

Power GCD and power LCM matrices defined on GCD-closed sets over unique factorization domains

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Abstract: Let $T = \{t_1, t_2, \dots, t_m\}$ be a well ordered set of m distinct positive integers with $t_1 < t_2 < \dots < t_m$. The *GCD* matrix on T is defined as $(T)_{m \times m} = (t_i, t_j)$, where (t_i, t_j) is the greatest common divisor of t_i and t_j , and the power *GCD* matrix on T is $(T^r)_{m \times m} = (t_i, t_j)^r$, where r is any real number. The *LCM* matrix on T is defined as $[T]_{m \times m} = [t_i, t_j]$, where $[t_i, t_j]$ is the least common multiple of t_i and t_j , and the power *LCM* matrix on T is $[T^r]_{m \times m} = [t_i, t_j]^r$. Set $T = \{t_1, t_2, \dots, t_m\}$ is said to be gcd-closed if $(t_i, t_j) \in T$ for every t_i and t_j in T . In this paper, we give a generalization for the power *GCD* and *LCM* matrices defined on gcd-closed sets over unique factorization domains (UFDs). Moreover, we present a speculation for a generalization of Bourque–Ligh conjecture to UFDs which states that the least common multiple matrix defined on a gcd-closed P -ordered set in any UFD is nonsingular. Some examples that show what is done are additionally given in $Z[i]$ and $Z_p[x]$.

Keywords: Power *GCD* P -matrix, Power *LCM* P -Matrix, P -ordering, gcd-closed sets, Prime residue system, Unique factorization domains.

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1 Introduction

Let $T = \{t_1, t_2, \dots, t_m\}$ be a well-ordered set of m distinct positive integers such that $t_1 < t_2 < \dots < t_m$. The *GCD* matrix on T is defined as $(T)_{m \times m} = (t_i, t_j)$, where (t_i, t_j) is the greatest common divisor of t_i and t_j , and the *LCM* matrix on T is defined as $[T]_{m \times m} = [t_i, t_j]$, where $[t_i, t_j]$ is the least common multiple of t_i and t_j . Moreover, set T is said to be:

- factor-closed if for every $t_i \in T$, all the divisors of t_i are also in T .
- gcd-closed if the greatest common divisor $(t_i, t_j) \in T$ for all elements t_i and t_j in T .
- lcm-closed if the least common multiple $[t_i, t_j] \in T$ for all elements t_i and t_j in T .

In 1875/76, Smith [18] showed that if $T = \{1, 2, \dots, m\}$, then the determinant of the *GCD* matrix (T) defined on T is equal to the product $\phi(1)\phi(2)\cdots\phi(m)$ and the determinant of the *LCM* matrix $[T]$ defined on T is also equal to $\phi(1)\pi(1)\phi(2)\pi(2)\cdots\phi(m)\pi(m)$, where ϕ is the Euler's totient function and π is a multiplicative function for prime powers. Also, he showed that if $T = \{t_1, t_2, \dots, t_m\}$ is a factor-closed set of distinct positive integers, then $\det(T) = \phi(t_1)\phi(t_2)\cdots\phi(t_m)$ and $\det[T] = \phi(t_1)\pi(t_1)\phi(t_2)\pi(t_2)\cdots\phi(t_m)\pi(t_m)$. Moreover, Smith showed that if f is an arithmetical multiplicative function, then $\det[f(i, j)] = (f * \mu)(1) (f * \mu)(2) \cdots (f * \mu)(m)$, where μ is the Möbius function and $(f * \mu)$ is the Dirichlet convolution of f and μ . In 1988/89, S. Beslin and S. Ligh [2, 4, 17], showed that if (T) is defined on a factor-closed set T , then it could be written as a product of a matrix B and its corresponding transpose B^T . Moreover, they conjectured that if $T = \{t_1, t_2, \dots, t_m\}$ is a set of distinct positive integers, then $\det(T) = \phi(t_1)\phi(t_2)\cdots\phi(t_m)$ if and only if T is factor-closed. In 1989, Beslin and El-Kassar [1] exposed *GCD* matrices over unique factorization domains (UFDs). In 1990, Z. Li [16] gave a generalization for Smith's determinant by obtaining the value of (T) defined on an arbitrary ordered set of distinct positive integers, and he showed that the conjecture of Beslin–Ligh is true. In 1991/92, Beslin and Ligh [3, 5] proved that if $T = \{t_1, t_2, \dots, t_m\}$ is a gcd-closed set, then (T) could be factored as a product of two triangular matrices and they calculated its determinant. In addition, they gave a structure theorem for the *LCM* matrix $[T]$ and showed that it is non-singular. In the same year, K. Bourque and S. Ligh [6] provided a formula for the inverse of (T) and $[T]$ on factor-closed sets, and showed that $\det(T)$ divides $\det[T]$. In addition, they also conjectured that if $[T]$ is defined on a gcd-closed set, then it is non-singular (Bourque–Ligh Conjecture). In 1996/97, P. Haukkanen et al. [10, 11] presented a brief review of papers relating to Smith's determinant and pointed out a common structure that could be found in many extensions and analogues of Smith's determinant. Further, they gave a counterexample for the conjecture of Bourque and Ligh that the least common multiple matrix on any gcd-closed set is invertible. Moreover, they calculated the *GCD* and *LCM* matrices for lcm-closed and gcd-closed sets. In 1998, Hong [12, 13, 14] showed that the Bourque–Ligh conjecture is true for a certain class of gcd-closed sets. In addition, he showed new bounds for $\det[f(x_i, x_j)]$ for a certain class of arithmetical functions and semi-multiplicative functions, which improve the results obtained by Bourque and Ligh [6]. Later in 2009/10, El-Kassar et al. [8, 9] presented generalizations for the

GCD and *LCM* matrices defined on both factor-closed and gcd-closed sets in principal ideal domains (PIDs).

Now, let r be a real number and let $T = \{t_1, t_2, \dots, t_m\}$ be a well ordered set of m distinct positive integers such that $t_1 < t_2 < \dots < t_m$. The power *GCD* matrix defined on T is the $m \times m$ matrix $(T^r) = (t_{ij})_{m \times m}$ such that $t_{ij} = (t_i, t_j)^r$, where (t_i, t_j) is the greatest common divisor of t_i and t_j . The power *LCM* defined on T is the $m \times m$ matrix $[T^r] = (t_{ij})_{m \times m}$ such that $t_{ij} = [t_i, t_j]^r$, where $[t_i, t_j]$ is the least common multiple of t_i and t_j . In 1996, S. Z. Chun [7] introduced the concept of r -th power of the *GCD* and *LCM* matrices (T^r) and $[T^r]$ respectively, where r is a real number. Moreover, he calculated $\det(T^r)$ and $\det[T^r]$ over the factor-closed set T . In addition, he discussed (T^r) and $[T^r]$ over gcd-closed sets and obtained the structures for their inverses and reciprocals over the domain of natural numbers. In 2009, S. Hong et al. [15] presented the structures, determinants, and the non-singularity of the power *GCD* matrices for a UFD by considering the results of Beslin and El-Kassar [1].

In this paper, we give a generalization for the power *GCD* and *LCM* P -matrices characterized on gcd-closed sets over unique factorization domains (UFDs). Additionally, we make a speculation for Bourque–Ligh conjecture to UFDs. A few precedents on $Z[i]$ and $Z_p[x]$ that show what as been done are additionally given. The article comprises of six particular sections. In the first section, we present the essentials and the definitions required all through our work. In the second and the third sections, we present structure hypotheses and the determinants of the power *GCD* P -matrices and their reciprocals characterized on gcd-closed P -ordered sets in unique factorization domains. In addition, a few precedents were given in $Z_2[x]$. In the fourth section, we study the inverses of the power *GCD* P -matrices on gcd-closed sets over UFDs. Finally, in the fifth and the sixth sections, the structure hypotheses and the determinants of the power *LCM* P -matrices characterized on gcd-closed P -ordered sets in UFDs are presented. Additionally, a few precedents were given in $Z_2[x]$ and $Z[i]$.

2 Preliminaries and definitions

It is well-known from literature that a zero-divisor is a non-zero element s of a ring R such that there exists another non-zero element t in R with $st = 0$. An integral domain D is a commutative ring with unity and no zero-divisors. A subring A of a ring R with unity is called two-sided ideal if for every $s \in R$ and $a \in A$ both sa and as are also in A . Moreover, two non-zero elements s and t of an integral domain D are said to be associates if $s \approx ut$ where u is a unit in D .

Definition 1. A well-ordered set $P \approx \{p_1, p_2, \dots, p_n\}$ of non-zero and non-associate prime elements of an integral domain R is called a prime residue system of R if P is complete and every prime element of R either belongs to P , or is an associate to an element of P .

Definition 2. A unique factorization domain (UFD) is defined to be an integral domain R in which every non-zero element t of R can be written uniquely up to order and associates as a product (an empty product if t is a unit) of prime elements p_i of R and a unit u .

That is, R is said to be a unique factorization domain of prime residue system $P \approx \{p_1, p_2, \dots, p_n\}$ if any non-zero element $t \in R$ could be written as $t \approx up_1^{\alpha_1} p_2^{\alpha_2} \dots p_m^{\alpha_m}$, where p_j are distinct elements of P , the α_j are positive integers and u is a unit in R .

Definition 3. An element $t \in R$ is called a P -number if t is a product of elements from the prime residue system P .

In our work, we denote by \overline{P} to be the set of 1 and all P -numbers. Moreover, we call d a P -divisor of t in R if d divides t and $d \in \overline{P}$. If t and s are both non-zero elements in R , then we use $(t, s)_p$ to denote the greatest common P -divisor of t and s in R .

Definition 4. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . The generalized Jordan totient function $\tilde{s}_{k,R}$ over a unique factorization domain R is defined to be the multiplicative function defined on $R - \{0\}$ such that

$$\tilde{s}_{k,R}(t) \approx \prod_{i \approx 1}^m p_i^{k(\alpha_i - 1)} (p_i^k - 1).$$

Note that if t is a unit then $\tilde{s}_{k,R}(t) \approx 1$. Moreover, $\tilde{s}_{k,R}(t_i t_j) \approx \tilde{s}_{k,R}(t_i) \tilde{s}_{k,R}(t_j)$ for any two relatively prime elements t_i and t_j in T .

Theorem 1. Let P be a prime residue system of a UFD R , and let t be an element of T , and let k be a positive integer. If $E(t)$ is a complete set of distinct non-associate divisors d of t in T , then

$$t^k \approx \sum_{d \in E(t)} \tilde{s}_{k,R}(d)$$

Proof. Since $\tilde{s}_{k,s}$ is multiplicative, then the function $f(t) \approx \sum_{d \in E(t)} \tilde{s}_{k,R}(d)$ is also multiplicative.

Hence, for any $p_j \in P$,

$$\begin{aligned} f(p_i^{n_i}) &\approx \sum_{d \in E(p_i^{n_i})} \tilde{s}_{k,R}(d) \\ &\approx 1 + (p_i)^{k(1-1)}(p_i^k - 1) + (p_i)^{k(2-1)}(p_i^k - 1) + \dots + (p_i)^{k(n_i-1)}(p_i^k - 1) \\ &\approx 1 + (p_i)^k - 1 + (p_i)^{k(2-1)} - (p_i)^k + \dots + (p_i)^{kn_i} - (p_i)^{k(n_i-1)} \\ &\approx (p_i)^{kn_i} \\ &\approx (p_i^{n_i})^k. \end{aligned}$$

Now, if t is a product of non-zero non-associate elements from P , then $f(t) \approx \sum_{d \in E(t)} \tilde{s}_{k,R}(d)$.

The proof is complete. □

Definition 5. (P -Ordering in UFDs) The P -ordering in a UFD R is defined via the following scheme: $t_i <_p t_j$ if for some m , $k_{i_m} < k_{j_m}$ and $k_{i_c} = k_{j_c}$ for each $c < m$ and this relation $<_p$ is considered as a well-defined linear ordering on T .

Hence, if the ordered set T is such that $t_1 <_p t_2 <_p \dots <_p t_m$, we say that T is P -ordered and the matrix defined on T is a P -matrix.

1992, K. Bourque and S. Ligh [6] calculated the determinant of $[T]$ when T is gcd-closed, and they made the following conjecture.

Bourque–Ligh Conjecture. The LCM matrix on a gcd-closed set is invertible.

3 Power GCD matrices on GCD-closed sets over a UFD

Let R be a UFD and P be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_n\}$ is a P -ordered set of non-zero non-associate elements in R . In the following, we present structure theorems of the power GCD P -matrices, and their determinants on gcd-closed P -ordered sets over a UFD. Moreover, some examples are given in $Z_2[x]$.

Definition 6. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . Let $T \approx \{t_1, t_2, \dots, t_m\}$ be a P -ordered set of non-zero non-associate elements in R . Then, T is said to be gcd-closed if for every t_i and t_j in T their greatest common divisor $(t_i, t_j)_p$ is an associate to some element $t_k \in T$.

Definition 7. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R and $r \geq 1$ is an integer, then the powers GCD P -matrix defined on T is the $m \times m$ matrix $[T^r] = (t_{ij})_{m \times m}$ such that $t_{ij} \approx (t_i, t_j)_p^r$.

3.1 Factorizations of power GCD P -matrices on GCD-closed sets over a UFD

Theorem 2. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , then

$$(T^r)_p \approx EA_r E^T.$$

Proof. Let $D = \{y_1, y_2, \dots, y_n\}$ be the smallest gcd-closed set containing T in R , and let $E(x)$ be a complete set of distinct non-associate divisors d of t in R . Define the $n \times n$ diagonal matrix A_r as:

$$A_r = \text{diag} \left(\sum_{\substack{d \in E(y_1), d \notin E(y_u) \\ y_u <_p y_1}} \tilde{s}_{r,s}(d), \sum_{\substack{d \in E(y_2), d \notin E(y_u) \\ y_u <_p y_2}} \tilde{s}_{r,s}(d), \dots, \sum_{\substack{d \in E(y_n), d \notin E(y_u) \\ y_u <_p y_n}} \tilde{s}_{r,s}(d) \right)$$

and $E = (e_{ij})_{m \times n}$ such that

$$e_{ij} = \begin{cases} 1, & \text{if } y_j \in E(t_i) \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$(EA_r E^T)_{ij} \approx \sum_{k=1}^n \left(e_{ik} \left(\sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u <_p y_k}} \tilde{s}_{r,s}(d) \right) e_{jk} \right) \approx \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \left(\sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u <_p y_k}} \tilde{s}_{r,s}(d) \right)$$

$$\approx \sum_{y_k \in E((t_i, t_j)_p)} \left(\sum_{\substack{d \in E(y_k), d \notin E(y_u), \\ y_u <_p y_k}} \tilde{s}_{r,s}(d) \right) \approx (t_i, t_j)_p^r.$$

□

Example 1. $T = \{1, 1 + x, 1 + x^3\}$ is an ordered set in $\mathbb{Z}_2[x]$. Then,

$$\begin{aligned} EA_2E^T &\approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & x^2 & 0 \\ 0 & 0 & x^2 + x^6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 + x^2 & 1 + x^2 \\ 1 & 1 + x^2 & 1 + x^6 \end{bmatrix} \approx (T^2)_p. \end{aligned}$$

Theorem 3. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , then

$$(T^r)_p \approx A_r B_r,$$

where $[A_r]_{n \times m}$ and $[B_r]_{m \times n}$ for some positive integer $n \geq m$.

Proof. Let $D = \{y_1, y_2, \dots, y_n\}$ be the smallest gcd-closed set containing T in R , and $E(x)$ be a complete set of distinct non-associate divisors d of t in R . Define the matrix $A_r = (a_{ij})_{m \times n}$ as:

$$a_{ij} = \begin{cases} \sum_{\substack{d \in E(y_j), d \notin E(y_u), \\ y_u <_p y_j}} \tilde{s}_{r,s}(d), & \text{if } y_j \in E(t_i) \\ 0, & \text{otherwise} \end{cases}$$

and $B_r = (b_{ij})$ as:

$$b_{ij} = \begin{cases} 1, & \text{if } a_{ji} \neq 0 \\ 0, & \text{if } a_{ji} = 0 \end{cases}.$$

Therefore,

$$\begin{aligned} (A_r B_r)_{ij} &\approx \sum_{k=1}^n (a_{ik} b_{kj}) \approx \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \left(\sum_{\substack{d \in E(y_k), d \notin E(y_u), \\ y_u <_p y_k}} \tilde{s}_{r,s}(d) \right) \\ &\approx \sum_{y_k \in E((t_i, t_j)_p)} \left(\sum_{\substack{d \in E(y_k), d \notin E(y_u), \\ y_u <_p y_k}} \tilde{s}_{r,s}(d) \right) \approx (t_i, t_j)_p^r. \end{aligned}$$

□

Example 2. Let $T \approx \{1, 1+x, 1+x^3\}$ in $\mathbb{Z}_2[x]$, then

$$\begin{aligned} A_2 B_2 &\approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & x^2 & 0 \\ 1 & x^2 & x^2+x^6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+x^2 & 1+x^2 \\ 1 & 1+x^2 & 1+x^6 \end{bmatrix} \approx (T^2)_p. \end{aligned}$$

Theorem 4. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , then

$$(T^r)_p \approx A_r A_r^T.$$

Proof. Let \bar{F} be an extension of the field of fractions F of R in which $\tilde{s}_{r,s}(t)$ has square roots for every $t_i \in T$, $D = \{y_1, y_2, \dots, y_n\}$ be the smallest gcd-closed set containing T in R , and $E(x)$ be a complete set of distinct non-associate divisors d of t in R . Define the matrix A_r as:

$$a_{ij} = \begin{cases} \sqrt{\begin{matrix} \sum_{\substack{d \in E(y_j) \\ d \notin E(y_u) \\ y_u <_p y_j}} \tilde{s}_{r,s}(d), & \text{if } y_j \in E(t_i) \\ 0, & \text{otherwise} \end{matrix}} \end{cases}.$$

Then,

$$\begin{aligned} (A_r A_r^T)_{ij} &\approx \sum_{k=1}^n (a_{ik} a_{jk}) \approx \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \left(\sqrt{\begin{matrix} \sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u <_p y_k}} \tilde{s}_{r,s}(d)} \sqrt{\begin{matrix} \sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u <_p y_k}} \tilde{s}_{r,s}(d)} \right) \\ &\approx \sum_{y_k \in E((t_i, t_j)_p)} \left(\sqrt{\begin{matrix} \sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u <_p y_k}} \tilde{s}_{r,s}(d)} \right) \approx (t_i, t_j)_p^r. \end{aligned}$$

□

Example 3. If $T \approx \{1, 1+x, 1+x^3\}$ in $\mathbb{Z}_2[x]$, then

$$\begin{aligned} A_2 A_2^T &\approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{x^2} & 0 \\ 1 & \sqrt{x^2} & \sqrt{x^2+x^6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{x^2} & \sqrt{x^2} \\ 0 & \sqrt{x^2} & \sqrt{x^2+x^6} \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1+x^2 & 1+x^2 \\ 1 & 1+x^2 & 1+x^6 \end{bmatrix} \approx (T^2)_p. \end{aligned}$$

3.2 Determinants of power GCD P-matrices on GCD-closed sets over a UFD

Theorem 5. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered gcd-closed set of non-zero non-associate elements in R , then

$$\det(T^r)_p \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u <_p t_i}} \tilde{s}_{r,s}(d) \right).$$

Proof. First Proof: By Theorem 1, we have $(T^r)_p \approx EA_rE^T$ and since T is a gcd-closed set, then $T \approx D \approx \{y_1, y_2, \dots, y_m\}$ and E is a lower triangular matrix with diagonal entries $e_{ij} \approx 1$ so that $\det[E] = 1$. Thus,

$$\det(T^r)_p \approx \det(EA_rE^T) \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u <_p t_i}} \tilde{s}_{r,s}(d) \right).$$

Second Proof: By Theorem 2, we have $(T^r)_p \approx A_rB_r$ and since T is gcd-closed in R , then A_r and B_r are lower and upper triangular matrices, respectively, with respective diagonal entries

$$a_{ii} = \sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u <_p t_i}} \tilde{s}_{r,s}(d) \text{ and } b_{ii} = 1. \text{ Therefore,}$$

$$\det(T^r)_p \approx \det(A_r) \det(B_r) \approx \det(A_r) \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u <_p t_i}} \tilde{s}_{r,s}(d) \right).$$

Third Proof: By Theorem 3, we have $(T^r)_p \approx A_rA_r^T$ and since T is a gcd-closed set in R , then A_r is a lower matrix with diagonal entries $a_{ii} = \sqrt{\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u <_p t_i}} \tilde{s}_{r,s}(d)}$ for all $1 \leq i \leq m$. Then,

$$\begin{aligned} \det(T^r)_p &\approx \det(A_r) \det(A_r^T) \approx \left(\prod_{i=1}^m \sqrt{\sum_{\substack{d \in E(t_i), \\ d \notin E(t_u), \\ t_u < t_i}} \tilde{s}_{r,s}(d)} \right) \left(\prod_{i=1}^m \sqrt{\sum_{\substack{d \in E(t_i), \\ d \notin E(t_u), \\ t_u < t_i}} \tilde{s}_{r,s}(d)} \right) \\ &\approx \prod_{i=1}^m \left(\sqrt{\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u < t_i}} \tilde{s}_{r,s}(d)} \right)^2 \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i), d \notin E(t_u), \\ t_u < t_i}} \tilde{s}_{r,s}(d) \right). \end{aligned}$$

□

Corollary 1. (Beslin–Ligh Result) *If $T = \{t_1, t_2, \dots, t_m\}$ is a gcd-closed set of positive integers, then*

$$\det(T^r)_p = \prod_{i=1}^m \left(\sum_{d/t_i, d|t_i, t_i < t_i} (\phi(d)) \right).$$

Proof. Using Theorem 4 by taking $r = 1$, $R = \mathbb{Z}$, then $<_p$ is the usual $<$ in \mathbb{Z} and $\tilde{s}_{1,\mathbb{Z}} = \phi$. Therefore,

$$\begin{aligned} \det(T^r)_p &= \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i) \\ d \notin E(t_u) \\ t_u <_p t_i}} \tilde{s}_{r,s}(d) \right) = \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i) \\ d \notin E(t_u) \\ t_u <_p t_i}} \tilde{s}_{1,\mathbb{Z}}(d) \right) \\ &= \prod_{i=1}^m \left(\sum_{d/t_i, d|t_i, t_i < t_i} (\phi(d)) \right). \end{aligned}$$

□

Definition 8. *Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_m\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R and $t \in S$ such that $t \approx up_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, where u is a unit in S , p_i are distinct elements of P , and α_i are positive integers for all $1 \leq i \leq m$, then we define the multiplicative function Φ_s on $R/\{0\}$ as:*

$$\Phi_s(t) \approx \prod_{i=1}^m p_i^{\alpha_i - 1} (p_i - 1).$$

We note that $\Phi_s(t) \approx 1$ if and only if t is a unit in R . Moreover, $\Phi_s(xy) \approx \Phi_s(x)\Phi_s(y)$ for any two relatively prime elements x and y in R .

Theorem 6. *Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_m\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , and $E(t)$ is a complete set of distinct non-associate divisors d of t in R , then*

$$t \approx \sum_{d \in E(t)} \Phi_s(d).$$

Proof. Since Φ_s is multiplicative, then

$$f(t) \approx \sum_{d \in E(t)} \Phi_s(d)$$

is also multiplicative. So, for any p_j in the prime residue system P in R we have

$$f(p_i^{n_i}) \approx \sum_{d \in E(p_i^{n_i})} \Phi_s(d) \approx 1 + (p_i - 1) + (p_i^2 - p_i) + \cdots + (p_i^{n_i} - p_i^{n_i - 1}) \approx p_i^{n_i}.$$

Now, if $t \approx up_1^{\alpha_1} p_2^{\alpha_2} \cdots p_m^{\alpha_m}$, then

$$\sum_{d \in E(t)} \Phi_s(d) \approx f(t) \approx \prod_{i=1}^m \left(\sum_{d \in E(p_i^{\alpha_i})} \Phi_s(d) \right) \approx u \prod_{i=1}^m p_i^{\alpha_i} \approx t.$$

□

Remark 1. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_m\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , then it is clear that if we take $r = 1$ and $\tilde{s}_{1,R} \approx \Phi_R$ we have a generalization for Beslin–Ligh Result [5] over a UFD. That is;

$$\det(T)_p \approx \prod_{i=1}^m \left(\sum_{d \in E(t_i), d \notin E(t_u), t_u < t_i} \Phi_s(d) \right).$$

4 Reciprocal power GCD P -matrices on GCD-closed sets over a UFD

Definition 9. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , the $m \times m$ matrix $1/(T^r)_p = (t_{ij})_{m \times m}$ such that $(t_{ij}) \approx (t_i, t_j)_p^{-r} \approx \frac{1}{[(t_i, t_j)_p]^r}$ is called the reciprocal power GCD P -matrix defined on T over a UFD R .

4.1 Factorizations of reciprocal power GCD P -matrices on GCD-closed sets over a UFD

Theorem 7. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R . If $T \approx \{t_1, t_2, \dots, t_m\}$ is a P -ordered set of non-zero non-associate elements in R , then

$$1/(T^r)_p \approx E A_{-r} E^T.$$

Proof. Let $D = \{y_1, y_2, \dots, y_n\}$ be the smallest gcd-closed set containing T in the principle ideal domain S and $E(t)$ be a complete set of distinct non-associate divisors d of t in R . Define the $n \times n$ diagonal matrix A_{-r} as:

$$A_{-r} = \text{diag} \left(\begin{array}{ccc} \sum_{\substack{d \in E(y_1) \\ d \notin E(y_u) \\ y_u < y_1}} \tilde{s}_{-r,s}(d), & \sum_{\substack{d \in E(y_2) \\ d \notin E(y_u) \\ y_u < y_2}} \tilde{s}_{-r,s}(d), & \dots, & \sum_{\substack{d \in E(y_n) \\ d \notin E(y_u) \\ y_u < y_n}} \tilde{s}_{-r,s}(d) \end{array} \right)$$

and $E = (e_{ij})_{m \times n}$ so that;

$$e_{ij} = \begin{cases} 1, & \text{if } y_j \in E(t_i) \\ 0, & \text{otherwise} \end{cases}.$$

Then,

$$\begin{aligned}
(EA_{-r}E^T)_{ij} &\approx \sum_{k=1}^n \left(e_{ik} \left(\sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u < y_k}} \tilde{s}_{-r,s}(d) \right) e_{jk} \right) \approx \sum_{\substack{y_k \in E(t_i) \\ y_k \in E(t_j)}} \left(\sum_{\substack{d \in E(y_k) \\ d \notin E(y_u) \\ y_u < y_k}} \tilde{s}_{-r,s}(d) \right) \\
&\approx \sum_{y_k \in E((t_i, t_j)_p)} \left(\sum_{\substack{d \in E(y_k) \\ d \in E(y_u) \\ y_u < y_k}} \tilde{s}_{-r,s}(d) \right) \approx (t_i, t_j)_p^{-r}.
\end{aligned}$$

□

Note: The reciprocal power *GCD P*-matrices could be easily written as $A_{-r}B_{-r}$ and $A_{-r}A_{-r}^T$ in a similar manner done above.

Example 4. If $T \approx \{1, 1+x, 1+x^3\}$ is a *P*-ordered gcd-closed set over $\mathbb{Z}_2[x]$, then the reciprocal of $(T^2)_p$ is

$$1/(T^2)_p \approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix}$$

and is factored as follows:

$$EA_{-2}E^T \approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{-x^2}{1+x^2} & 0 \\ 0 & 0 & \frac{x^4-x^2}{1+x^6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix}$$

or,

$$A_{-2}B_{-2} \approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & \frac{-x^2}{1+x^2} & 0 \\ 1 & \frac{-x^2}{1+x^2} & \frac{x^4-x^2}{1+x^6} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix}$$

or,

$$A_{-2}A_{-2}^T \approx \begin{bmatrix} 1 & 0 & 0 \\ 1 & \sqrt{\frac{-x^2}{1+x^2}} & 0 \\ 1 & \sqrt{\frac{-x^2}{1+x^2}} & \sqrt{\frac{x^4-x^2}{1+x^6}} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & \sqrt{\frac{-x^2}{1+x^2}} & \sqrt{\frac{-x^2}{1+x^2}} \\ 0 & 0 & \sqrt{\frac{x^4-x^2}{1+x^6}} \end{bmatrix} \approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix}.$$

4.2 Determinant of reciprocal power *GCD P*-matrices on *GCD*-closed sets over a UFD

Theorem 8. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R , and $T \approx \{t_1, t_2, \dots, t_m\}$ is a gcd-closed *P*-ordered set of non-zero non-associate elements

in R . Then,

$$\det \left(1/(T^r)_p \right) \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i), d \notin E(t_u) \\ t_u < t_i}} \tilde{s}_{-r,s}(d) \right)$$

Proof. Let T be a P -ordered set in S . Since T is a gcd-closed set, then $T \approx D = \{y_1, y_2, \dots, y_m\}$ and $1/(T^r)_p \approx EA_{-r}E^T$ and E is a lower triangular matrix with $\det[E] = 1$. Thus,

$$\det \left(1/(T^r)_p \right) \approx \det(EA_{-r}E^T) \approx \det(A_{-r}) \approx \prod_{i=1}^m \left(\sum_{\substack{d \in E(t_i) \\ d \notin E(t_u) \\ t_u < t_i}} \tilde{s}_{-r,s}(d) \right).$$

□

Note that we may prove the above theorem by using the factorizations $1/(T^r)_p \approx A_{-r}B_{-r}$ and $1/(T^r)_p \approx A_{-r}A_{-r}^T$.

5 Inverses of power GCD P -matrices on GCD -closed sets over a UFD

Theorem 9. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R , and $T \approx \{t_1, t_2, \dots, t_m\}$ is a gcd-closed P -ordered set of non-zero non-associate elements in R . Then, the inverse of $(T^r)_p$ is $(T^r)_p^{-1} = (t_{ij})_{m \times m}$ such that

$$t_{ij} = \sum_{t_i \in E(t_k), t_j \in E(t_k)} \left(\frac{\mu_p \left(\frac{t_k}{t_i} \right) \mu_p \left(\frac{t_k}{t_j} \right)}{\sum_{\substack{d \in E(t_k), d \notin E(t_u) \\ t_u < t_k}} \tilde{s}_{r,s}(d)} \right).$$

Proof. Let $F \approx E^{-1}$, such that

$$f_{ij} = \begin{cases} \mu \left(\frac{t_i}{t_j} \right), & \text{if } t_j \in E(t_i) \\ 0, & \text{otherwise} \end{cases},$$

then

$$\begin{aligned} t_{ij} &\approx (EA_rE^T)_{ij}^{-1} \approx ((E^T)^{-1}(A_r)^{-1}(E)^{-1})_{ij} \approx (F^T(A_r)^{-1}F)_{ij} \approx \sum_{k=1}^m f_{ki} \frac{1}{a_{kk}} f_{kj} \\ &\approx \sum_{\substack{t_i \in E(t_k) \\ t_j \in E(t_k)}} \left(\mu_p \left(\frac{t_k}{t_i} \right) \frac{1}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u)}} \tilde{s}_{r,s}(d)} \mu_p \left(\frac{t_k}{t_j} \right) \right) \approx \sum_{\substack{t_i \in E(t_k) \\ t_j \in E(t_k)}} \left(\frac{\mu_p \left(\frac{t_k}{t_i} \right) \mu_p \left(\frac{t_k}{t_j} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u)}} \tilde{s}_{r,s}(d)} \right) \end{aligned}$$

□

Example 5. If $T \approx \{1, 1+i, 2\}$ is a P -ordered gcd-closed set of the $\mathbb{Z}[i]$, then, by computing the entries of $(T^2)_p^{-1}$ we get:

$$t_{11} = \sum_{1 \in E(t_k)} \left(\frac{\mu_p \left(\frac{t_k}{1} \right) \mu_p \left(\frac{t_k}{1} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < t_k}} \tilde{s}_{2,s}(d)} \right) = \frac{4}{5} - \frac{2}{5}i = (T^2)_{11}^{-1},$$

$$t_{12} = \sum_{\substack{1 \in E(t_k) \\ (1+i) \in E(t_k)}} \left(\frac{\mu_p \left(\frac{t_k}{1} \right) \mu_p \left(\frac{t_k}{1+i} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = \frac{1}{5} + \frac{2}{5}i = (T^2)_{12}^{-1},$$

$$t_{13} = \sum_{\substack{1 \in E(t_k) \\ (2) \in E(t_k)}} \left(\frac{\mu_p \left(\frac{t_k}{1} \right) \mu_p \left(\frac{t_k}{2} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = 0 = (T^2)_{13}^{-1},$$

$$t_{21} = \sum_{\substack{2 \in E(t_k) \\ 1 \in E(t_k)}} \left(\frac{\mu_p \left(\frac{t_k}{2} \right) \mu_p \left(\frac{t_k}{1} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = \frac{1}{5} + \frac{2}{5}i = (T^2)_{21}^{-1},$$

$$t_{22} = \sum_{(1+i) \in E(t_k)} \left(\frac{\mu_p \left(\frac{t_k}{(1+i)} \right) \mu_p \left(\frac{t_k}{(1+i)} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = -\frac{3}{10}i = (T^2)_{22}^{-1},$$

$$t_{23} = \sum_{\substack{(1+i) \in E(t_k) \\ 2 \in E(t_k)}} \left(\frac{\mu_p \left(\frac{t_k}{(1+i)} \right) \mu_p \left(\frac{t_k}{2} \right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = -\frac{1}{5} - \frac{1}{10}i = (T^2)_{23}^{-1},$$

$$t_{31} = \sum_{\substack{2 \in E(t_k) \\ 1 \in E(t_k)}} \left(\frac{\mu_p\left(\frac{t_k}{1}\right) \mu_p\left(\frac{t_k}{2}\right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = 0 = (T^2)_{31}^{-1},$$

$$t_{32} = \sum_{\substack{2 \in E(t_k) \\ (1+i) \in E(t_k)}} \left(\frac{\mu_p\left(\frac{t_k}{2}\right) \mu_p\left(\frac{t_k}{(1+i)}\right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = -\frac{1}{5} - \frac{1}{10}i = (T^2)_{32}^{-1},$$

and

$$t_{33} = \sum_{2 \in E(t_k)} \left(\frac{\mu_p\left(\frac{t_k}{2}\right) \mu_p\left(\frac{t_k}{2}\right)}{\sum_{\substack{d \in E(t_k) \\ d \notin E(t_u) \\ t_u < (t_k)}} \tilde{s}_{2,s}(d)} \right) = \frac{1}{5} + \frac{1}{10}i = (T^2)_{33}^{-1}.$$

Hence,

$$(T^2)_p (T^2)_p^{-1} \approx \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2i & 2i \\ 1 & 2i & 4 \end{bmatrix} \begin{bmatrix} \frac{4}{5} - \frac{2}{5}i & \frac{1}{5} + \frac{2}{5}i & 0 \\ \frac{1}{5} + \frac{2}{5}i & -\frac{3}{10}i & -\frac{1}{5} - \frac{1}{10}i \\ 0 & -\frac{1}{5} - \frac{1}{10}i & \frac{1}{5} + \frac{1}{10}i \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6 Power LCM P -matrices on GCD-closed sets over a UFD

6.1 Factorizations of power LCM P -matrices on GCD-closed sets over a UFD

Theorem 10. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R , and $T \approx \{t_1, t_2, \dots, t_m\}$ be a gcd-closed P -ordered set of non-zero non-associate elements in R . The power LCM matrix can be decomposed, up to associates, as

$$[T^r]_p \approx D_r E A_{-r} E^T D_r.$$

Proof. Let $D = \{y_1, y_2, \dots, y_n\}$ be the smallest gcd-closed set containing T in R , and $E(x)$ be a complete set of distinct non-associate divisors d of t in R . Define the following matrices: $(A_{-r})_{n \times n}$ is a diagonal matrix such that:

$$A_{-r} \approx \text{diag} \left(\sum_{\substack{d \in E(y_1) \\ d \notin E(y_u) \\ y_u < y_1}} \tilde{s}_{-r,s}(y_1), \sum_{\substack{d \in E(y_2) \\ d \notin E(y_u) \\ y_u < y_2}} \tilde{s}_{-r,s}(y_2), \dots, \sum_{\substack{d \in E(y_n) \\ d \notin E(y_u) \\ y_u < y_n}} \tilde{s}_{-r,s}(y_n) \right)$$

$E = (e_{ij})_{m \times n}$ such that:

$$e_{ij} = \begin{cases} 1, & \text{if } y_j \in E(t_i) \\ 0, & \text{otherwise} \end{cases}.$$

and $D_r \approx \text{diag}(t_1^r, t_2^r, \dots, t_m^r)$. So,

$$(D_r E A_{-r} E^T D_r)_{ij} \approx (D_r (T^{-r}) D_r)_{ij} \approx t_i^r (T^{-r})_{ij} t_j^r \approx \frac{t_i^r t_j^r}{(t_i, t_j)^r} \approx [[t_i, t_j]_p]^r.$$

□

Example 6. If $T \approx \{1, 1 + x, 1 + x^3\}$ is a P -ordered gcd-closed set over $\mathbb{Z}_2[x]$, then the LCM power P -matrix ($r = 2$) is:

$$[T^2]_p = \begin{bmatrix} 1 & 1 + x^2 & 1 + x^6 \\ 1 + x^2 & 1 + x^2 & 1 + x^6 \\ 1 + x^6 & 1 + x^6 & 1 + x^6 \end{bmatrix}$$

and the reciprocal GCD power P -matrix is:

$$1/(T^2)_p = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix}$$

and

$$D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & 0 \\ 0 & 0 & 1 + x^6 \end{bmatrix}.$$

Then,

$$\begin{aligned} D_2 \left(1/(T^2)_p \right) D_2 &\approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & 0 \\ 0 & 0 & 1 + x^6 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^2} \\ 1 & \frac{1}{1+x^2} & \frac{1}{1+x^6} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 + x^2 & 0 \\ 0 & 0 & 1 + x^6 \end{bmatrix} \\ &\approx \begin{bmatrix} 1 & 1 + x^2 & 1 + x^6 \\ 1 + x^2 & 1 + x^2 & 1 + x^6 \\ 1 + x^6 & 1 + x^6 & 1 + x^6 \end{bmatrix} \approx [T^2]_p. \end{aligned}$$

6.2 Determinants of power LCM P -matrices on GCD-closed sets over a UFD

Theorem 11. Let R be a UFD and $P \approx \{p_1, p_2, \dots, p_i, \dots\}$ be a complete prime residue system of R , and $T \approx \{t_1, t_2, \dots, t_m\}$ be a gcd-closed P -ordered set of non-zero non-associate elements in R . Then,

$$\det[T^r]_p \approx \prod_{i=1}^m \left(\left(\sum_{\substack{d \in E(y_i) \\ d \notin E(y_u) \\ y_u < y_i}} \tilde{s}_{-r,s}(t_i) \right) t_i^{2r} \right).$$

Proof. Since T is a gcd-closed set, then $T \approx D = \{y_1, y_2, \dots, y_m\}$ and $[T^r] \approx D_r E A_{-r} E^T D_r$, where E is a lower triangular matrix with diagonal entry $e_{ii} = 1$, i.e., $\det[E] = 1$. Thus,

$$\begin{aligned} \det[T^r]_p &\approx \det(D_r E A_{-r} E^T D_r) \\ &\approx \det(D_r) \det(E) \det(A_{-r}) \det(E^T) \det(D_r) \\ &\approx \prod_{i=1}^m t_i^r \times \det(A_{-r}) \times \prod_{i=1}^m t_i^r \approx \prod_{i=1}^m \left(\left(\sum_{\substack{d \in E(y_i) \\ d \notin E(y_u) \\ y_u < y_i}} \tilde{s}_{-r,s}(t_i) \right) t_i^{2r} \right). \end{aligned}$$

□

Example 7. If $T = \{1, 1 + x, 1 + x^3\}$ is a P -ordered gcd-closed set over $\mathbb{Z}_2[x]$, then the determinant of the power LCM P -matrix with $r = 2$ is

$$\begin{aligned} \det[T^2]_p &\approx \prod_{i=1}^3 \left(\left(\sum_{\substack{d \in E(y_i) \\ d \notin E(y_u) \\ y_u < y_i}} \tilde{s}_{-2,s}(t_i) \right) t_i^{2 \times 2} \right) \\ &\approx -x^{14} - 2x^{13} - 3x^{10} + 3x^8 + 2x^5 + x^4 \\ &\approx x^{14} - x^{10} + x^{18} - x^4. \end{aligned}$$

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