Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 1, 122–127
DOI: 10.7546/nntdm.2019.25.1.122-127

Averages of the Dirichlet convolution of the binary digital sum

Teerapat Srichan

Department of Mathematics, Faculty of Science
Kasetsart University, Bangkok, Thailand
e-mail: fscitrp@ku.ac.th

Received: 17 April 2018 **Revised:** 22 November 2018 **Accepted:** 4 January 2019

Abstract: We derive some averages of the Dirichlet convolution of the binary digital sum $s_2(n)$, the sum of digits ofthe expansion of n in base 2. The Trollope–Delange formula is used in our proof. It provides an explicit asymptotic formula for the total number of digits '1' in the binary expansions of the integers between 1 and $n-1$ in term of the continuous, nowhere differentiable Takagi function. Moreover, we also extend the result to averages of the k-th convolution of the binary digital sum, for $k \geq 2$.

Keywords: Binary digital sum, Dirichlet convolution.
2010 Mathematics Subject Classification: 11A63.

1 Introduction and statement of results

Each non-negative integer can be written uniquely in base 2 as $n = \sum\limits_{i\geq 0} a_i 2^i$, where the integer $a_i \in \{0,1\}$. The binary digital sum is defined by $s_2(n) = \sum\limits_{i \geq 0} a_i$. The sum of digits function $s(n)$ appears in many different fields of mathematics. In 1948, Bellman and Shapiro [1] proved that

$$\sum_{0\leq n<x} s_2(n) = \frac{x \log x}{2\log 2} + O(x \log\log x). \tag{1}$$

L. Mirsky [6] improved the error term in (1) to $O(x)$. There are many related results to (1). We refer the reader to the monograph [7, Chapter 4, section 4.3] for more details. The classical results on the sum (1) is the Trollope–Delange formula [8, Trollope] and [3, Delange] which is stated as the following theorem.

Theorem 1.1. *[Trollope [8] and Delange [3]] Let T be the Takagi function:*

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} \Psi(2^{n-1}x),\ 0 \le x \le 1,$$

where $\psi(x) = |2x - 2\left\lfloor x + \frac{1}{2} \right\rfloor|$. And let $F : \mathbb{R} \to \mathbb{R}$ be periodic of period 1, continuous and nowhere differentiable defined by

$$F(x) = 1 - x - 2^{1-x} T\left(\frac{1}{2^{1-x}}\right),\ 0 \le x \le 1.$$

Then

$$\sum_{0 \le n < x} s_2(n) = \frac{x \log x}{2 \log 2} + \frac{x}{2} F\left(\frac{\log x}{\log 2}\right). \tag{2}$$

The Trollope–Delange formula is a powerful tool for attacking problems involving the binary digital sum. In particular, the moments of the sum of digits function is an interesting question. Coquet [2] gave the moments of the binary sum of digits function,

$$\sum_{n<x} s_2(n)^2 = \left(\frac{\log_2 x}{2}\right)^2 x + x \log_2 x \nu_1(\log_2 x) + x\nu_2(\log_2 x),$$

where $\nu_1(x)$ and $\nu_2(x)$ are continuous nowhere differentiable functions of period 1. Recently, Grabner, Kirschenhofer, Prodinger and Tichy [4] considered the s-th moment in the binary number system by using Delange's approach and proved that, for a given integer $s \ge 1$,

$$\sum_{n<x} s_2(n)^s = \left(\frac{\log_2 x}{2}\right)^s x + x \sum_{j=1}^{s-1} (\log_2 x)^j \nu_j(\log_2 x),$$

where $\nu_j(x)$ are continuous nowhere differentiable functions of period 1.

We define a Dirichlet convolution for binary digital sum as

$$s_2^{(2)}(n) := s_2 * s_2(n) = \sum_{d|n} s_2(d) s_2(n/d).$$

In this paper, we shall investigate the asymptotic behaviour for

$$\sum_{n \le x} s_2^{(2)}(n), \qquad x > 1. \tag{3}$$

The Trollope–Delange formula (2) is the main ingredient in the proof. Our result is:

Theorem 1.2. *Let $x > 1$, we have*

$$\sum_{0 \le n \le x} s_2^{(2)}(n) \le \frac{x\log^3 x}{24\log^2 2} + \frac{x\log^2 x}{\log^2 2}\left(\frac{1}{8} + F_M(\sqrt{x})\log 2\right)$$
$$+ \frac{x\log x}{\log 2}\left(\frac{1}{4}F\left(\frac{\log x}{\log 2}\right) - \frac{1}{2}G(\sqrt{x}) + \frac{1}{4}F_M(\sqrt{x})\right)$$
$$- \frac{x}{2}\left(\frac{G(\sqrt{x}) + H(\sqrt{x})}{\log 2} - F_M(\sqrt{x})F\left(\frac{\log x}{\log 2}\right) + \frac{G(\sqrt{x})F_M(\sqrt{x})}{2} + F^2\left(\frac{\log x}{\log 2}\right)\right),$$
$$\sum_{0 \le n \le x} s^{(2)}(n) \ge \frac{x\log^3 x}{24\log^2 2} + \frac{x\log^2 x}{8\log^2 2} + \frac{x\log x}{\log 2}\left(\frac{1}{4}F\left(\frac{\log x}{\log 2}\right) - \frac{1}{2}G(\sqrt{x})\right)$$
$$- \frac{x}{2}\left(\frac{G(\sqrt{x}) + H(\sqrt{x})}{\log 2} - F\left(\frac{\log x}{\log 2}\right) + \frac{G(\sqrt{x})}{2} + F^2\left(\frac{\log x}{\log 2}\right)\right),$$

where $F_M(x)$, $G(x)$ and $H(x)$ are continuous nowhere differentiable functions of period 1 that can be expressed explicitly from $F(x)$.

The consequence of Theorem 1.2 is:

Corollary 1.2.1. *As $x \to \infty$, we have*

$$\sum_{0 \le n \le x} s_2^{(2)}(n) = \frac{x\log^3 x}{24\log^2 2} + O\left(x\log^2 x\right).$$

Moreover, we extend Corollary 1.2.1 to the k-th convolution of the binary digital sum, for any integer $k \ge 2$. Namely, for $k \ge 2$, we define the k-th convolution for binary digital sum as

$$s_2^{(k)}(n) := s_2 * s_2^{(k-1)}(n) = \sum_{d|n} s_2(d)s_2^{(k-1)}(n/d)$$

and by using the mathematical induction, we obtain the following result.

Theorem 1.3. *For any integer $k \ge 2$, we have*

$$\sum_{0 \le n \le x} s_2^{(k)}(n) = \frac{a(k)}{24\log^2 2}x\log^{2k-1} x + O\left(x\log^{2k-2} x\right), \tag{4}$$

where

$$a(2) = 1 \text{ and } a(k) = \frac{a(k-1)}{2\log 2}\sum_{j=0}^{2k-3}\binom{2k-3}{j}\frac{(-1)^j}{j+2}.$$

In the proofs of our results, Theorem 1.1 plays an important role. Dirichlet's hyperbolic method and Abel's identity are used to derive Theorem 1.2.

2 Lemmas on the binary digital sum

The following lemmas are used in the proofs of Theorem 1.2 and 1.3.

Lemma 2.1. *As $x \to \infty$, we have*

$$\sum_{n \le x} \frac{s_2(n)}{n} = \frac{\log x}{2\log 2} + \frac{\log^2 x}{4\log 2} + \frac{1}{2}F\left(\frac{\log x}{\log 2}\right) + \frac{1}{2}G\left(x\right), \tag{5}$$

$$\sum_{n \le x} \frac{s_2(n)\log n}{n} = \frac{\log^3 x}{6\log 2} + \frac{\log^2 x}{4\log 2} + \frac{\log x}{2}F\left(\frac{\log x}{2\log 2}\right) + \left(\frac{\log x}{2} - \frac{1}{2}\right)G(x) - \frac{1}{2}H(x), \tag{6}$$

where

$$G(x) = \int_1^x F\left(\frac{\log u}{\log 2}\right)\frac{du}{u}$$

and

$$H(x) = \int_1^x \int_1^u F\left(\frac{\log v}{\log 2}\right)\frac{dvdu}{uv}.$$

Proof. By Abel's identity and (2), we have

$$\begin{aligned}\sum_{n \le x} \frac{s_2(n)}{n} &= \frac{1}{x}\left(\frac{x\log x}{2\log 2} + \frac{x}{2}F\left(\frac{\log x}{\log 2}\right)\right) + \int_1^x \left(\frac{u\log u}{2\log 2} + \frac{u}{2}F\left(\frac{\log u}{\log 2}\right)\right)\frac{du}{u^2} \\ &= \frac{\log x}{2\log 2} + \frac{1}{2}F\left(\frac{\log x}{\log 2}\right) + \frac{\log^2 x}{4\log 2} + \frac{1}{2}G(x).\end{aligned}$$

Using Abel's identity again and (5), we have

$$\begin{aligned}\sum_{n \le x} \frac{s_2(n)\log n}{n} &= \log x\left(\frac{\log x}{2\log 2} + \frac{1}{2}F\left(\frac{\log x}{\log 2}\right) + \frac{\log^2 x}{4\log 2} + \frac{1}{2}G(x)\right) \\ &\quad - \int_1^x \left(\frac{\log u}{2\log 2} + \frac{1}{2}F\left(\frac{\log u}{\log 2}\right) + \frac{\log^2 u}{4\log 2} + \frac{1}{2}G(u)\right)\frac{du}{u} \\ &= \frac{\log^3 x}{6\log 2} + \frac{\log^2 x}{4\log 2} + \frac{\log x}{2}F\left(\frac{\log x}{\log 2}\right) + \left(\frac{\log x}{2} - \frac{1}{2}\right)G(x) - \frac{1}{2}H(x).\end{aligned}$$

□

Lemma 2.2.

$$\sum_{n \le y} \frac{s_2(n)}{n}F\left(\frac{\log(x/n)}{\log 2}\right) \le F_M(y)\sum_{n \le y}\frac{s_2(n)}{n}, \tag{7}$$

where $F_M(y) := \max\{F\left(\frac{\log(x/n)}{\log 2}\right), n \in [1, y]\}$.

Proof. This follows from the boundness of the Takagi function (see [5]) □

Lemma 2.3. *For any positive integer j, we have*

$$\sum_{n \le x} \frac{s_2(n)}{n}\log^j n = \frac{\log^{j+2} x}{2(j+2)\log 2} + O\left(\log^{j+1} x\right).$$

Proof. This follows from using Abel's identity and (5). □

3 Proofs of Theorems

Proof of Theorem 1.2. Let $x > 1$. In view of Dirichlet's hyperbolic method, we have

$$\begin{aligned}\sum_{0\le n\le x} s_2^{(2)}(n) &= \sum_{0\le n<x}\sum_{d\mid n} s_2(d)s_2(n/d)\\ &= 2\sum_{n\le\sqrt{x}} s_2(n)\sum_{m\le x/n} s_2(m) - (\sum_{n\le\sqrt{x}} s_2(n))^2. \end{aligned}\tag{8}$$

Inserting (2) in the right hand of (8), we have

$$\begin{aligned}\sum_{0\le n\le x} s_2^{(2)}(n) &= 2\sum_{n\le\sqrt{x}} s_2(n)\left(\frac{x\log(x/n)}{2n\log 2}+\frac{x}{2n}F\left(\frac{\log(x/n)}{\log 2}\right)\right)-\left(\frac{\sqrt{x}\log x}{4\log 2}+\frac{\sqrt{x}}{2}F\left(\frac{\log x}{\log 2}\right)\right)^2\\ &= \frac{x\log x}{\log 2}\sum_{n\le\sqrt{x}}\frac{s_2(n)}{n}-\frac{x}{\log 2}\sum_{n\le\sqrt{x}}\frac{s_2(n)\log n}{n}+x\sum_{n\le\sqrt{x}}\frac{s_2(n)}{n}F\left(\frac{\log(x/n)}{\log 2}\right)\\ &\quad-\frac{x\log^2 x}{16\log^2 2}-\frac{x\log x}{4\log 2}F\left(\frac{\log x}{\log 2}\right)-\frac{x}{4}F^2\left(\frac{\log x}{\log 2}\right).\end{aligned}$$

We apply (5) and (6) in Lemma 2.1 to the first and second sum, respectively, and Lemma 2.2 to the third sum. This completes Theorem 1.2. □

Proof of Theorem 1.3. We use induction on k. The relation (4) holds for $k = 2$ with $a(2) = 1$. Let us assume that the relation (4) holds for all $k < l$. We have

$$\begin{aligned}\sum_{0\le n\le x} s_2^{(l)}(n) &= \sum_{0\le n<x}\sum_{d\mid n} s_2(d)s_2^{(l-1)}(n/d)\\ &= \sum_{n\le x} s_2(n)\sum_{m\le x/n} s_2^{(l-1)}(m).\end{aligned}\tag{9}$$

Inserting the hypothesis in the right hand of (9), we have

$$\begin{aligned}\sum_{0\le n\le x} s_2^{(l)}(n) &= \sum_{n\le x} s_2(n)\left(\frac{a(l-1)}{24\log^2 2}\frac{x}{n}(\log x-\log n)^{2l-3}+O\left(\frac{x}{n}(\log x-\log n)^{2l-4}\right)\right)\\ &= \frac{a(l-1)x}{24\log^2 2}\sum_{n\le x}\frac{s_2(n)}{n}(\log x-\log n)^{2l-3}+O\left(x\left|\sum_{n\le x}\frac{s_2(n)}{n}(\log x-\log n)^{2l-4}\right|\right).\end{aligned}$$

By Abel's identity and (5), we have

$$\begin{aligned}\sum_{0\le n\le x} s_2^{(l)}(n) &= \frac{a(l-1)x}{24\log^2 2}\sum_{n\le x}\frac{s_2(n)}{n}\sum_{j=0}^{2l-3}\binom{2l-3}{j}(-1)^j\log^{2l-3-j}x\log^j n+O\left(x\log^{2l-2}x\right)\\ &= \frac{a(l-1)x}{24\log^2 2}\log^{2l-3}x\sum_{j=0}^{2l-3}\binom{2l-3}{j}(-1)^j\log^{-j}x\sum_{n\le x}\frac{s_2(n)}{n}\log^j n+O\left(x\log^{2l-2}x\right).\end{aligned}$$

In view of Lemma 2.3, we have

$$\sum_{0\le n\le x} s_2^{(l)}(n) = \frac{a(l-1)x}{24\log^2 2}\log^{2l-3} x\sum_{j=0}^{2l-3}\binom{2l-3}{j}(-1)^j\log^{-j} x\left(\frac{\log^{j+2} x}{2(j+2)\log 2}+O\left(\log^{j+1} x\right)\right)$$
$$+O\left(x\log^{2l-2} x\right)$$
$$=\frac{a(l-1)x}{24\log^2 2}\log^{2l-1} x\sum_{j=0}^{2l-3}\binom{2l-3}{j}\left(\frac{(-1)^j}{2(j+2)\log 2}\right)+O\left(x\log^{2l-2} x\right).$$

This gives

$$a(l)=\frac{a(l-1)}{2\log 2}\sum_{j=0}^{2l-3}\binom{2l-3}{j}\frac{(-1)^j}{j+2}.$$

Thus, the relation (4) holds at rank l, and Theorem 1.3 is proved. □

Acknowledgements

We thank the referees for their time and comments. This work was supported by Thailand Research Fund (Grant No. MRG6080210).

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