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On dual Horadam octonions

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Abstract: In this paper, we introduce the dual Horadam octonions, we give the Binet formula, generating function, exponential generating function, summation formula, Catalan's identity, Cassini's identity and d'Ocagne's identity of dual Horadam octonions. Employing these results, we present the Binet formula, generating function, summation formula, Catalan, Cassini and d'Ocagne identities for dual Fibonacci, dual Lucas, dual Jacobsthal, dual Jacobsthal–Lucas, dual Pell and dual Pell–Lucas octonions. So we generalize results that were obtained earlier by scientists. Finally, we introduce the matrix generator for dual Horadam octonions and this generator gives the Cassini formula for the dual Horadam octonions.
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1 Introduction

The octonions were discovered independently by Graves and Cayley in 1843. Octonions are used in mathematics, physics and geometry. In particular, they are used in string theory [12], quantum logic [6] and quantum mechanics [2].

The octonions are 8-dimensional algebra over the real numbers with basis $\{1, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ and denoted by $\mathcal{O}$. Octonions are noncommutative and nonassociative. Every octonion b can be written in the form

$$b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7 \tag{1}$$

with the real coefficient $\{b_i\}$, $i = 0, 1, 2, ..., 7$. Addition and subtraction of octonions is done by adding and subtracting corresponding terms. The multiplication rules between e_l's are given in [13], where $l \in \{1, 2, ..., 7\}$.

From [13], we have the following results;

• $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ are square roots of -1,

• e_l and e_k are noncommutative when $l \neq k$ and $l, k \in \{1, 2, ..., 7\}$.

The conjugate of octonion $b = b_0 + b_1e_1 + b_2e_2 + b_3e_3 + b_4e_4 + b_5e_5 + b_6e_6 + b_7e_7$ is given by

$$\bar{b} = b_0 - b_1e_1 - b_2e_2 - b_3e_3 - b_4e_4 - b_5e_5 - b_6e_6 - b_7e_7. \tag{2}$$

For more knowledge about octonions, one can see [1, 4, 11, 13, 14].

Now we give some basic definitions and properties of Horadam sequences which are used in this paper.

Horadam [9] defined the following sequence;

$$\{w_m\} = \{w_m(a, b; p, q)\} : w_m = pw_{m-1} + qw_{m-2}; w_0 = a, w_1 = b(m \geq 2) \tag{3}$$

where w_0, w_1, p, q are integers. Here, w_m is the m-th Horadam number. For more details on this topic we refer to [9, 10] and references therein. The solutions of the characteristic equation $y^2 - py - q = 0$ associated with the recurrence relation (3) are $\lambda = \frac{p+\sqrt{p^2+4q}}{2}$ and $\gamma = \frac{p-\sqrt{p^2+4q}}{2}$. So the Binet formula for the Horadam sequences is given by

$$w_m = \frac{N\lambda^m - K\gamma^m}{\lambda - \gamma} \quad \text{where} \quad N = b - a\gamma \quad \text{and} \quad K = b - a\lambda. \tag{4}$$

Note that

$$\lambda + \gamma = p, \quad \lambda.\gamma = -q \quad \text{and} \quad \lambda - \gamma = \sqrt{p^2 + 4q}. \tag{5}$$

Remark 1.1. *Consider Eq (3);*

• *If* $p = 1$, $q = 1$, $a = 0$ *and* $b = 1$, *the Fibonacci sequence is obtained,*

• *If* $p = 1$, $q = 1$, $a = 2$ *and* $b = 1$, *the Lucas sequence is obtained,*

• *If* $p = 2$, $q = 1$, $a = 0$ *and* $b = 1$, *the Pell sequence is obtained,*

• *If* $p = 1$, $q = 2$, $a = 0$ *and* $b = 1$, *the Jacobsthal sequence is obtained,*

• *If* $p = 1$, $q = 2$, $a = 2$ *and* $b = 1$, *the Jacobsthal–Lucas sequence is obtained,*

• *If* $p = 2$, $q = 1$, $a = 2$ *and* $b = 2$, *the Pell–Lucas sequence is obtained.*

Hence, the Horadam sequence is the generalization of sequences above.

In [5], Clifford extented the real numbers and defined the dual number B gave by $B = b + \varepsilon b^*$, where $b, b^* \in \mathbb{R}$ and ε is called by the dual unit, it satisfies the following rules; $\varepsilon \neq 0$, $0\varepsilon = \varepsilon 0$, $1\varepsilon = \varepsilon 1 = \varepsilon$ and $\varepsilon^2 = 0$.

Let $\mathfrak{D}$ denote the set of dual numbers, where $\mathfrak{D}$ is a commutative ring having the εb^* as a divisor of zero. A dual octonion $\widetilde{o}$ can be defined as $\widetilde{o} = o + \varepsilon o^*$, where $o, o^* \in \mathcal{O}$. The set of dual octonions can be denoted by $\widetilde{\mathcal{O}}$ and any $\widetilde{o} \in \widetilde{\mathcal{O}}$ can be written as

$$\widetilde{o} = B_0e_0 + B_1e_1 + B_2e_2 + B_3e_3 + B_4e_4 + B_5e_5 + B_6e_6 + B_7e_7, \tag{6}$$

where $B_k \in \mathfrak{D}$, $B_k = b_k + \varepsilon {b_k}^*$, $\quad b_k, {b_k}^* \in \mathbb{R}$, $k = 0, 1, 2, 3, 4, 5, 6, 7$.

Recently, a lot of research has focused on dual quaternions and octonions. Some of them are as follows.

Nurkan and Güven [15] defined the dual Fibonacci quaternion and the Lucas quaternion and they gave the Binet and Cassini formulas for them. In [3], authors study the dual k-Pell, the dual k-Pell–Lucas and the dual modified k-Pell quaternions and octonions, and investigate some fundamental algebric properties of the quaternions and octonions. Halıcı [7] considers the dual Fibonacci octonions and the dual Fibonacci quaternions, she investigates some properties of them. In [16], authors investigate properties of dual Fibonacci and dual Lucas octonions.

In this paper, we introduce the dual Horadam octonions, we give the Binet formula, generating function and exponential generating function of the dual Horadam octonions. Also, we obtain some identities for dual Horadam octonions including Catalan, Cassini and d'Ocagne identities. Since the dual Horadam octonions are the generalization of the dual Fibonacci, dual Lucas, dual Jacobsthal, dual Jacobsthal–Lucas, dual Pell, dual Pell–Lucas octonions, using the definition of these six dual octonions, we present the Binet formula, generating function, summation formula, Catalan, Cassini, d'Ocagnes identities for six dual octonions. Finally, we introduce the matrix generator for dual Horadam octonions and this generator gives the Cassini formula for the dual Horadam octonions.

2 Some properties of dual Horadam octonions

Definition 2.1. *For $m \geq 0$, any m-th Horadam octonions are defined by*

$$\mathcal{OH}_m = \sum_{s=0}^{7} w_{m+s} e_s, \tag{7}$$

where w_m is the m-th Horadam number and $\{e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$ is the standard octonion basis.

Throughout this paper, we take $e_0 = 1$.

The m-th dual Fibonacci quaternion is given [15] by Nurkan and Güven as:

$$\widetilde{\mathcal{Q}_m} = \mathcal{Q}_m + \varepsilon \mathcal{Q}_{m+1}. \tag{8}$$

Here, $\mathcal{Q}_m = F_m + iF_{m+1} + jF_{m+2} + kF_{m+3}$ is the m-th dual Fibonacci quaternion i, j and k are the standard orthonormal basis in $\mathbb{R}^3$ and satisfy the following rules:

$$i^2 = j^2 = k^2 = ijk = -1$$
$$ij = -ji = k, \qquad ki = -ik = j, \qquad jk = -kj = i$$

The m-th dual Horadam number is defined by $\widetilde{w_m} = w_m + \varepsilon w_{m+1}$, where ε is the dual unit $\varepsilon^2 = 0$, $\varepsilon \neq 0$. Here w_m is the m-th Horadam number.

With the same idea we can define dual Horadam octonions as follows.

Definition 2.2. *For $m \geq 0$, any m-th dual Horadam octonions are defined by*

$$\widetilde{\mathcal{OH}_m} = \mathcal{OH}_m + \varepsilon \mathcal{OH}_{m+1}, \tag{9}$$

where $\mathcal{OH}_m$ is m-th Horadam octonion.

Using Eqs. (3), (9) and Remark 1.1, we can easily define dual Fibonacci, dual Lucas, dual Jacobsthal, dual Jacobsthal–Lucas, dual Pell and dual Pell–Lucas octonions.

The dual Horadam octonions $\widetilde{\mathcal{OH}_m}$ consist of 8 dual elements and can be represented as

$$\begin{aligned}\widetilde{\mathcal{OH}_m} &= (w_m + \varepsilon w_{m+1}) + (w_{m+1} + \varepsilon w_{m+2})e_1 + (w_{m+2} + \varepsilon w_{m+3})e_2 \\ &+ (w_{m+3} + \varepsilon w_{m+4})e_3 + (w_{m+4} + \varepsilon w_{m+5})e_4 + (w_{m+5} + \varepsilon w_{m+6})e_5 \\ &+ (w_{m+6} + \varepsilon w_{m+7})e_6 + (w_{m+7} + \varepsilon w_{m+8})e_7.\end{aligned}$$

By the definition of dual Horadam numbers, we have

$$\widetilde{\mathcal{OH}_m} = \widetilde{w_m} + \widetilde{w_{m+1}}e_1 + \widetilde{w_{m+2}}e_2 + \widetilde{w_{m+3}}e_3 + \widetilde{w_{m+4}}e_4 + \widetilde{w_{m+5}}e_5 + \widetilde{w_{m+6}}e_6 + \widetilde{w_{m+7}}e_7.$$

The scalar part and vector part of $\widetilde{\mathcal{OH}_m}$ are given as

$$S_{\widetilde{\mathcal{OH}_m}} = w_m + \varepsilon w_{m+1}$$

$$\begin{aligned}V_{\widetilde{\mathcal{OH}_m}} &= (w_{m+1} + \varepsilon w_{m+2})e_1 + (w_{m+2} + \varepsilon w_{m+3})e_2 + (w_{m+3} + \varepsilon w_{m+4})e_3 \\ &+ (w_{m+4} + \varepsilon w_{m+5})e_4 + (w_{m+5} + \varepsilon w_{m+6})e_5 + (w_{m+6} + \varepsilon w_{m+7})e_6 \\ &+ (w_{m+7} + \varepsilon w_{m+8})e_7,\end{aligned}$$

respectively.

Let $\widetilde{\mathcal{OH}_m} = \mathcal{OH}_m + \varepsilon\mathcal{OH}_{m+1}$ and $\widetilde{\mathcal{OH}_m}' = \mathcal{OH}_m' + \varepsilon\mathcal{OH}_{m+1}'$ be two dual Horadam octonions. Then, addition and multiplication of them are given by

$$\widetilde{\mathcal{OH}_m} + \widetilde{\mathcal{OH}_m}' = (\mathcal{OH}_m + \mathcal{OH}_m') + \varepsilon(\mathcal{OH}_{m+1} + \mathcal{OH}_{m+1}')$$

$$\widetilde{\mathcal{OH}_m}\widetilde{\mathcal{OH}_m}' = (\mathcal{OH}_m\mathcal{OH}_m') + \varepsilon(\mathcal{OH}_m\mathcal{OH}_{m+1}' + \mathcal{OH}_{m+1}\mathcal{OH}_m'),$$

respectively. Here, $\mathcal{OH}_m = \sum_{l=0}^{7} w_{m+l}e_l$, $\mathcal{OH}_m' = \sum_{l=0}^{7} w_{m+l}'e_l$, $\mathcal{OH}_{m+1} = \sum_{l=1}^{8} w_{m+l}e_{l-1}$ and $\mathcal{OH}_{m+1}' = \sum_{l=1}^{8} w_{m+l}'e_{l-1}$ are Horadam octonions.

The conjugate of $\widetilde{\mathcal{OH}_m}$ is given by

$$\overline{\widetilde{\mathcal{OH}_m}} = S_{\widetilde{\mathcal{OH}_m}} - V_{\widetilde{\mathcal{OH}_m}} \tag{10}$$

$$= (w_m + \varepsilon w_{m+1})e_0 - \sum_{l=1}^{7}(w_{m+l} + \varepsilon w_{m+l+1})e_l. \tag{11}$$

We now introduce the Binet formula for dual Horadam octonions.

Theorem 2.3. *(Binet formula) [8] For $m \geq 1$, the Binet formula for dual Horadam octonion is*

$$\widetilde{\mathcal{OH}_m} = \frac{N\lambda'\lambda^m(1+\lambda\varepsilon) - K\gamma'\gamma^m(1+\gamma\varepsilon)}{\lambda - \gamma} \tag{12}$$

where $\lambda' = \sum_{s=0}^{7} \lambda^s e_s$, $\gamma' = \sum_{s=0}^{7} \gamma^s e_s$, $N = b - a\gamma$ and $K = b - a\lambda$.

Proof. Using the definition (9) and the Binet formula for the Horadam numbers, we obtain

$$\begin{aligned}\widetilde{\mathcal{OH}_m} &= \sum_{s=0}^{7} w_{m+s}e_s + \varepsilon \sum_{s=0}^{7} w_{m+s+1}e_s \\ &= w_m e_0 + \dots + w_{m+7}e_7 + \varepsilon(w_{m+1}e_0 + w_{m+2}e_1 + \dots + w_{m+8}e_7) \\ &= (\frac{N\lambda^m - K\gamma^m}{\lambda-\gamma})e_0 + \dots + (\frac{N\lambda^{m+7} - K\gamma^{m+7}}{\lambda-\gamma})e_7 \\ &+ \varepsilon\Big[(\frac{N\lambda^{m+1} - K\gamma^{m+1}}{\lambda-\gamma})e_0 + \dots + (\frac{N\lambda^{m+8} - K\gamma^{m+8}}{\lambda-\gamma})e_7\Big] \\ &= \frac{N\lambda^m}{\lambda-\gamma}\Big(e_0 + \lambda e_1 + \dots + \lambda^7 e_7\Big)(1+\varepsilon\lambda) \\ &- \frac{K\gamma^m}{\lambda-\gamma}\Big(e_0 + \gamma e_1 + \dots + \gamma^7 e_7\Big)(1+\varepsilon\gamma) \\ &= \frac{N\lambda^{'}\lambda^m(1+\varepsilon\lambda) - K\gamma^{'}\gamma^m(1+\varepsilon\gamma)}{\lambda-\gamma}.\end{aligned}$$

□

Throughout this paper $\widetilde{\mathfrak{F}_m}$, $\widetilde{\mathfrak{L}_m}$, $\widetilde{\mathfrak{J}_m}$, $\widetilde{\mathfrak{j}_m}$, $\widetilde{\mathfrak{P}_m}$ and $\widetilde{\mathfrak{p}_m}$ denote the m-th dual Fibonacci, dual Lucas, dual Jacosthal, dual Jacobsthal–Lucas, dual Pell and dual Pell–Lucas octonions, respectively.

In the Binet formula for dual Horadam octonions, if we put $N = K = 1$, $\lambda = \frac{1+\sqrt{5}}{2}$ and $\gamma = \frac{1-\sqrt{5}}{2}$, we obtain the Binet formula for dual Fibonacci octonions as follows;

$$\widetilde{\mathfrak{F}_m} = \frac{\lambda^{'}\lambda^m(1+\lambda\varepsilon) - \gamma^{'}\gamma^m(1+\gamma\varepsilon)}{\sqrt{5}}$$

which is given in [16], it is given in the second row of the Table 1 (see below).

In the Eq. (12), if we take $N = \sqrt{5}$, $K = -\sqrt{5}$, $\lambda = \frac{1+\sqrt{5}}{2}$ and $\gamma = \frac{1-\sqrt{5}}{2}$, we have the Binet formula for dual Lucas octonions $\widetilde{\mathfrak{L}_m} = (1+\varepsilon\lambda)\lambda^{'}\lambda^m + (1+\varepsilon\gamma)\gamma^{'}\gamma^m$ which is given in [16]. It can be seen in the third row of the Table 1.

In the following table we give the Binet formula for dual Fibonacci, dual Lucas, dual Jacobsthal, dual Jacobsthal– Lucas, dual Pell and dual Pell–Lucas octonions.

Remark 2.4. *The results of Theorem 2.3 are given in the following Table 1.*

Dual Oc	$w_m(a,b;p,q)$	N	K	λ	γ	Binet formulas
D F	$w_m(0,1;1,1)$	1	1	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\widetilde{\mathfrak{F}_m} = \frac{\lambda^{'}\lambda^m(1+\lambda\varepsilon)-\gamma^{'}\gamma^m(1+\gamma\varepsilon)}{\sqrt{5}}$
D L	$w_m(2,1;1,1)$	$\sqrt{5}$	$-\sqrt{5}$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$\widetilde{\mathfrak{L}_m} = \lambda^{'}\lambda^m(1+\lambda\varepsilon)+\gamma^{'}\gamma^m(1+\gamma\varepsilon)$
D J	$w_m(0,1;1,2)$	1	1	2	-1	$\widetilde{\mathfrak{J}_m} = \frac{\lambda^{'}\lambda^m(1+\lambda\varepsilon)-\gamma^{'}\gamma^m(1+\gamma\varepsilon)}{3}$
D J–L	$w_m(2,1;1,2)$	3	-3	2	-1	$\widetilde{\mathfrak{j}_m} = \lambda^{'}\lambda^m(1+\lambda\varepsilon)+\gamma^{'}\gamma^m(1+\gamma\varepsilon)$
D P	$w_m(0,1;2,1)$	1	1	$1+\sqrt{2}$	$1-\sqrt{2}$	$\widetilde{\mathfrak{P}_m} = \frac{\lambda^{'}\lambda^m(1+\lambda\varepsilon)-\gamma^{'}\gamma^m(1+\gamma\varepsilon)}{2\sqrt{2}}$
D P–L	$w_m(2,2;2,1)$	$2\sqrt{2}$	$-2\sqrt{2}$	$1+\sqrt{2}$	$1-\sqrt{2}$	$\widetilde{\mathfrak{p}_m} = \lambda^{'}\lambda^m(1+\lambda\varepsilon)+\gamma^{'}\gamma^m(1+\gamma\varepsilon)$

Table 1. Binet formulas for dual octonions.

Proposition 2.5. *For $t \geq 0$, the following identity holds*

$$\mathcal{OH}_{t+2} = p\mathcal{OH}_{t+1} + q\mathcal{OH}_t.$$

Proof. By Eqs. (3) and (7), we obtain the result. □

In the following proposition we give some properties, which are used throughout the paper.

Proposition 2.6. *For $m \geq 0$, the following identities hold;*

$(i)\quad \widetilde{\mathcal{OH}_{m+2}} = p\widetilde{\mathcal{OH}_{m+1}} + q\widetilde{\mathcal{OH}_m},$

$(ii)\quad \widetilde{\mathcal{OH}_m} + \overline{\widetilde{\mathcal{OH}_m}} = 2(w_m + \varepsilon w_{m+1}),$

$(iii)\quad \widetilde{\mathcal{OH}_m}.\overline{\widetilde{\mathcal{OH}_m}} = \mathcal{OH}_m\overline{\mathcal{OH}_m} + 2\varepsilon \sum_{l=0}^{7} w_{m+l}w_{m+1+l},$

$(iv)\quad \widetilde{\mathcal{OH}_m}^2 = 2{w_m}^2 - \mathcal{OH}_m\overline{\mathcal{OH}_m} + 2\{\varepsilon(w_m w_{m+1} - \sum_{l=1}^{7} w_{m+l}w_{m+1+l})$
$+ \sum_{l=1}^{7}[w_m\widetilde{w_{m+l}} + w_{m+1}w_{m+l}\varepsilon]e_l\},$

$(v)\quad \overline{\widetilde{\mathcal{OH}_m}}^2 = 2{w_m}^2 - \mathcal{OH}_m\overline{\mathcal{OH}_m} + 2\{\varepsilon(w_m w_{m+1} - \sum_{l=1}^{7} w_{m+l}w_{m+1+l})$
$- \sum_{l=1}^{7}[w_m\widetilde{w_{m+l}} + w_{m+1}w_{m+l}\varepsilon]e_l\},$

$(vi)\quad \widetilde{\mathcal{OH}_m}^2 + \overline{\widetilde{\mathcal{OH}_m}}^2 = 4{w_m}^2 - 2\mathcal{OH}_m\overline{\mathcal{OH}_m} + 4w_m w_{m+1}\varepsilon - 4\varepsilon\sum_{l=1}^{7} w_{m+l}w_{m+1+l}.$

Proof. (i) By the Eq. (9) and Proposition 2.5, we obtain

$$\begin{aligned}\widetilde{\mathcal{OH}_{m+2}} &= \mathcal{OH}_{m+2} + \varepsilon\mathcal{OH}_{m+3} \\ &= p\mathcal{OH}_{m+1} + q\mathcal{OH}_m + \varepsilon(p\mathcal{OH}_{m+2} + q\mathcal{OH}_{m+1}) \\ &= p(\mathcal{OH}_{m+1} + \varepsilon\mathcal{OH}_{m+2}) + q(\mathcal{OH}_m + \varepsilon\mathcal{OH}_{m+1}) \\ &= p\widetilde{\mathcal{OH}_{m+1}} + q\widetilde{\mathcal{OH}_m}.\end{aligned}$$

(ii) From the definition of $\overline{\widetilde{\mathcal{OH}_m}}$, equations (7) and (9), we have the result.

(iii) Employing Definitions (7), (10) and the multiplication Table in ([16]), we can write

$$\begin{aligned}&\widetilde{\mathcal{OH}_m}.\overline{\widetilde{\mathcal{OH}_m}} \\ &= \Big[(w_m + \varepsilon w_{m+1})e_0 + (w_{m+1} + \varepsilon w_{m+2})e_1 + ... + (w_{m+7} + \varepsilon w_{m+8})e_7)\Big] \\ &\times \Big[(w_m + \varepsilon w_{m+1})e_0 - (w_{m+1} + \varepsilon w_{m+2})e_1 - ... - (w_{m+7} + \varepsilon w_{m+8})e_7\Big] \\ &= {w_m}^2 + {w_{m+1}}^2 + ... + {w_{m+7}}^2 \\ &+ 2\varepsilon(w_m w_{m+1} + w_{m+1}w_{m+2} + ... + w_{m+7}w_{m+8}) \\ &= \mathcal{OH}_m\overline{\mathcal{OH}_m} + 2\varepsilon\sum_{l=0}^{7} w_{m+l}w_{m+1+l}.\end{aligned}$$

(iv) From the definition (7), the multiplication Table in [16] and the definition of dual Horadam sequences, we obtain

$$
\begin{aligned}
\widetilde{\mathcal{OH}_m}^2 &= (\mathcal{OH}_m + \varepsilon \mathcal{OH}_{m+1})^2 \\
&= [(w_m + w_{m+1}\varepsilon)e_0 + (w_{m+1} + w_{m+2}\varepsilon)e_1 + \cdots + (w_{m+7} + w_{m+8}\varepsilon)e_7]^2 \\
&= {w_m}^2 - {w_{m+1}}^2 - \cdots - {w_{m+7}}^2 + 2\varepsilon(w_m w_{m+1} - w_{m+1}w_{m+2} - \cdots - w_{m+7}w_{m+8}) \\
&+ 2[w_m w_{m+1} + w_m w_{m+2}\varepsilon + {w_{m+1}}^2\varepsilon]e_1 + 2[w_m w_{m+2} + w_m w_{m+3}\varepsilon + w_{m+1}w_{m+2}\varepsilon]e_2 \\
&+ 2[w_m w_{m+3} + w_m w_{m+4}\varepsilon + w_{m+1}w_{m+3}\varepsilon]e_3 + 2[w_m w_{m+4} + w_{m+1}w_{m+4}\varepsilon + w_m w_{m+5}\varepsilon]e_4 \\
&+ 2[w_m w_{m+5} + w_m w_{m+6}\varepsilon + w_{m+1}w_{m+5}\varepsilon]e_5 + 2[w_m w_{m+6} + w_m w_{m+7}\varepsilon + w_{m+1}w_{m+6}\varepsilon]e_6 \\
&+ 2[w_m w_{m+7} + w_m w_{m+8}\varepsilon + w_{m+1}w_{m+7}\varepsilon]e_7 \\
&= {w_m}^2 - ({w_{m+1}}^2 + \ldots + {w_{m+7}}^2) + 2\varepsilon(w_m w_{m+1} - \sum_{l=1}^{7} w_{m+l}w_{m+1+l}) \\
&+ 2\sum_{l=1}^{7}[w_{m+l}\widetilde{w_{m+l}} + w_{m+1}w_{m+l}\varepsilon]e_l \\
&= 2{w_m}^2 - \mathcal{OH}_m\overline{\mathcal{OH}_m} + 2\big\{\varepsilon(w_m w_{m+1} - \sum_{l=1}^{7} w_{m+l}w_{m+1+l}) \\
&+ \sum_{l=1}^{7}[w_m\widetilde{w_{m+l}} + w_{m+1}w_{m+l}\varepsilon]e_l\big\}.
\end{aligned}
$$

(v) Using the definition of $\overline{\widetilde{\mathcal{OH}_m}}$ and the multiplication Table in [16], it is easy to get the identity required.
(vi) By employing (iv) and (v), the result immediately follows. □

Theorem 2.7. *[8] The generating function for the dual Horadam octonion is*

$$h(x) = \frac{\widetilde{\mathcal{OH}_0} + (\widetilde{\mathcal{OH}_1} - p\widetilde{\mathcal{OH}_0})x}{1 - px - qx^2}.$$

Proof. Since the proof is similar to that of Theorem 2.3 in [13], it will be omitted. □

Remark 2.8. *The results of Theorem 2.7 are listed in the following Table 2:*

Dual Octonions	$w_m(a,b;p,q)$	λ	γ	Generating functions
Dual Fibonacci	$w_m(0,1;1,1)$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$f(x) = \frac{\widetilde{\mathfrak{F}_0}+(\widetilde{\mathfrak{F}_1}-\widetilde{\mathfrak{F}_0})x}{1-x-x^2}$
Dual Lucas	$w_m(2,1;1,1)$	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$	$l(x) = \frac{\widetilde{\mathfrak{L}_0}+(\widetilde{\mathfrak{L}_1}-\widetilde{\mathfrak{L}_0})x}{1-x-x^2}$
Dual Jacobsthal	$w_m(0,1;1,2)$	2	-1	$J(x) = \frac{\widetilde{\mathfrak{J}_0}+(\widetilde{\mathfrak{J}_1}-\widetilde{\mathfrak{J}_0})x}{1-x-2x^2}$
Dual Jacobsthal–Lucas	$w_m(2,1;1,2)$	2	-1	$j(x) = \frac{\widetilde{\mathfrak{j}_0}+(\widetilde{\mathfrak{j}_1}-\widetilde{\mathfrak{j}_0})x}{1-x-2x^2}$
Dual Pell	$w_m(0,1;2,1)$	$1+\sqrt{2}$	$1-\sqrt{2}$	$P(x) = \frac{\widetilde{\mathfrak{P}_0}+(\widetilde{\mathfrak{P}_1}-2\widetilde{\mathfrak{P}_0})x}{1-2x-x^2}$
Dual Pell–Lucas	$w_m(2,2;2,1)$	$1+\sqrt{2}$	$1-\sqrt{2}$	$p(x) = \frac{\widetilde{\mathfrak{p}_0}+(\widetilde{\mathfrak{p}_1}-2\widetilde{\mathfrak{p}_0})x}{1-2x-x^2}$

Table 2. Generating functions for dual octonions.

Proposition 2.9. *The following identities hold:*
(i) $\widetilde{\mathcal{OH}}_1 - \lambda\widetilde{\mathcal{OH}}_0 = K\gamma^{'}(1+\gamma\varepsilon)$,
(ii) $\widetilde{\mathcal{OH}}_1 - \gamma\widetilde{\mathcal{OH}}_0 = N\lambda^{'}(1+\lambda\varepsilon)$.

Proof. (i) Using the Binet formula for dual Horadam octonions, we get

$$\begin{aligned}
\widetilde{\mathcal{OH}}_1 - \lambda\widetilde{\mathcal{OH}}_0 &= \Big(\frac{N\lambda^{'}\lambda(1+\lambda\varepsilon) - K\gamma^{'}\gamma(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)\\
&\quad - \lambda\Big(\frac{N\lambda^{'}(1+\lambda\varepsilon) - K\gamma^{'}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)\\
&= K\gamma^{'}(1+\gamma\varepsilon).
\end{aligned}$$

(ii) The proof of the second equation is similar to the proof of (i), so it is omitted. □

Theorem 2.10. *For $t \in \mathbb{Z}^+$, $u \in \mathbb{N}$, the generating function of the dual Horadam octonion $\widetilde{\mathcal{OH}}_{u+t}$ is*

$$\sum_{u=0}^{\infty} \widetilde{\mathcal{OH}}_{u+t} x^u = \frac{\widetilde{\mathcal{OH}}_t + qx\widetilde{\mathcal{OH}}_{t-1}}{1-px-qx^2}.$$

Proof. By considering Theorem 2.3 and Eq. (5), we obtain

$$\begin{aligned}
&\sum_{u=0}^{\infty} \widetilde{\mathcal{OH}}_{u+t} x^u\\
&= \sum_{u=0}^{\infty} \Big(\frac{N\lambda^{u+t}\lambda^{'}(1+\lambda\varepsilon) - K\gamma^{u+t}\gamma^{'}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)x^u\\
&= \frac{1}{\lambda-\gamma}\Big\{N\lambda^{'}\lambda^t(1+\lambda\varepsilon)\sum_{u=0}^{\infty}(\lambda x)^u - K\gamma^{'}\gamma^t(1+\gamma\varepsilon)\sum_{u=0}^{\infty}(\gamma x)^u\Big\}\\
&= \frac{1}{\lambda-\gamma}\Big\{N\lambda^{'}\lambda^t(1+\lambda\varepsilon)(\frac{1}{1-\lambda x}) - K\gamma^{'}\gamma^t(1+\gamma\varepsilon)(\frac{1}{1-\gamma x})\}\\
&= \frac{1}{1-px-qx^2}\Big\{\frac{N\lambda^{'}\lambda^t(1+\lambda\varepsilon) - K\gamma^{'}\gamma^t(1+\gamma\varepsilon)}{\lambda-\gamma}\\
&+ \frac{-N\lambda^{'}\lambda^t\gamma x(1+\lambda\varepsilon) + K\gamma^{'}\gamma^t\lambda x(1+\gamma\varepsilon)}{\lambda-\gamma}\Big\}\\
&= \frac{1}{1-px-qx^2}\Big\{\widetilde{\mathcal{OH}}_t - \Big[\frac{N\lambda^{'}\lambda^{t-1}x(1+\lambda\varepsilon) - K\gamma^{'}\gamma^{t-1}\lambda x(1+\gamma\varepsilon)}{\lambda-\gamma}\Big]\lambda\gamma\Big\}\\
&= \frac{\widetilde{\mathcal{OH}}_t + qx\widetilde{\mathcal{OH}}_{t-1}}{1-px-qx^2}.
\end{aligned}$$

□

Theorem 2.11. *The exponential generating function for the dual Horadam octonion is*

$$\sum_{l=0}^{\infty} \frac{\widetilde{\mathcal{OH}}_l}{l!} y^l = \frac{N\lambda^{'}(1+\lambda\varepsilon)e^{\lambda y} - K\gamma^{'}(1+\gamma\varepsilon)e^{\gamma y}}{\lambda-\gamma}.$$

Proof. Using the Binet formula for dual Horadam octonions, we have

$$\begin{aligned}\sum_{l=0}^{\infty}\frac{\widetilde{\mathcal{OH}_l}}{l!}y^l &= \sum_{l=0}^{\infty}\Big(\frac{N\lambda^{'}\lambda^l(1+\lambda\varepsilon)-K\gamma^{'}\gamma^l(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)\frac{y^l}{l!}\\ &= \frac{N\lambda^{'}(1+\lambda\varepsilon)}{\lambda-\gamma}\sum_{l=0}^{\infty}\frac{\lambda^l y^l}{l!}-\frac{K\gamma^{'}(1+\gamma\varepsilon)}{\lambda-\gamma}\sum_{l=0}^{\infty}\frac{\gamma^l y^l}{l!}\\ &= \frac{N\lambda^{'}(1+\lambda\varepsilon)e^{\lambda y}-K\gamma^{'}(1+\gamma\varepsilon)e^{\gamma y}}{\lambda-\gamma}.\end{aligned}$$

□

We next give a proposition for the dual Horadam octonions. The proof of proposition can be done by using Binet's formula for $\widetilde{\mathcal{OH}_m}$.

Proposition 2.12. *For $m \geq 0$, the following identities hold:*

$(i)[8]\quad \sum\limits_{l=0}^{m}\widetilde{\mathcal{OH}_l}=\frac{1}{1-p-q}\Big(\widetilde{\mathcal{OH}_0}-\widetilde{\mathcal{OH}_{m+1}}-q\widetilde{\mathcal{OH}_m}-[\frac{N\lambda'\gamma(1+\lambda\epsilon)-K\gamma'\lambda(1+\gamma\epsilon)}{\lambda-\gamma}]\Big)$;

$(ii)\quad \sum\limits_{l=0}^{m}\widetilde{\mathcal{OH}_{2l}}=\frac{1}{1-p^2-2q+q^2}\Big(\widetilde{\mathcal{OH}_0}-\widetilde{\mathcal{OH}_{2m+2}}+q^2\widetilde{\mathcal{OH}_{2m}}-[\frac{N\lambda'\gamma^2(1+\lambda\epsilon)-K\gamma'\lambda^2(1+\gamma\epsilon)}{\lambda-\gamma}]\Big)$;

$(iii)\quad \sum\limits_{l=1}^{m}\widetilde{\mathcal{OH}_{2l-1}}=\frac{1}{1-p^2-2q+q^2}\Big(\widetilde{\mathcal{OH}_1}-\widetilde{\mathcal{OH}_{2m+1}}+q^2\widetilde{\mathcal{OH}_{2m-1}}+q[\frac{N\lambda'\gamma(1+\lambda\epsilon)-K\gamma'\lambda(1+\gamma\epsilon)}{\lambda-\gamma}]\Big)$.

We next give Catalan's, Cassini's and d'Ocagne's identities for dual Horadam octonions.

Theorem 2.13. *(Catalan's Identities) For every nonnegative integer number u and t such that $t \leq u$, we have*

$$(i)\quad \widetilde{\mathcal{OH}_{u+t}}\widetilde{\mathcal{OH}_{u-t}}-\widetilde{\mathcal{OH}_u}^2=\frac{(-q)^{u-t}}{(p^2+4q)}NK(1+p\varepsilon)(\lambda^t-\gamma^t)\Big\{\gamma^{'}\lambda^{'}\gamma^t-\lambda^{'}\gamma^{'}\lambda^t\Big\},$$

$$(ii)\quad \widetilde{\mathcal{OH}_{u-t}}\widetilde{\mathcal{OH}_{u+t}}-\widetilde{\mathcal{OH}_u}^2=\frac{(-q)^{u-t}}{(p^2+4q)}NK(1+p\varepsilon)(\lambda^t-\gamma^t)\Big\{\lambda^{'}\gamma^{'}\gamma^t-\gamma^{'}\lambda^{'}\lambda^t\Big\}.$$

Proof. (i) Using the (5) and (12), we obtain

$$\begin{aligned}&\widetilde{\mathcal{OH}_{u+t}}\widetilde{\mathcal{OH}_{u-t}}-\widetilde{\mathcal{OH}_u}^2\\ &=\Big(\frac{N\lambda^{'}\lambda^{u+t}(1+\lambda\varepsilon)-K\gamma^{'}\gamma^{u+t}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)\\ &\times\Big(\frac{N\lambda^{'}\lambda^{u-t}(1+\lambda\varepsilon)-K\gamma^{'}\gamma^{u-t}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)-\Big(\frac{N\lambda^{'}\lambda^{u}(1+\lambda\varepsilon)-K\gamma^{'}\gamma^{u}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big)^2\\ &=\frac{-NK\lambda^{'}\gamma^{'}\lambda^{u+t}\gamma^{u-t}[1+(\lambda+\gamma)\varepsilon]-NK\gamma^{'}\lambda^{'}\gamma^{u+t}\lambda^{u-t}[1+(\lambda+\gamma)\varepsilon]}{(\lambda-\gamma)^2}\\ &+\frac{NK\lambda^{'}\gamma^{'}\lambda^{u}\gamma^{u}[1+(\lambda+\gamma)\varepsilon]+NK\gamma^{'}\lambda^{'}\lambda^{u}\gamma^{u}[1+(\lambda+\gamma)\varepsilon]}{(\lambda-\gamma)^2}\\ &=\frac{NK[1+(\lambda+\gamma)\varepsilon](\lambda\gamma)^u\big\{-\lambda^{'}\gamma^{'}\lambda^t\gamma^{-t}+\lambda^{'}\gamma^{'}-\gamma^{'}\lambda^{'}\gamma^t\lambda^{-t}+\gamma^{'}\lambda^{'}\big\}}{(\lambda-\gamma)^2}\\ &=\frac{NK(1+p\varepsilon)(-q)^u}{(\lambda-\gamma)^2}\Big\{\frac{-\lambda^{'}\gamma^{'}\lambda^t(\lambda^t-\gamma^t)+\gamma^{'}\lambda^{'}\gamma^t(\lambda^t-\gamma^t)}{(\lambda\gamma)^t}\Big\}\\ &=\frac{NK(1+p\varepsilon)(-q)^{u-t}(\lambda^t-\gamma^t)}{(p^2+4q)}\Big\{\gamma^{'}\lambda^{'}\gamma^t-\lambda^{'}\gamma^{'}\lambda^t\Big\}.\end{aligned}$$

The proof of (ii) is similar to the proof of (i) and will be omitted. □

If, in Theorem 2.13(i), we write the values of N, K, λ, γ and $w_n(a,b;p,q)$ as in Table 2, then we obtain the Catalan identity for the corresponding dual octonion as in the following table;

Remark 2.14. *In Table 3, we show the results of Theorem 2.13(i):*

Dual Oc	Catalan's identities
D F	$\widetilde{\mathfrak{F}_{u+t}}\widetilde{\mathfrak{F}_{u-t}}-\widetilde{\mathfrak{F}_u}^2=\frac{(-1)^{u-t}(1+\varepsilon)}{5}\Big\{(\frac{1+\sqrt{5}}{2})^t-(\frac{1-\sqrt{5}}{2})^t\Big\}\Big\{\gamma'\lambda'(\frac{1-\sqrt{5}}{2})^t-\lambda'\gamma'(\frac{1+\sqrt{5}}{2})^t\Big\}$
D L	$\widetilde{\mathfrak{L}_{u+t}}\widetilde{\mathfrak{L}_{u-t}}-\widetilde{\mathfrak{L}_u}^2=-(-1)^{u-t}(1+\varepsilon)\Big\{(\frac{1+\sqrt{5}}{2})^t-(\frac{1-\sqrt{5}}{2})^t\Big\}\Big\{\gamma'\lambda'(\frac{1-\sqrt{5}}{2})^t-\lambda'\gamma'(\frac{1+\sqrt{5}}{2})^t\Big\}$
D J	$\widetilde{\mathfrak{J}_{u+t}}\widetilde{\mathfrak{J}_{u-t}}-\widetilde{\mathfrak{J}_u}^2=\frac{(-2)^{u-t}}{9}(1+\varepsilon)\Big\{2^t-(-1)^t\Big\}\Big\{\gamma'\lambda'(-1)^t-\lambda'\gamma'2^t\Big\}$
D J–L	$\widetilde{\mathfrak{j}_{u+t}}\widetilde{\mathfrak{j}_{u-t}}-\widetilde{\mathfrak{j}_u}^2=-(-2)^{u-t}(1+\varepsilon)\Big\{2^t-(-1)^t\Big\}\Big\{\gamma'\lambda'(-1)^t-\lambda'\gamma'2^t\Big\}$
D P	$\widetilde{\mathfrak{P}_{u+t}}\widetilde{\mathfrak{P}_{u-t}}-\widetilde{\mathfrak{P}_u}^2=\frac{(-1)^{u-t}}{8}(1+2\varepsilon)\Big\{(1+\sqrt{2})^t-(1-\sqrt{2})^t\Big\}\Big\{\gamma'\lambda'(1-\sqrt{2})^t-\lambda'\gamma'(1+\sqrt{2})^t\Big\}$
D P–L	$\widetilde{\mathfrak{p}_{u+t}}\widetilde{\mathfrak{p}_{u-t}}-\widetilde{\mathfrak{p}_u}^2=-(-1)^{u-t}(1+2\varepsilon)\{(1+\sqrt{2})^t-(1-\sqrt{2})^t\}\{\gamma'\lambda'(1-\sqrt{2})^t-\lambda'\gamma'(1+\sqrt{2})^t\}$

Table 3. Catalan's identities for dual octonions.

For $t=1$ in the Catalan identity we have the Cassini identity for dual Horadam octonions in the next Theorem:

Theorem 2.15. *(Cassini's identities) For $u\geq 1$, we have*

(i) $\widetilde{\mathcal{OH}_{u+1}}\widetilde{\mathcal{OH}_{u-1}}-\widetilde{\mathcal{OH}_u}^2=\frac{NK(1+p\varepsilon)(-q)^{u-1}}{\sqrt{p^2+4q}}\Big[\gamma'\lambda'\gamma-\lambda'\gamma'\lambda\Big]$;

(ii)[8] $\widetilde{\mathcal{OH}_{u-1}}\widetilde{\mathcal{OH}_{u+1}}-\widetilde{\mathcal{OH}_u}^2=\frac{NK(1+p\varepsilon)(-q)^{u-1}}{\sqrt{p^2+4q}}\Big[\lambda'\gamma'\gamma-\gamma'\lambda'\lambda\Big]$.

If, in Theorem 2.15(i), we put the values of N, K, λ, γ and $w_n(a,b;p,q)$ as in Table 2, then we get the Cassini identity for the corresponding dual octonion as in the Table 4.

Remark 2.16. *The results of Theorem 2.15(i) are listed in the following Table 4:*

Dual Octonions	Cassini's identities
Dual Fibonacci	$\widetilde{\mathfrak{F}_{u+1}}\widetilde{\mathfrak{F}_{u-1}}-\widetilde{\mathfrak{F}_u}^2=\frac{(-1)^{u-1}}{\sqrt{5}}(1+\varepsilon)\Big\{\gamma'\lambda'(\frac{1-\sqrt{5}}{2})-\lambda'\gamma'(\frac{1+\sqrt{5}}{2})\Big\}$
Dual Lucas	$\widetilde{\mathfrak{L}_{u+1}}\widetilde{\mathfrak{L}_{u-1}}-\widetilde{\mathfrak{L}_u}^2=-(-1)^{u-1}\sqrt{5}(1+\varepsilon)\Big\{\gamma'\lambda'(\frac{1-\sqrt{5}}{2})-\lambda'\gamma'(\frac{1+\sqrt{5}}{2})\Big\}$
Dual Jacobshtal	$\widetilde{\mathfrak{J}_{u+1}}\widetilde{\mathfrak{J}_{u-1}}-\widetilde{\mathfrak{J}_u}^2=\frac{(-2)^{u-1}}{3}(1+\varepsilon)\Big\{\gamma'\lambda'(-1)-\lambda'\gamma'2\Big\}$
Dual Jacobsthal–Lucas	$\widetilde{\mathfrak{j}_{u+1}}\widetilde{\mathfrak{j}_{u-1}}-\widetilde{\mathfrak{j}_u}^2=-3(-2)^{u-1}(1+\varepsilon)\Big\{\gamma'\lambda'(-1)-\lambda'\gamma'2\Big\}$
Dual Pell	$\widetilde{\mathfrak{P}_{u+1}}\widetilde{\mathfrak{P}_{u-1}}-\widetilde{\mathfrak{P}_u}^2=(1+2\varepsilon)\frac{(-1)^{u-1}}{2\sqrt{2}}\Big\{\gamma'\lambda'(1-\sqrt{2})-\lambda'\gamma'(1+\sqrt{2})\Big\}$
Du Pell–Lucas	$\widetilde{\mathfrak{p}_{u+1}}\widetilde{\mathfrak{p}_{u-1}}-\widetilde{\mathfrak{p}_u}^2=-2\sqrt{2}(-1)^{u-1}(1+2\varepsilon)\{\gamma'\lambda'(1-\sqrt{2})-\lambda'\gamma'(1+\sqrt{2})\}$

Table 4: Cassini's identities for dual octonions.

Theorem 2.17. *(d'Ocagne's identities) For $t \in \mathbb{Z}^+$ and $u \in \mathbb{N}$, such that $u > t+1$. Then*
(i) $\widetilde{\mathcal{OH}_{t+1}}\widetilde{\mathcal{OH}_u} - \widetilde{\mathcal{OH}_t}\widetilde{\mathcal{OH}_{u+1}} = \frac{NK(1+p\varepsilon)(-q)^t}{\sqrt{p^2+4q}}\Big[-\lambda'\gamma'\gamma^{u-t} + \gamma'\lambda'\lambda^{u-t}\Big]$,

(ii) $\widetilde{\mathcal{OH}_u}\widetilde{\mathcal{OH}_{t+1}} - \widetilde{\mathcal{OH}_t}\widetilde{\mathcal{OH}_{u+1}} = \frac{NK(1+p\varepsilon)(-q)^t(\lambda^{u-t}-\gamma^{u-t})}{p^2+4q}\Big[\gamma'\lambda'\lambda - \lambda'\gamma'\gamma\Big]$,

(iii) $\widetilde{\mathcal{OH}_u}\widetilde{\mathcal{OH}_{t+1}} - \widetilde{\mathcal{OH}_{u+1}}\widetilde{\mathcal{OH}_t} = \frac{NK(1+p\varepsilon)(-q)^t}{\sqrt{p^2+4q}}\Big[\lambda'\gamma'\lambda^{u-t} - \gamma'\lambda'\gamma^{u-t}\Big]$.

Proof. From the Binet formula for dual Horadam octonions and Eq. (5), we have

$$
\begin{aligned}
&(i) \quad \widetilde{\mathcal{OH}_{t+1}}\widetilde{\mathcal{OH}_u} - \widetilde{\mathcal{OH}_t}\widetilde{\mathcal{OH}_{u+1}} \\
&= \Big(\frac{N\lambda'\lambda^{t+1}(1+\lambda\varepsilon) - K\gamma'\gamma^{t+1}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big) \times \Big(\frac{N\lambda'\lambda^{u}(1+\lambda\varepsilon) - K\gamma'\gamma^{u}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big) \\
&- \Big(\frac{N\lambda'\lambda^{t}(1+\lambda\varepsilon) - K\gamma'\gamma^{t}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big) \times \Big(\frac{N\lambda'\lambda^{u+1}(1+\lambda\varepsilon) - K\gamma'\gamma^{u+1}(1+\gamma\varepsilon)}{\lambda-\gamma}\Big) \\
&= NK(1+p\varepsilon)\Big\{\frac{-\lambda'\gamma'\lambda^t\gamma^u(\lambda-\gamma) + \gamma'\lambda'\lambda^t\gamma^u(\lambda-\gamma)}{(\lambda-\gamma)^2}\Big\} \\
&= \frac{NK(1+p\varepsilon)(-q)^t}{\sqrt{p^2+4q}}\Big[-\lambda'\gamma'\gamma^{u-t} + \gamma'\lambda'\lambda^{u-t}\Big].
\end{aligned}
$$

The proofs of (ii) and (iii) are similar to the proof of (i), so they are omitted. □

If, in Theorem 2.17 (i), we take the values of N, K, λ, γ and $w_n(a,b;p,q)$ as in Table 2, then we obtain the d'Ocagne identity for the corresponding dual octonion as in the following table.

Remark 2.18. *The results of Theorem 2.17(i) are given in the following table (Table 5);*

Dual Oc.	d'Ocagne's identities
D F	$\widetilde{\mathfrak{F}_{t+1}}\widetilde{\mathfrak{F}_u} - \widetilde{\mathfrak{F}_t}\widetilde{\mathfrak{F}_{u+1}} = \frac{(-1)^t(1+\varepsilon)}{\sqrt{5}}\Big\{\gamma'\lambda'(\frac{1+\sqrt{5}}{2})^{u-t} - \lambda'\gamma'(\frac{1-\sqrt{5}}{2})^{u-t}\Big\}$
D L	$\widetilde{\mathfrak{L}_{t+1}}\widetilde{\mathfrak{L}_u} - \widetilde{\mathfrak{L}_t}\widetilde{\mathfrak{L}_{u+1}} = -(-1)^t\sqrt{5}(1+\varepsilon)\Big\{\gamma'\lambda'(\frac{1+\sqrt{5}}{2})^{u-t} - \lambda'\gamma'(\frac{1-\sqrt{5}}{2})^{u-t}\Big\}$
D J	$\widetilde{\mathfrak{J}_{t+1}}\widetilde{\mathfrak{J}_u} - \widetilde{\mathfrak{J}_t}\widetilde{\mathfrak{J}_{u+1}} = \frac{(-2)^t(1+\varepsilon)}{3}\Big\{-\lambda'\gamma'(-1)^{u-t} + \gamma'\lambda'2^{u-t}\Big\}$
D J–L	$\widetilde{\mathfrak{j}_{t+1}}\widetilde{\mathfrak{j}_u} - \widetilde{\mathfrak{j}_t}\widetilde{\mathfrak{j}_{u+1}} = -3(-2)^t(1+\varepsilon)\{-\lambda'\gamma'(-1)^{u-t} + \gamma'\lambda'2^{u-t}\}$
D P	$\widetilde{\mathfrak{P}_{t+1}}\widetilde{\mathfrak{P}_u} - \widetilde{\mathfrak{P}_t}\widetilde{\mathfrak{P}_{u+1}} = \frac{(-1)^t(1+2\varepsilon)}{2\sqrt{2}}\Big\{\gamma'\lambda'(1+\sqrt{2})^{u-t} - \lambda'\gamma'(1-\sqrt{2})^{u-t}\Big\}$
D P–L	$\widetilde{\mathfrak{p}_{t+1}}\widetilde{\mathfrak{p}_u} - \widetilde{\mathfrak{p}_t}\widetilde{\mathfrak{p}_{u+1}} = -2\sqrt{2}(-1)^t(1+2\varepsilon)\{\gamma'\lambda'(1+\sqrt{2})^{u-t} - \lambda'\gamma'(1-\sqrt{2})^{u-t}\}$

Table 5. d'Ocagne's identities for dual octonions.

3 The matrix generator for dual Horadam octonions

In this section, we introduce the matrix generator for dual Horadam octonions.
Let

$$\widetilde{\mathcal{OH}(m)} = \begin{bmatrix} \widetilde{\mathcal{OH}_m} & \widetilde{\mathcal{OH}_{m-1}} \\ \widetilde{\mathcal{OH}_{m-1}} & \widetilde{\mathcal{OH}_{m-2}} \end{bmatrix}$$

be a matrix with entries dual Horadam octonions. Then we give the following theorem:

Theorem 3.1. *For $v \geq 2$, we have*

$$\begin{bmatrix} \widetilde{\mathcal{OH}_v} & \widetilde{\mathcal{OH}_{v-1}} \\ \widetilde{\mathcal{OH}_{v-1}} & \widetilde{\mathcal{OH}_{v-2}} \end{bmatrix} = \begin{bmatrix} \widetilde{\mathcal{OH}_2} & \widetilde{\mathcal{OH}_1} \\ \widetilde{\mathcal{OH}_1} & \widetilde{\mathcal{OH}_0} \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}^{v-2}.$$

Proof. We will use the induction method to prove this theorem. If $v = 2$, then the result immediately follows.
If $v = 3$, then using Proposition 2.6(i), we obtain

$$\begin{aligned} \begin{bmatrix} \widetilde{\mathcal{OH}_2} & \widetilde{\mathcal{OH}_1} \\ \widetilde{\mathcal{OH}_1} & \widetilde{\mathcal{OH}_0} \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} &= \begin{bmatrix} p\widetilde{\mathcal{OH}_2} + q\widetilde{\mathcal{OH}_1} & \widetilde{\mathcal{OH}_2} \\ p\widetilde{\mathcal{OH}_1} + q\widetilde{\mathcal{OH}_0} & \widetilde{\mathcal{OH}_1} \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\mathcal{OH}_3} & \widetilde{\mathcal{OH}_2} \\ \widetilde{\mathcal{OH}_2} & \widetilde{\mathcal{OH}_1} \end{bmatrix}. \end{aligned}$$

Thus, it is true for $v = 3$.
Suppose that it is true for all $v \leq k$. We must show it holds for $v = k + 1$.
By Proposition 2.6(i), we obtain

$$\begin{aligned} \begin{bmatrix} \widetilde{\mathcal{OH}_2} & \widetilde{\mathcal{OH}_1} \\ \widetilde{\mathcal{OH}_1} & \widetilde{\mathcal{OH}_0} \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}^{k-1} &= \left(\begin{bmatrix} \widetilde{\mathcal{OH}_2} & \widetilde{\mathcal{OH}_1} \\ \widetilde{\mathcal{OH}_1} & \widetilde{\mathcal{OH}_0} \end{bmatrix} \cdot \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix}^{k-2} \right) \cdot \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\mathcal{OH}_k} & \widetilde{\mathcal{OH}_{k-1}} \\ \widetilde{\mathcal{OH}_{k-1}} & \widetilde{\mathcal{OH}_{k-2}} \end{bmatrix} \begin{bmatrix} p & 1 \\ q & 0 \end{bmatrix} \\ &= \begin{bmatrix} p\widetilde{\mathcal{OH}_k} + q\widetilde{\mathcal{OH}_{k-1}} & \widetilde{\mathcal{OH}_k} \\ p\widetilde{\mathcal{OH}_{k-1}} + q\widetilde{\mathcal{OH}_{k-2}} & \widetilde{\mathcal{OH}_{k-1}} \end{bmatrix} \\ &= \begin{bmatrix} \widetilde{\mathcal{OH}_{k+1}} & \widetilde{\mathcal{OH}_k} \\ \widetilde{\mathcal{OH}_k} & \widetilde{\mathcal{OH}_{k-1}} \end{bmatrix}. \end{aligned}$$

So it is true for $v = k + 1$. Thus, the proof is complete. □

The determinant of $\widetilde{\mathcal{OH}(m)} = \begin{bmatrix} \widetilde{\mathcal{OH}_m} & \widetilde{\mathcal{OH}_{m-1}} \\ \widetilde{\mathcal{OH}_{m-1}} & \widetilde{\mathcal{OH}_{m-2}} \end{bmatrix}$ gives the Cassini formula for dual Horadam octonions.

Corollary 3.1.1. *Let $m \geq 2$ be an integer. Then*

$$\widetilde{\mathcal{OH}_m}\widetilde{\mathcal{OH}_{m-2}} - \widetilde{\mathcal{OH}_{m-1}}^2 = \left(\widetilde{\mathcal{OH}_2}\widetilde{\mathcal{OH}_0} - \widetilde{\mathcal{OH}_1}^2\right)(-q)^{m-2}.$$

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