Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 4, 45–58
DOI: 10.7546/nntdm.2018.24.4.45-58

A new symmetric endomorphism operator for some generalizations of certain generating functions

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Received: 28 February 2018 **Accepted:** 24 October 2018

Abstract: In this article, we introduce new symmetric endomorphism operators by making use of appropriate infinite product series. The main results show that after direct calculations, the proposed operators are qualified to obtain generating functions for k-Jacobsthal numbers and Tchebychev polynomials of the first and second kind.
Keywords: Symmetric functions, Mersenne numbers, k-Jacobsthal numbers.
2010 Mathematics Subject Classification: 05A15, 05E05, 11B39.

1 Introduction and preliminaries

It is well-known in the theory of orthogonal polynomials that a positive definite inner product generates a unique set of real orthogonal polynomials. When the inner product is not Hermitian, the existence of the corresponding orthogonal polynomials is not guaranteed. A generalization of orthogonal polynomials in the sense that they satisfy $r \in \mathbb{N}$ orthogonality conditions leads to the concept of multiple orthogonal polynomials. These orthogonal polynomials arise naturally in the theory of simultaneous rational approximation, in particular in Hermite–Padé approximation of a system of r (Markov) functions, see [24].

Recently, Baeder et al. [3] used relations and series rearrangement to generalize generating functions for several higher continuous orthogonal polynomials in the Askey scheme, namely the Wilson, continuous dual Hahn, continuous Hahn, and Meixner–Pollaczek polynomials. For different works in the field, readers may refer to [9, 11, 12, 14, 16, 17, 19, 21–23].

In this contribution, we define a new useful operator denoted by $\delta_{e_1e_2}^{-k}$, for which we can formulate, extend and prove new results based on our previous ones [5–7]. In order to determine generating functions for k-Jacobsthal numbers and Mersenne numbers and Tchebychev polynomials of the first and second kind, we combine between our indicated past techniques and these presented polishing approaches.

Definition 1.1 ([1]). *Given two sets of indeterminate A and E (called alphabets), we define $S_j(A-E)$ as follows:*

$$\frac{\Pi_{e\epsilon B}(1-ze)}{\Pi_{a\epsilon A}(1-za)} = \sum_{j=0}^{\infty} S_j(A-E)z^j, \tag{1}$$

with $S_j(A-E) = 0$ for $j < 0$.

Remark 1.1. *By taking $A = 0$ in* (1)*, we obtain*

$$\Pi_{e\epsilon B}(1-ze) = \sum_{j=0}^{\infty} S_j(-E)z^j. \tag{2}$$

Proposition 1.1 ([2]). *Considering successively the case of $A = 0$ or $E = 0$, we can derive the following factorization*

$$\sum_{j=0}^{\infty} S_j(A-E)z^j = \sum_{j=0}^{\infty} S_j(A)z^j \sum_{j=0}^{\infty} S_j(-E)z^j. \tag{3}$$

Thus,

$$S_n(A-E) = \sum_{k=0}^{n} S_{n-k}(A)S_k(-E). \tag{4}$$

The summation is in fact limited to a finite number of nonzero terms. In particular, we have $\Pi_{e\epsilon B}(x-e) = S_n(x-E) = S_0(-E)x^n + S_1(-E)x^{n-1} + S_2(-E)x^{n-2} + \cdots + S_n(-E)$, where $S_j(-E)$ are the coefficients of polynomials $S_n(x-E)$ for $0 < j < n$. We note that $S_j(-E) = 0$ for $j > n$.

For $E = \{e, e, \ldots\}$ (we denote $E = ne$), we have

$$S_n(x-ne) = (x-e)^n.$$

Thus, the special case of $E = \{1, \ldots, 1\}$ gives the two binomial coefficients

$$S_j(-n) = (-1)^j \binom{n}{j} \text{ and } S_j(n) = \binom{n+j-1}{j}. \tag{5}$$

By combining (4) and (5), we obtain the following expression

$$S_j(A-jx) = S_j(A) - \binom{j}{1} S_{j-1}(A)x + \cdots + (-1)^j \binom{j}{j} x^j.$$

Definition 1.2 ([20]). *Given a function g on $\mathbb{R}^n$, the divided difference operator is defined as follows:*

$$\partial_{x_i x_{i+1}}(g) = \frac{g(x_1,\ldots,x_i,x_{i+1},\ldots,x_n) - g(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)}{x_i - x_{i+1}}.$$

Definition 1.3 ([4]). *The symmetrizing operator $\delta^k_{e_1e_2}$ is defined by*

$$\delta^k_{e_1e_2}(e_1^j) = \frac{e_1^{k+j} - e_2^{k+j}}{e_1 - e_2} = S_{k+j-1}(e_1+e_2), \text{ for all } k,j \in \mathbb{N}.$$

Definition 1.4 ([5]). *The symmetrizing operator $\delta^{-k}_{e_1e_2}$ is defined by*

$$\delta^{-k}_{e_1e_2} f = \frac{e_2^k f(x) - e_1^k f(y)}{(e_1e_2)^k (e_1 - e_2)}, \text{ for all } k \in \mathbb{N}.$$

2 Main results

Lemma 2.1 ([5]). *Let $E = \{e_1, e_2\}$, we define the operator $\delta^{-k}_{e_1e_2}$ as follows:*

$$\delta^{-k}_{e_1e_2} f\,(e_1) = \frac{-S_{k-1}(E)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1e_2} f(e_1), \text{ for all } k \in \mathbb{N}.$$

Theorem 2.1. *Given an alphabet $E = \{e_1, e_2\}$, two sequences $\sum_{j=0}^{+\infty} a_j z^j$ and $\sum_{j=0}^{+\infty} b_j z^j$ such that $\left(\sum_{j=0}^{+\infty} a_j z^j\right)\left(\sum_{j=0}^{+\infty} b_j z^j\right) = 1$, then*

$$\frac{\sum\limits_{j=0}^{\infty} b_j\ \delta^k_{e_1e_2}(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} = \sum_{j=0}^{k-1} a_j e_1^j e_2^j \delta^{k-j}_{e_1e_2}(e_1) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} a_{j+k+1} \delta^{k+1}_{e_1e_2}(e_1) z^j. \quad (6)$$

Proof. Let $\sum_{j=0}^{\infty} a_j z^j$ and $\sum_{j=0}^{\infty} b_j z^j$ be two sequences such that $\sum_{j=0}^{\infty} a_j z^j \sum_{j=0}^{\infty} b_j z^j = 1$. The left-hand side of (6) can be written as

$$\frac{-S_{k-1}(E)}{e_1^k e_2^k} f(e_1) + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1e_2} f(e_1) = \frac{-S_{k-1}(E)}{e_1^k e_2^k} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j} + \frac{e_1^k}{e_1^k e_2^k} \partial_{e_1e_2} \frac{1}{\sum\limits_{j=0}^{\infty} b_j e_1^j z^j}$$

$$
\begin{aligned}
&= \frac{1}{e_1^k e_2^k}\left(\frac{-\sum\limits_{j=0}^{\infty} b_j e_2^j S_{k-1}(E) z^j + \frac{e_1^k}{e_1-e_2}\sum\limits_{j=0}^{\infty} b_j\left(e_2^j - e_1^j\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}\right)\\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum\limits_{j=0}^{\infty} b_j\left(e_2^j \frac{e_1^k-e_2^k}{e_1-e_2} + e_1^k\frac{e_1^j-e_2^j}{e_1-e_2}\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}\right)\\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum\limits_{j=0}^{\infty} b_j\left(\frac{e_1^{k+j}-e_2^{k+j}}{e_1-e_2}\right) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}\right)\\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum\limits_{j=0}^{\infty} b_j\ \delta_{e_1e_2}^k(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)}\right).
\end{aligned}
$$

The right-hand side can be expressed as

$$
\begin{aligned}
\delta_{e_1e_2}^{-k} f(e_1) &= \delta_{e_1e_2}^{-k}\left(\sum_{j=0}^{\infty} a_j e_1^j z^j\right)\\
&= \frac{e_2^k \sum\limits_{j=0}^{\infty} a_j e_1^j z^j - e_1^k \sum\limits_{j=0}^{\infty} a_j e_2^j z^j}{e_1^k e_2^k (e_1 - e_2)}\\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{j=0}^{\infty} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j\right)\\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{j=0}^{k-1} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j + \sum_{j=k+1}^{\infty} a_j \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j\right)\\
&= \frac{-1}{e_1^k e_2^k}\left(\sum_{j=0}^{k-1} a_j e_1^j e_2^j \delta_{e_1e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1}\sum_{j=0}^{\infty} a_{j+k+1}\delta_{e_1e_2}^{k+1}(e_1) z^j\right),
\end{aligned}
$$

therefore

$$
\frac{\sum\limits_{j=0}^{\infty} b_j\ \delta_{e_1e_2}^k(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} b_j e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} b_j e_2^j z^j\right)} = \sum_{j=0}^{k-1} a_j e_1^j e_2^j \delta_{e_1e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1}\sum_{j=0}^{\infty} a_{j+k+1}\delta_{e_1e_2}^{k+1}(e_1) z^j.
$$

□

Theorem 2.2. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, \ldots, a_n\}$, we have*

$$\frac{\sum\limits_{j=0}^{\infty} S_j(-A)\, \delta_{e_1e_2}^{k}(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_2^j z^j\right)} = \sum_{j=0}^{k-1} S_j(A)\, e_1^j e_2^j \delta_{e_1e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} S_{j+k+1}(A)\, \delta_{e_1e_2}^{k+1}(e_1) z^j, \tag{7}$$

for all $k \in \mathbb{N}$.

Proof. The action of the operator $\delta_{e_1e_2}^{-k}$ on the series $f(e_1) = \left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^j z^j\right)^{-1}$ gives the left member of equality (7), then

$$\begin{aligned}
\delta_{e_1e_2}^{-k} f(e_1) &= \delta_{e_1e_2}^{-k}\left(\frac{1}{\sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^j z^j}\right)\\
&= \frac{\frac{e_2^k}{\sum_{j=0}^{\infty} S_j(-A) e_1^j z^j} - \frac{e_1^k}{\sum_{j=0}^{\infty} S_j(-A) e_2^j z^j}}{e_1^k e_2^k (e_1 - e_2)}\\
&= \frac{\sum\limits_{j=0}^{\infty} S_j(-A)\, e_2^{j+k} z^j - \sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^{j+k} z^j}{e_1^k e_2^k (e_1 - e_2)\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_2^j z^j\right)}\\
&= \frac{-1}{e_1^k e_2^k}\left(\frac{\sum\limits_{j=0}^{\infty} S_j(-A)\, \delta_{e_1e_2}^{k}(e_1^j) z^j}{\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_1^j z^j\right)\left(\sum\limits_{j=0}^{\infty} S_j(-A)\, e_2^j z^j\right)}\right).
\end{aligned}$$

The second member of the formula (7) is written as:

$$\begin{aligned}
\delta_{e_1e_2}^{-k} f(e_1) &= \delta_{e_1e_2}^{-k}\left(\sum_{j=0}^{\infty} S_j(A)\, e_1^j z^j\right)\\
&= \frac{e_2^k \sum\limits_{j=0}^{\infty} S_j(A)\, e_1^j z^j - e_1^k \sum\limits_{j=0}^{\infty} S_j(A)\, e_2^j z^j}{e_1^k e_2^k (e_1 - e_2)}\\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{j=0}^{\infty} S_j(A)\, \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j\right)\\
&= \frac{1}{e_1^k e_2^k}\left(\sum_{j=0}^{k-1} S_j(A)\, \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j + \sum_{j=k+1}^{\infty} S_j(A)\, \frac{e_2^k e_1^j - e_1^k e_2^j}{e_1 - e_2} z^j\right)
\end{aligned}$$

$$= \frac{-1}{e_1^k e_2^k}\left(\sum_{j=0}^{k-1} S_j\left(A\right) e_1^j e_2^j \delta_{e_1 e_2}^{k-j}(e_1) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} S_{j+k+1}\left(A\right) \delta_{e_1 e_2}^{k+1} z^j\right).$$

The two quantities are equal. □

3 Applications

3.1 Applications of Theorem 2.1

3.1.1 The case of $\frac{1}{1-z} = \sum_{j=0}^{\infty} z^j$

Corollary 3.1. *Given an alphabet $E = \{e_1, e_2\}$, we have*

$$\frac{S_{k-1}\left(E\right) - S_k\left(E\right)}{\left(1-e_1 z\right)\left(1-e_2 z\right)} = \sum_{j=0}^{k-1} e_1^j e_2^j S_{k-j-1}\left(E\right) z^j - e_1^k e_2^k z^{k+1} \sum_{j=0}^{\infty} S_j\left(E\right) z^j, \textit{ for all } k \in \mathbb{N}. \tag{8}$$

For $\mathbf{k} = \mathbf{1}$ in (8), we have

$$\sum_{j=0}^{\infty} S_j\left(E\right) z^j = \frac{1}{\left(1-e_1 z\right)\left(1-e_2 z\right)} = \frac{1}{1-(e_1+e_2)z + e_1 e_2 z^2}. \tag{9}$$

We replace e_2 by $(-e_2)$ in (9), we obtain

$$\sum_{j=0}^{\infty} S_j(e_1 + [-e_2]) z^j = \frac{1}{1-(e_1-e_2)z - e_1 e_2 z^2}. \tag{10}$$

Setting $\begin{cases} e_1 e_2 = 2 \\ e_1 - e_2 = k \end{cases}$ in (10), and this gives

$$\sum_{j=0}^{\infty} S_{j-1}(e_1 + [-e_2]) z^j = \frac{z}{1-kz-2z^2} = \sum_{j=0}^{\infty} J_{k,j} z^j, \tag{11}$$

which represents a generating function for k-Jacobsthal numbers, such that

$$J_{k,j} = S_{j-1}(e_1 + [-e_2]),$$

with $e_{1,2} = \frac{k \pm \sqrt{k^2+8}}{2}$.

Again, we put $\begin{cases} e_1 e_2 = -2 \\ e_1 - e_2 = 3 \end{cases}$ in (10), we get

$$\sum_{j=0}^{\infty} S_{j-1}(e_1 + [-e_2]) z^j = \frac{z}{1-3z+2z^2}, \tag{12}$$

which represents a generating function for Mersenne numbers, such that $M_j = S_{j-1}(e_1 + [-e_2])$, with $e_1 = 2,\ e_2 = 1$.

Theorem 3.1. *We have the following new generating function of the k-Jacobsthal numbers at negative indices as*

$$\sum_{j=0}^{\infty} J_{k,-j} z^j = \frac{z}{1+kz-2z^2}.$$

Theorem 3.2. *We have the following new generating function of the Mersenne numbers at negative indices as*

$$\sum_{j=0}^{\infty} M_{-j} z^j = \frac{z}{1+3z+2z^2}.$$

Furthermore, we replace e_1 by $2e_1$ and e_2 by $(-2e_2)$ in (9), and under the condition $4e_1e_2 = -1$, we obtain

$$\sum_{j=0}^{+\infty} S_j(2e_1 + [-2e_2]) z^j = \frac{1}{1-2(e_1-e_2)z+z^2}, \tag{13}$$

which represents a generating function for Tchebychev polynomials of the second kind given by Boussayoud et al. [4, 8], such that $U_j = S_j(2e_1 + [-2e_2])$. Moreover, from formula (13), we can deduce that

$$\sum_{j=0}^{+\infty} \left[S_j(2e_1 + [-2e_2]) - (e_1 - e_2) S_{j-1}(2e_1 + [-2e_2])\right] z^j = \frac{1-(e_1-e_2)z}{1-2(e_1-e_2)z+z^2},$$

which represents a generating function for Tchebychev polynomials of the first kind given in [4, 7], such that

$$T_j(e_1 - e_2) = \left[S_j(2e_1 + [-2e_2]) - (e_1 - e_2) S_{j-1}(2e_1 + [-2e_2])\right].$$

For $\mathbf{k} = \mathbf{2}$ in (8), we have

$$\sum_{j=0}^{\infty} S_{j+1}(E) z^j = \frac{e_1 + e_2 - e_1e_2z}{(1-ze_1)(1-ze_2)} = \frac{e_1 + e_2 - e_1e_2z}{1-(e_1+e_2)z+e_1e_2z^2}. \tag{14}$$

Also, by replacing e_2 with $(-e_2)$ in (14), we obtain

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2]) z^j = \frac{(e_1 - e_2) + e_1e_2z}{1-(e_1-e_2)z-e_1e_2z^2}. \tag{15}$$

Putting $\begin{cases} e_1e_2 = 2 \\ e_1 - e_2 = k \end{cases}$ in (15), it gives

$$\sum_{j=0}^{\infty} S_{j+1}(e_1 + [-e_2]) z^j = \frac{k+2z}{1-kz-2z^2}, \tag{16}$$

which represents a new generating function.

Substituting $\left\{\begin{array}{l} e_1e_2 = 1 \\ e_1 - e_2 = k \end{array}\right.$ in (15), yields

$$\sum_{j=0}^{\infty} S_{j+1}\left(e_1 + [-e_2]\right) z^j = \frac{k+z}{1-kz-z^2}, \tag{17}$$

which represents a new generating function.

Multiplying the equation (10) by (2) and subtracting it from (11) by (k), we obtain

$$\sum_{j=0}^{\infty} \left(2S_j(e_1 + [-e_2]) - kS_{j-1}(e_1 + [-e_2])\right) z^j = \frac{2-kz}{1-kz-2z^2},$$

which represents a generating function for k-Jacobsthal–Lucas numbers such that

$$j_{k,j} = 2S_j(e_1 + [-e_2]) - kS_{j-1}(e_1 + [-e_2]).$$

Theorem 3.3. *We have the following new generating function of the k-Jacobsthal numbers at negative indices as*

$$\sum_{j=0}^{\infty} j_{k,-j} z^j = \frac{2+kz}{1+kz-2z^2}.$$

3.2 Applications of Theorem 2.2

In this section, we attempt to give results for some well-known generating functions. In fact, we will use Theorem 2.2 to derive k-Jacobsthal numbers and Tchebychev polynomials of second kind. If $k = 1$, the next Corollary gives a new generating function.

Corollary 3.2. *Given two alphabets $E = \{e_1, e_2\}$ and $A = \{a_1, a_2, \}$, we have*

$$\begin{aligned}&\sum_{j=0}^{\infty} S_{j+2}(a_1 + a_2)S_j(e_1 + e_2)z^j \\ &= \frac{e_1e_2a_1^2a_2^2z^2 - a_1a_2(e_1+e_2)(a_1+a_2)z + (a_1+a_2)^2 - a_1a_2}{\prod\limits_{a\in A_2}(1-ae_1z)\prod\limits_{a\in A_2}(1-ae_2z)}.\end{aligned} \tag{18}$$

3.2.1 The case $E = \{1, y\}$, $A = \{1, x\}$

Substituting $e_1 = a_1 = 1$, $e_2 = x$ and $a_2 = y$ in (18), we obtain

$$\sum_{j=0}^{\infty} S_{j+2}(1+x)S_j(1+y)z^j = \frac{xy^2z^2 - x(1+x)(1+y)z + (1+x)^2 - x}{(1-z)(1-xz)(1-yz)(1-xyz)}, \tag{19}$$

which represents a new generating function.

Proposition 3.1. *For all $j \in \mathbb{N}$, we have*

$$S_{j+2}(1+x) = S_j(1+x) + x(x+1)S_j(x).$$

Proof. From (19), we have

$$\begin{aligned}\sum_{j=0}^{\infty} S_{j+2}(1+x)S_j(1+y)z^j &= \frac{xy^2z^2 - x(1+x)(1+y)z + (1+x)^2 - x}{(1-z)(1-zx)(1-zy)(1-zxy)} \\ &= \frac{1-xyz^2}{(1-z)(1-zx)(1-zy)(1-zxy)} + \frac{x(1+x)}{(1-zx)(1-zxy)},\end{aligned}$$

after that, as in [7, 18], we have $\sum_{j=0}^{\infty} S_j(1+x)S_j(1+y)z^j = \frac{1-xyz^2}{(1-z)(1-zx)(1-zy)(1-zxy)}$, then

$$\sum_{j=0}^{\infty} S_{j+2}(1+x)S_j(1+y)z^j = \sum_{j=0}^{\infty} S_j(1+x)S_j(1+y)z^j + \sum_{j=0}^{\infty} x(1+x)S_j(x)S_j(1+y)z^j,$$

therefore

$$S_{j+2}(1+x) = S_j(1+x) + x(x+1)S_j(x).$$

□

3.2.2 The case $E = \{e_1, -e_2\}$, $A = \{a_1, -a_2\}$

Replacing e_2 by $(-e_2)$ and a_2 by $(-a_2)$ in (18) yields

$$\begin{aligned}&\sum_{j=0}^{\infty} S_{j+1}(a_1 + [-a_2])S_{j-1}(e_1 + [-e_2])z^j \\ &= \frac{-e_1e_2a_1^2a_2^2z^3 + a_1a_2(e_1-e_2)(a_1-a_2)z^2 + ((a_1-a_2)^2 + a_1a_2)z}{(1-a_1e_1z)(1+a_2e_1z)(1+a_1e_2z)(1-a_2e_2z)}.\end{aligned} \tag{20}$$

This case consists of three related parts.

Firstly, the substitutions of

$$\begin{cases} a_1 - a_2 = k, \\ a_1a_2 = 2, \end{cases} \quad \text{and} \quad \begin{cases} e_1 - e_2 = k, \\ e_1e_2 = 2, \end{cases}$$

in (20) give

$$\begin{aligned}&\sum_{j=0}^{\infty} S_{j+1}(a_1 + [-a_2])S_{j-1}(e_1 + [-e_2])z^j \\ &= \frac{(k^2+2)z + 2k^2z^2 - 8z^3}{1 - k^2z - 4(k^2+2)z^2 - -4k^2z^3 + 16z^4} \\ &= \sum_{j=0}^{\infty} J_{k,j+2}J_{k,j}z^j,\end{aligned} \tag{21}$$

which represents a new generating function of k-Jacobsthal numbers $J_{k,j}$, such that

$$J_{k,j+2}J_{k,j} = S_{j+1}(a_1 + [-a_2])S_{j-1}(e_1 + [-e_2]).$$

We have the following theorem.

Theorem 3.4. *For $n, k \in \mathbb{N}$, the new generating function of the product of k-Jacobsthal numbers is given by*

$$\sum_{j=0}^{\infty} J_{k,j+1}J_{k,j}z^j = \frac{kz+2kz^2}{1-k^2z-4(k^2+2)z^2--4k^2z^3+16z^4}.$$

Proof. We have

$$\begin{aligned}\sum_{j=0}^{\infty} J_{k,j+2}J_{k,j}z^j &= \sum_{j=0}^{\infty}(kJ_{k,j+1}+2J_{k,j})J_{k,j}z^j \\ &= k\sum_{j=0}^{\infty} J_{k,j+1}J_{k,j}z^j + 2\sum_{j=0}^{\infty} J_{k,j}^2 z^j.\end{aligned}$$

since

$$\sum_{j=0}^{\infty} J_{k,j}^2 z^j = \frac{z-4z^3}{1-k^2z-4(k^2+2)z^2--4k^2z^3+16z^4}, \text{ (see [10])}$$

we have

$$\begin{aligned}\sum_{j=0}^{\infty} J_{k,j+2}J_{k,j}z^j = k&\frac{kz+2kz^2}{1-k^2z-4(k^2+2)z^2--4k^2z^3+16z^4} \\ &+ \frac{2z-8z^3}{1-k^2z-4(k^2+2)z^2--4k^2z^3+16z^4},\end{aligned}$$

therefore

$$\sum_{j=0}^{\infty} J_{k,j+1}J_{k,j}z^j = \frac{kz+2kz^2}{1-k^2z-4(k^2+2)z^2--4k^2z^3+16z^4}.$$

Thus, this completes the proof. □

Secondly, by making the following restrictions: $e_1 - e_2 = 1$, $e_1e_2 = 1$, $4a_1a_2 = -1$, and by replacing $(a_1 - a_2)$ by $2(a_1 - a_2)$ in (20), we get a new generating function, involving the product of k-Jacobsthal numbers with Tchebychev polynomial of second kind as follows:

$$\begin{aligned}\sum_{j=0}^{\infty} S_{j+1}(2a_1+[-2a_2])S_{j-1}(e_1+[-e_2])z^j &= \frac{(4(a_1-a_2)^2-1)z-2k(a_1-a_2)z^2-2z^3}{P_{UJ}} \\ &= \sum_{j=0}^{\infty} U_{j+1}(a_1-a_2)J_{k,j}z^j,\end{aligned}$$

with $P_{UJ} = 1 - 2k\left(a_1 - a_2\right)z + (8\left(a_1 - a_2\right)^2 - k^2 - 4)z^2 + 4k\left(a_1 - a_2\right)z^3 + 4z^4$.

Thirdly, choose a_i and e_i $(i = 1, 2)$ such that $4e_1e_2 = -1$, $4a_1a_2 = -1$, and by replacing $(a_1 - a_2)$ with $2(a_1 - a_2)$, and $(e_1 - e_2)$ with $2(e_1 - e_2)$ in (20), we get a new generating function, involving the square of Tchebychev polynomials of second kind given by

$$\begin{aligned}\sum_{j=0}^{\infty} S_{j+2}(2a_1+[-2a_2])S_j(2e_1+[-2e_2])z^j &= \frac{z^2-1+4(a_1-a_2)^2-4(a_1-a_2)(e_1-e_2)z}{P_{UU}} \\ &= \sum_{j=0}^{\infty} U_{j+2}(a_1-a_2)U_j(e_1-e_2)z^j,\end{aligned}$$

with $P_{UU} = 1-4(e_1-e_2)(a_1-a_2)z+(4(a_1-a_2)^2+4(e_1-e_2)^2-2)z^2-4(e_1-e_2)(a_1-a_2)z^3+z^4$. Thus, we get the following theorem.

Theorem 3.5. *We have a new generating function of the product of Tchebychev polynomials of second kind as*

$$\sum_{j=0}^{\infty} U_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j = \frac{2(a_1-a_2)-2(e_1-e_2)z}{P_{UU}}.$$

Proof. We have,

$$\sum_{j=0}^{\infty} U_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j = \sum_{j=0}^{\infty}[2(a_1-a_2)U_{j+1}(a_1-a_2)-U_j(a_1-a_2)]U_j(e_1-e_2)z^j$$
$$= 2(a_1-a_2)\sum_{j=0}^{\infty} U_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j - \sum_{j=0}^{\infty} U_j(a_1-a_2)U_j(e_1-e_2)z^j.$$

Since

$$\sum_{j=0}^{\infty} U_j(a_1-a_2)U_j(e_1-e_2)z^j = \frac{1-z^2}{P_{UU}} \quad \text{(see [6,7])}$$

so

$$\sum_{j=0}^{\infty} U_{j+2}(a_1-a_2)U_j(e_1-e_2)z^j = 2(a_1-a_2)\frac{2(a_1-a_2)-2(e_1-e_2)z}{P_{UU}} - \frac{1-z^2}{P_{UU}},$$

therefore,

$$\sum_{j=0}^{\infty} U_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j = \frac{2(a_1-a_2)-2(e_1-e_2)z}{P_{UU}}.$$

This completes the proof. □

Finally, according the formulas (9) and (13) in [4,7], and to the fact that

$$S_{j+1}(2a_1+[-2a_2]) = \frac{(2a_1)^{j+2}-(-2a_2)^{j+2}}{2a_1+2a_2}$$

we have

$$\sum_{j=0}^{\infty} T_{j+2}(a_1-a_2)U_j(e_1-e_2)z^j = \frac{-1+2(a_1-a_2)^2-2(a_1-a_2)(e_1-e_2)z+z^2}{P_{TU}},$$

with $P_{TU} = 1-4(e_1-e_2)(a_1-a_2)z+(4(a_1-a_2)^2+4(e_1-e_2)^2-2)z^2-4(e_1-e_2)(a_1-a_2)z^3+z^4$, which represents a new generating function for the combined Tchebychev polynomials of the second and first kinds.

On the other hand, we have

$$\sum_{j=0}^{\infty} T_{j+2}(a_1-a_2)U_j(e_1-e_2)z^j = \sum_{j=0}^{\infty}[2(a_1-a_2)T_{j+1}(a_1-a_2)-T_j(a_1-a_2)]U_j(e_1-e_2)z^j$$
$$= 2(a_1-a_2)\sum_{j=0}^{\infty} T_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j$$
$$-\sum_{j=0}^{\infty} T_j(a_1-a_2)U_j(e_1-e_2)z^j.$$

Further, the identity [6, 7]: $\sum\limits_{j=0}^{\infty} T_j(a_1-a_2)U_j(e_1-e_2)z^j = \frac{1-2(a_1-a_2)(e_1-e_2)z+(2(a_1-a_2)^2-1)z^2}{P_{TU}}$, then

$$\sum_{j=0}^{\infty} T_{j+2}(a_1-a_2)U_j(e_1-e_2)z^j = \frac{2(a_1-a_2)\left[(a_1-a_2)(1+z^2)-2(e_1-e_2)z\right]}{P_{TU}} - \frac{1-2(a_1-a_2)(e_1-e_2)z+(2(a_1-a_2)^2-1)z^2}{P_{TU}}.$$

We get a new generating function defined as follows:

$$\sum_{j=0}^{\infty} T_{j+1}(a_1-a_2)U_j(e_1-e_2)z^j = \frac{(a_1-a_2)-2(e_1-e_2)z+(a_1-a_2)z^2}{P_{TU}}.$$

4 Conclusion

This research proposes new developments to determine generating functions. The suggested techniques are based on symmetric functions. The main results are consistent with some ideas, obtained in other previous works [5, 7, 8, 13]. The first results of this work seem promising, but further investigations in the field should be continued. Future research can be done around the extension of alphabet E elements and the treats of k parameter values.

Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions.

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