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Inequalities between arithmetic functions φ, ψ and σ. Part 1

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Abstract: For three of the basic arithmetic functions φ, ψ and σ are proved the inequalities $\psi(n)^n > \sigma(n)^{\varphi(n)}$ and $\sigma(n)^n < \psi(n)^{\sigma(n)}$ for each natural number $n \geq 2$.
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1 Introduction

One of the most interesting areas of the number theory is related to the arithmetic functions. Some properties of them are discussed in a series of papers of the authors [1–4, 7]. In the paper, two new inequalities will be formulated and proved.

For the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i}, \tag{1}$$

where $k, \alpha_1, ..., \alpha_k, k \geq 1$ are natural numbers and $p_1, ..., p_k$ are different primes, the following arithmetic functions are defined by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1),\ \psi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1},\ \sigma(1) = 1$$

(see, e.g. [5, 6]).

Also, we use the following notation for the above n:

$$\underline{set}(1) = \emptyset,\ \underline{set}(n) = \{p_1, ..., p_k\}.$$

One of the interesting inequalities containing the arithmetic functions φ and ψ is:

$$\psi(n)^{\varphi(n)} < n^n < \varphi(n)^{\psi(n)} \tag{2}$$

(see [2]).

It can be easily seen that the following inequalities

$$\psi(n)^{\varphi(n)} < \sigma(n)^{\varphi(n)} < n^n < \varphi(n)^{\psi(n)} < \varphi(n)^{\sigma(n)} \tag{3}$$

are valid, too.

In the present paper, two new inequalities related to (2) and (3), will be defined and proved, using different methods.

2 Main result

Theorem 1. For each natural number $n \geq 2$:

$$\psi(n)^n > \sigma(n)^{\varphi(n)}. \tag{4}$$

Proof: Let n be a prime number. Then from (4) we obtain:

$$\psi(n)^n - \sigma(n)^{\varphi(n)} = (n+1)^n - (n+1)^{n-1} > 0.$$

Let for the natural number $n \geq 2$ of the form (1), the inequality (4) be valid. Let p be a prime number. For it there are two possibilities.

Case 1. Let $p \notin \underline{set}(n)$. Then,

$$\psi(np)^{np} - \sigma(np)^{\varphi(np)} = (\psi(n)(p+1))^{np} - (\sigma(n)(p+1))^{\varphi(n)(p-1)}$$

$$= \psi(n)^{np}.(p+1)^{np} - \sigma(n)^{\varphi(n)(p-1)}.(p+1)^{\varphi(n)(p-1)}$$

$$= (\psi(n)^n)^p.(p+1)^{np} - (\sigma(n)^{\varphi(n)})^{p-1}.(p+1)^{\varphi(n)(p-1)}$$

(by the induction assumption)

$$= (\psi(n)^n)^p.(p+1)^{np} - (\psi(n)^n)^{p-1}.(p+1)^{\varphi(n)(p-1)} > 0.$$

<u>Case 2.</u> Let $p \in \underline{set}(n)$. Then $n = p^a m$ for the natural numbers $a, m \geq 1$, where $(m, p) = 1$.

First, obviously, for $q \geq 3$

$$q > (1 + \frac{1}{q})^{q-1} > (1 + \frac{1}{q^2})^{q-1}. \tag{5}$$

Therefore, for each prime number $q \geq 3$:

$$q^{2q-1} = q.q^{2q-2} > q^{2q-2}(1 + \frac{1}{q})^{q-1} = (q^2(1 + \frac{1}{q}))^{q-1}$$

Second, we see that for $q \geq 3$ and for $a \geq 1$:

$$q + \frac{1}{q} - \frac{q^{a+2} - 1}{q^{a+1} - 1} = \frac{q^2 + 1}{q} - \frac{q^{a+2} - 1}{q^{a+1} - 1}$$

$$= \frac{1}{q(q^{a+1} - 1)}((q^{a+3} + q^{a+1} - q^2 - 1) - (q^{a+3} + q))$$

$$= \frac{1}{q(q^{a+1} - 1)}(q^{a+1} - q^2 + q - 1) > 0. \tag{6}$$

Third,

$$\sigma(np) = \sigma(mp^{a+1}) = \sigma(m)\frac{p^{a+2} - 1}{p - 1} = \sigma(n)\frac{p^{a+2} - 1}{p^{a+1} - 1}.$$

Now, we obtain sequentially:

$$\psi(np)^{np} - \sigma(np)^{\varphi(np)} = (\psi(n)p)^{np} - \left(\sigma(n)\frac{p^{a+2} - 1}{p^{a+1} - 1}\right)^{\varphi(n)p}$$

$$= \psi(n)^{np}p^{np} - \sigma(n)^{\varphi(n)p}\left(\frac{p^{a+2} - 1}{p^{a+1} - 1}\right)^{\varphi(n)p}$$

(by the induction assumption)

$$> \psi(n)^{np}p^{np} - \psi(n)^{np}\left(\frac{p^{a+2} - 1}{p^{a+1} - 1}\right)^{\varphi(n)p}$$

$$= \psi(n)^{np}\left(p^{np} - \left(\frac{p^{a+2} - 1}{p^{a+1} - 1}\right)^{\varphi(n)p}\right)$$

$$\geq \psi(n)^{np}\left(p^{np} - \left(\frac{p^{a+2} - 1}{p^{a+1} - 1}\right)^{(n-1)p}\right)$$

(from (6))

$$\geq \psi(n)^{np}\left(p^{np} - \left(p + \frac{1}{p}\right)^{(n-1)p}\right)$$

$$= \psi(n)^{np} p^{(n-1)p} \left(p^p - \left(1 + \frac{1}{p^2}\right)^{(n-1)p} \right)$$

(from (5))

$$> 0.$$

This completes the proof. □

Theorem 2. For each natural number $n \geq 2$:

$$\sigma(n)^n < \psi(n)^{\sigma(n)}. \tag{7}$$

Proof: It is well-known that the function $f(x) = x^{\frac{1}{x}}$ is strictly decreasing for $x \geq e$ – Euler's number. As $3 > e$, particularly we get that

$$\sigma(n)^{\frac{1}{\sigma(n)}} \leq \psi(n)^{\frac{1}{\psi(n)}},$$

as $\sigma(n) \geq \psi(n) \geq 3$ for $n \geq 2$. Now, as $\psi(n) > n$ for $n \geq 2$, we get that

$$\sigma(n)^{\frac{1}{\sigma(n)}} < \psi(n)^{\frac{1}{n}},$$

which implies (7). This completes the proof. □

In near future, other inequalities related to φ, ψ, σ and other arithmetic functions will be discussed.

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