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Improving the estimates for a sequence involving prime numbers

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Abstract: Based on new explicit estimates for the prime counting function, we improve the currently known estimates for the particular sequence $C_n = np_n - \sum_{k \leq n} p_k$, $n \geq 1$, involving the prime numbers.

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1 Introduction

Let p_n denote the nth prime number. In this paper, we consider the sequence $(C_n)_{n \geq 1}$, where

$$C_n = np_n - \sum_{k \leq n} p_k.$$

In [1, Theorem 10], it is proved that the asymptotic formula

$$C_n = \sum_{k=1}^{m-1} (k-1)! \left(1 - \frac{1}{2^k}\right) \frac{p_n^2}{\log^k p_n} + O\left(\frac{p_n^2}{\log^m p_n}\right) \tag{1.1}$$

holds for each positive integer m. By setting $m = 9$ in (1.1), we get

$$C_n = \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \chi(n) + O\left(\frac{p_n^2}{\log^9 p_n}\right), \tag{1.2}$$

where

$$\chi(n) = \frac{45p_n^2}{8\log^4 p_n} + \frac{93p_n^2}{4\log^5 p_n} + \frac{945p_n^2}{8\log^6 p_n} + \frac{5715p_n^2}{8\log^7 p_n} + \frac{80325p_n^2}{16\log^8 p_n}.$$

In the direction of (1.2), the present author [1, Theorems 3 and 4] showed that

$$C_n \geq \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \Theta(n) \tag{1.3}$$

for every integer $n \geq 52\,703\,656$, where

$$\Theta(n) = \frac{43.6p_n^2}{8\log^4 p_n} + \frac{90.9p_n^2}{4\log^5 p_n} + \frac{927.5p_n^2}{8\log^6 p_n} + \frac{5620.5p_n^2}{8\log^7 p_n} + \frac{79075.5p_n^2}{16\log^8 p_n},$$

and that the upper bound

$$C_n \leq \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + \Omega(n) \tag{1.4}$$

holds for every positive integer n, where

$$\Omega(n) = \frac{46.4p_n^2}{8\log^4 p_n} + \frac{95.1p_n^2}{4\log^5 p_n} + \frac{962.5p_n^2}{8\log^6 p_n} + \frac{5809.5p_n^2}{8\log^7 p_n} + \frac{118848p_n^2}{16\log^8 p_n}.$$

Using new explicit estimates for the prime counting function $\pi(x)$, see [2, Propositions 3 and 5], we improve the inequalities (1.3) and (1.4) by showing the following two results.

Theorem 1.1. *For every integer $n \geq 440\,200\,309$, we have*

$$C_n \geq \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + L(n),$$

where

$$L(n) = \frac{44.4p_n^2}{8\log^4 p_n} + \frac{92.1p_n^2}{4\log^5 p_n} + \frac{937.5p_n^2}{8\log^6 p_n} + \frac{5674.5p_n^2}{8\log^7 p_n} + \frac{79789.5p_n^2}{16\log^8 p_n}.$$

Theorem 1.2. *For every positive integer n, we have*

$$C_n \leq \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + U(n),$$

where

$$U(n) = \frac{45.6p_n^2}{8\log^4 p_n} + \frac{93.9p_n^2}{4\log^5 p_n} + \frac{952.5p_n^2}{8\log^6 p_n} + \frac{5755.5p_n^2}{8\log^7 p_n} + \frac{116371p_n^2}{16\log^8 p_n}.$$

2 Preliminaries

Let $\pi(x)$ denote the number of primes not exceeding x. In 1793, Gauss [3] stated a conjecture concerning the asymptotic behaviour of $\pi(x)$, namely

$$\pi(x) \sim \mathrm{li}(x) \qquad (x \to \infty), \tag{2.1}$$

where the *logarithmic integral* $\mathrm{li}(x)$ is defined for every real $x \geq 0$ as

$$\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} = \lim_{\varepsilon \to 0} \left\{ \int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\log t} + \int_{1+\varepsilon}^x \frac{\mathrm{d}t}{\log t} \right\} = \int_2^x \frac{dt}{\log t} + 1.04516\ldots. \tag{2.2}$$

Let m be a positive integer. Using integration by parts, (2.2) implies that

$$\mathrm{li}(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right). \tag{2.3}$$

The asymptotic formula (2.1) was proved independently by Hadamard [4] and de la Vallée-Poussin [5], and is known as the *Prime Number Theorem*. In a later paper, where the existence of a zero-free region for the Riemann zeta-function $\zeta(s)$ to the left of the line $\mathrm{Re}(s) = 1$ was proved, de la Vallée-Poussin [6] also estimated the error term in the Prime Number Theorem by showing

$$\pi(x) = \mathrm{li}(x) + O(xe^{-a\sqrt{\log x}}), \tag{2.4}$$

where a is a positive absolute constant. Applying (2.3) to (2.4), we get

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right). \tag{2.5}$$

3 A proof of Theorem 1.1

In the following proof of Theorem 1.1, we use a new lower bound for $\pi(x)$.

Proof of Theorem 1.1. First, let m be an integer with $m \geq 2$, and let $a_2, \ldots, a_m$, and x_0 be real numbers so that

$$\pi(x) \geq \frac{x}{\log x} + \sum_{k=2}^{m} \frac{a_k x}{\log^k x} \tag{3.1}$$

for every $x \geq x_0$, and let y_0 be a real number such that

$$\mathrm{li}(x) \geq \sum_{j=1}^{m-1} \frac{(j-1)!x}{\log^j x} \tag{3.2}$$

for every $x \geq y_0$. The asymptotic formulae (2.5) and (2.3) guarantee the existence of such parameters. In [1, Theorem 13], it is proved that

$$C_n \geq d_0 + \sum_{k=1}^{m-1} \left(\frac{(k-1)!}{2^k}(1 + 2t_{k-1,1}) \right) \frac{p_n^2}{\log^k p_n} \tag{3.3}$$

for every integer $n \geq \max\{\pi(x_0) + 1, \pi(\sqrt{y_0}) + 1\}$, where $t_{i,j}$ is defined by

$$t_{i,j} = (j-1)! \sum_{l=j}^{i} \frac{2^{l-j} a_{l+1}}{l!} \tag{3.4}$$

and d_0 is given by

$$d_0 = d_0(m, a_2, \ldots, a_m, x_0) = \int_2^{x_0} \pi(x)\,dx - (1 + 2t_{m-1,1})\,\mathrm{li}(x_0^2) + \sum_{k=1}^{m-1} t_{m-1,k} \frac{x_0^2}{\log^k x_0}.$$

Now we choose $m = 9$, $a_2 = 1$, $a_3 = 2$, $a_4 = 5.85$, $a_5 = 23.85$, $a_6 = 119.25$, $a_7 = 715.5$, $a_8 = 5008.5$, $a_9 = 0$, $x_0 = 19\,027\,490\,297$, and $y_0 = 4171$. By [2, Proposition 5], we see that the inequality (3.1) holds for every $x \geq x_0$. By [1, Lemma 15], the inequality (3.2) holds for every $x \geq y_0$. Substituting these values into (3.3), we get

$$C_n \geq d_0 + \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + L(n)$$

for every $n \geq 841\,160\,648$, where $d_0 = d_0(9, 1, 2, 5.85, 23.85, 119.25, 715.5, 5008.5, 0, x_0)$ is given by

$$d_0 = \int_2^{x_0} \pi(x)\,dx - 253.3\,\mathrm{li}(x_0^2) + \frac{126.15x_0^2}{\log x_0} + \frac{62.575x_0^2}{\log^2 x_0} + \frac{61.575x_0^2}{\log^3 x_0} + \frac{89.4375x_0^2}{\log^4 x_0} + \frac{165.95x_0^2}{\log^5 x_0} + \frac{357.75x_0^2}{\log^6 x_0} + \frac{715.5x_0^2}{\log^7 x_0}. \tag{3.5}$$

Hence it suffices to show that $d_0 > 0$. By [1, Lemma 16], we have

$$\mathrm{li}(x) \leq \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \frac{120x}{\log^6 x} + \frac{900x}{\log^7 x} \tag{3.6}$$

for every $x \geq 10^{16}$. Applying (3.6) to (3.5), we get

$$d_0 \geq \int_2^{x_0} \pi(x)\,dx - \frac{x_0^2}{2\log x_0} - \frac{3x_0^2}{4\log^2 x_0} - \frac{7x_0^2}{4\log^3 x_0} - \frac{5.55x_0^2}{\log^4 x_0} - \frac{23.025x_0^2}{\log^5 x_0} - \frac{117.1875x_0^2}{\log^6 x_0} - \frac{1065.515625x_0^2}{\log^7 x_0}.$$

We can show by a straightforward calculation that $d_0 \geq 1.12 \cdot 10^{13} > 0$. For every integer n satisfying $440\,200\,309 \leq n \leq 841\,160\,647$, we check the inequality with a computer. □

4 A proof of Theorem 1.2

Next, we use a recent result concerning an upper bound for the prime counting function $\pi(x)$ to establish the required inequality stated in Theorem 1.2.

Proof of Theorem 1.2. Let m be an integer with $m \geq 2$, let $a_2, \ldots, a_m$, and x_1 be real numbers so that

$$\pi(x) \leq \frac{x}{\log x} + \sum_{k=2}^{m} \frac{a_k x}{\log^k x} \tag{4.1}$$

for every $x \geq x_1$ and let λ and y_1 be real numbers so that

$$\mathrm{li}(x) \leq \sum_{j=1}^{m-2} \frac{(j-1)!x}{\log^j x} + \frac{\lambda x}{\log^{m-1} x} \tag{4.2}$$

for every $x \geq y_1$. Again, the asymptotic formulae (2.5) and (2.3) guarantee the existence of such parameters. In [1, Theorem 14] it is proved that the inequality

$$\begin{aligned} C_n \leq d_1 &+ \sum_{k=1}^{m-2} \left(\frac{(k-1)!}{2^k} (1 + 2t_{k-1,1}) \right) \frac{p_n^2}{\log^k p_n} \\ &+ \left(\frac{(1+2t_{m-1,1})\lambda}{2^{m-1}} - \frac{a_m}{m-1} \right) \frac{p_n^2}{\log^{m-1} p_n} \end{aligned} \tag{4.3}$$

holds for every integer $n \geq \max\{\pi(x_1)+1, \pi(\sqrt{y_1})+1\}$, where $t_{i,j}$ is defined by (3.4) and

$$d_1 = d_1(m, a_2, \ldots, a_m, x_1) = \int_2^{x_1} \pi(x)\, dx - (1 + 2t_{m-1,1}) \operatorname{li}(x_1^2) + \sum_{k=1}^{m-1} t_{m-1,k} \frac{x_1^2}{\log^k x_1}.$$

Next we choose $m = 9$, $a_2 = 1$, $a_3 = 2$, $a_4 = 6.15$, $a_5 = 24.15$, $a_6 = 120.75$, $a_7 = 724.5$, $a_8 = 6601$, $a_9 = 0$, $\lambda = 6300$, $x_1 = 13$, and $y_1 = 10^{18}$. By [2, Proposition 3], we get that the inequality (4.1) holds for every $x \geq x_1$ and by [1, Lemma 19], we see that (4.2) holds for every $y \geq y_1$. By substituting these values into (4.3), we get

$$C_n \leq d_1 + \frac{p_n^2}{2\log p_n} + \frac{3p_n^2}{4\log^2 p_n} + \frac{7p_n^2}{4\log^3 p_n} + U(n) - \frac{0.375p_n^2}{16\log^8 p_n} \tag{4.4}$$

for every integer $n \geq 50\,847\,535$, where $d_1 = d_1(9, 1, 2, 6.15, 24.15, 120.75, 724.5, 6601, 0, x_1)$ is given by

$$\begin{aligned} d_1 = \int_2^{x_1} \pi(x)\, dx - \frac{26599}{90} \operatorname{li}(x_1^2) + \frac{26509x_1^2}{180\log x_1} + \frac{26329x_1^2}{360\log^2 x_1} + \frac{25969x_1^2}{360\log^3 x_1} \\ + \frac{25231x_1^2}{240\log^4 x_1} + \frac{11891x_1^2}{60\log^5 x_1} + \frac{5221x_1^2}{12\log^6 x_1} + \frac{943x_1^2}{\log^7 x_1}. \end{aligned}$$

A direct computation gives $d_1 \leq 453$. We define $f(x) = 0.375x^2/(16\log^8 x) - 453$. Since $f'(x) \geq 0$ for every $x \geq e^4$ and $f(9\,187\,322) > 0$, we get $f(p_n) \geq 0$ for every integer $n \geq \pi(9\,187\,322) + 1 = 614\,124$. Now we can use (4.4) to obtain the desired inequality for every integer $n \geq 50\,847\,535$. Finally, we check the remaining cases with a computer. □

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