effort was made to represent an even-dimensional rhotrix over a linear map. This representation showed that an even-dimensional rhotrix is a linear structure, and that it is a special type of rhotrix. All even-dimensional rhotrices are rhotrices except for the converse. Representing a rhotrix this way enables us to have by definition, even-dimensional rhotrices. Therefore, this work is an expansion and a contribution to rhotrix algebra.

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References

[1] Ajibade, A. O. (2003). The concept of Rhotrix in Mathematical Enrichment, *International Journal of Mathematical Education in Science and Technology*, 34 (2), 175–177.

[2] Aminu, A. (2009). On the Linear System over Rhotrices, *Notes on Number Theory and Discrete Mathematics*, 15 (4), 7–12.

[3] Aminu, A. & Michael, O. (2015). An introduction to the concept of paraletrix, a generalization of rhotrix, *Journal of the African Mathematical Union & Springer-Verlag*, 26 (5–6), 871–885.

[4] Atanassov, K. T. & Shannon, A. G. (1998). Matrix-Tertions and Matrix-Noitrets: Exercise for Mathematical Enrichment, *International Journal Mathematical Education in Science and Technology*, 29 (6), 898–903.

[5] Chinedu, M. P. (2012). Row-Wise Representation of Arbitrary Rhotrix, *Notes on Number Theory and Discrete Mathematics*, 18 (2), 1–27.

[6] Ezugwu, E. A., Ajibade, A. O. & Mohammed, A. (2011). Generalization of Heart-Oriented rhotrix Multiplication and its Algorithm Implementation, *International Journal of Computer Applications*, 13 (3), 5–11.

[7] Isere, A. O. (2016). Natural Rhotrix. *Cogent Mathematics*, 3 (1) , 1–10 (article 1246074).

[8] Isere, A. O. (2017). A note on the Classical and Non-Classical rhotrices, *The Journal of Mathematical Association of Nigeria (Abacus)*, 44(2), 119–124.

[9] Isere, A. O. (2018). Even-dimensional rhotrix, *Notes on Number Theory and Discrete Mathematics*, 24 (2), 125–133.

[10] Isere, A. O. & Adeniran, J. O. (2018). The Concept of Rhotrix Quasigroups and Rhotrix Loops, *The Journal of Nigerian Mathematical Society*, 37(3), 139–153.

[11] Mohammed, A. (2009). A remark on the classifications of rhotrices as abstract strutures, *International Journal of Physical Sciences*, 4 (9), 496–499.

[12] Mohammed, A. & Tella, Y. (2012). Rhotrix Sets and Rhotrix Spaces Category, *International Journal of Mathematics and computational methods in Science and Technology*, 2 (5), 2012.

[13] Mohammed, A., Balarabe, M. & Imam, A. T. (2012). Rhotrix Linear Transformation, *Advances in Linear Algebra & Matrix Theory*, 2, 43–47.

[14] Mohammed, A. (2014) A new expression for rhotrix, *Advances in Linear Algebra & Matrix Theory*, 4, 128–133.

[15] Sani, B. (2004). An alternative method for multiplication of rhotrices, *International Journal of Mathematical Education in Science and Technology*, 35, 777–781.

[16] Sani, B. (2007). The row-column multiplication for higher dimensional rhotrices, *International Journal of Mathematical Education in Science and Technology*, 38, 657–662.

[17] Sani, B. (2008). Conversion of a rhotrix to a coupled matrix, *International Journal of Mathematical Education in Science and Technology*, 39, 244–249.

[18] Tudunkaya, S. M. & Manjuola, S. O. (2010). Rhotrices and the Construction of Finite Fields, *Bull. Pure Appl Sci. Sect, E. Math. Stat.*, 29 (2), 225–229.

[19] Usaini, S. & Mohammed, L. (2012). On the rhotrix eigenvalues and eigenvectors, *Journal of the African Mathematical Union & Springer-Verlag*, 25, 223–235.