Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 25, 2019, No. 1, 48–49
DOI: 10.7546/nntdm.2019.25.1.48-49

New form of the Newton's binomial theorem

Mladen V. Vassilev-Missana

5, V. Hugo Str., Sofia-1124, Bulgaria
e-mail: missana@abv.bg

Received: 20 July 2018 **Accepted:** 14 January 2019

Abstract: A new version of the Newton's binomial theorem has been proposed and proved in the paper.
Keywords: Divisor, Multiplictive function, Prime number, Square-free number.
2010 Mathematics Classification Numbers: 11A25.

Let $n > 1$ be an integer. If $n = \prod_{i=1}^{k} p_i^{\alpha_i}$ is the canonical factorization of n (using primes), then we set

$$k = \omega(n) \tag{1}$$

and by convention

$$\omega(1) = 0. \tag{2}$$

It is a well-known fact (see, e.g. [1]) that for each real number x, function $f(n) = x^{\omega(n)}$ is multiplicative.

Theorem. For an arbitrary square-free number n, the identity

$$(x+1)^{\omega(n)} = \sum_{d/n} x^{\omega(d)} \tag{3}$$

holds, where $\sum_{d/n}$ means the sum over all divisors d of n.

Proof: Let n be an arbitrary positive integer. Since $f(n)$ is multiplicative, then the function

$$F(n) = \sum_{d/n} f(d)$$

is multiplicative, too. But for $m = p^{\alpha}$, where p is a prime number and $\alpha \geq 1$ is an integer, we have

$$F(m) = \sum_{d/m} f(d) = f(1) + f(p) + ... + f(p^{\alpha}) = x^0 + \underbrace{x^1 + ... + x^1}_{\alpha\text{-times}} = 1 + \alpha x.$$

Hence, for

$$n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$$

we have

$$F(n) = \prod_{i=1}^{\omega(n)} F(p_i^{\alpha_i}) = \prod_{i=1}^{\omega(n)} (1 + \alpha_i x). \tag{4}$$

When n is square-free, then $\alpha_i = 1$ for $i = 1, ..., \omega(n)$ and then from (4)

$$F(n) = (x+1)^{\omega(n)}$$

and (3) is proved.

Putting in (3) $x = \frac{a}{b}$, where a, b are real numbers and $b \neq 0$, we obtain

$$(a+b)^{\omega(n)} = \sum_{d/n} a^{\omega(d)} b^{\omega(n)-\omega(d)} = \sum_{d/n} a^{\omega(n)-\omega(d)} b^{\omega(d)}. \tag{5}$$

Equality (5) corresponds to the Newton's Binomial Theorem. For example, for $n = 6$ we have $\omega(6) = 2$ and (5) yields

$$(a+b)^2 = a^2 + 2ab + b^2.$$

For $n = 30$ we have $\omega(30) = 3$ and (5) yields

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

For $n = \prod_{i=1}^{k} p_i$ we have $\omega(n) = k$ and (5) yields

$$(a+b)^k = \sum_{i=0}^{k} \binom{k}{i} a^{k-i} b^i.$$

References

[1] Polya, G., & Szegö, G. (1998) *Problems and Theorems in Analysis. II*, Springer, Berlin, pp. 120–125.