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Enumeration of 3- and 4-Wilf classes of four 4-letter patterns

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Abstract: Let S_n be the symmetric group of all permutations of n letters. We show that there are precisely 27 (respectively, 15) Wilf classes consisting of exactly 3 (respectively, 4) symmetry classes of subsets of four 4-letter patterns.
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1 Introduction

This paper is a sequel to [2] (also, see [1]) and continues the investigaton of permutations avoiding a quadruple of (distinct) 4-letter patterns. In [2] we determined all 64 Wilf classes consisting of exactly 2 symmetry classes of quadruples of 4-letter patterns. Following the same terminology and notation as in that paper, here we establish the following results.

Theorem 1. *The number of Wilf classes consisting of exactly* 3 *symmetry classes of subsets of* 4 *patterns in* S_4 (3-*Wilf classes*) *is* 27.

Theorem 2. *The number of Wilf classes consisting of exactly* 4 *symmetry classes of subsets of* 4 *patterns in* S_4 (4-*Wilf classes*) *is* 15.

For Theorem 1, a perusal of the counting sequences $(|S_n(T)|)_{n=1,\dots,16}$ for a representative quadruple T in each symmetry class of 4 patterns in S_4 shows that there are at most 27 3-Wilf classes of subsets of 4 patterns in S_4, see Table 1 in the appendix below. We used the insertion encoding algorithm (INSENC) [7] on the symmetry classes in Table 1 and successful outcomes, always a rational generating function, are referenced by "INSENC". To prove Theorem 1, we find in Section 2 an explicit formula for the generating function $F_T(x) = \sum_{n\geq 0} |S_n(T)|x^n$ for each T that appears in one of the 27 candidate triples, whenever $F_T(x)$ is nonrational. The 13 cases where $F_T(x)$ is rational and INSENC fails are marked "EX" for Exercise in Table 1, and their proofs are omitted.

The analogous results for Theorem 2 are listed in Table 2 and we include a selection of proofs in Section 3.

2 Proof of Theorem 1

2.1 Case 656

The enumeration of the first two symmetry classes is obtained from [4]. Thus it remains to enumerate the last class $T = \{2341, 2314, 1243, 1234\}$. Define $a_n(i_1, i_2, \dots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$. Define $A_n(v) = \sum_{i=1}^{n} a_n(i)v^{i-1}$ and $A(x, v) = \sum_{n\geq 0} A_n(v)x^n$. To find an explicit formula for $A(x, v)$, we define $a_n^+(i) = \sum_{j=i+1}^{n} a_n(i, j)$, $A_n^+(v) = \sum_{i=1}^{n-1} a_n^+(i)v^{i-1}$ and $A^+(x, v) = \sum_{n\geq 2} A_n^+(v)x^n$.

Lemma 3. *We have*

$$A^+(x, v) = \frac{x^3}{1-2x} + x\big(A(x, v) - 1\big).$$

Proof. Since we avoid 2314 and 2341 we have $a_n(i, j) = 0$ for all $1 \leq i < j \leq n-1$ such that $(i, j) \neq (1, n-1)$. We have $|S_n(123, 132)| = 2^{n-1}$ [6], so $a_n(1, n-1) = 2^{n-3}$. Moreover, from the definitions, $a_n(i, n) = a_{n-1}(i)$. Therefore,

$$a_n^+(i) = 2^{n-3}\delta_{i=1} + a_{n-1}(i).$$

Multiplying by v^{i-1} and summing over $i = 1, 2, \dots, n$, we obtain for $n \geq 3$,

$$A_n^+(v) = 2^{n-3} + A_{n-1}(v)$$

with $A_2^+(v) = 1$, and the result follows by summing over $n \geq 2$. $\square$

Similarly, we define $a_n^-(i) = \sum_{j=1}^{i-1} a_n(i, j)$, $A_n^-(v) = \sum_{i=1}^{n} a_n^-(i)v^{i-1}$ and $A^-(x, v) = \sum_{n\geq 2} A_n^-(v)x^n$.

Lemma 4. *With $v = 1/C(x)$, we have*

$$A^-(x, 1) = (v^2x - v^2 + vx - x + 1)A(x, 1) - \frac{v^3x - v^2x^2 - v^3 + 2v^2x - x^3 + v - 2x}{v - 2x}.$$

Proof. For $2 \leq j < i \leq n-2$, if $\pi = ij\pi' \in S_n(T)$, then the leftmost letter of π' is either smaller than j or equal to n (the maximal letter), and so

$$a_n(i,j) = \sum_{k=1}^{j-1} a_n(i,j,k) + a_n(i,j,k).$$

Note that $\pi = ij(j-1)\pi'$ avoids T if and only if $(i-1)(j-1)\pi'$ avoids T, and for $k \leq j-2$, $\pi = ijk\pi'$ avoids T if and only if $jk\pi'$ avoids T, and $\pi = ijn\pi'$ avoids T if and only if $ij\pi'$ avoids T. Hence, for $2 \leq j < i \leq n-2$,

$$a_n(i,j) = \sum_{k=1}^{j-2} a_{n-1}(j,k) + a_{n-1}(i-1,j-1) + a_{n-1}(i,j),$$

which is equivalent to

$$a_n(i,j) = a^-_{n-1}(j) - a_{n-2}(j-1) + a_{n-1}(i-1,j-1) + a_{n-1}(i,j).$$

Summing over $j = 2,3,\ldots,i-2$ and using the fact $a_n(i,i-1) = a_{n-1}(i-1)$, we have

$$\begin{aligned}
&a^-_n(i) - a_n(i,1) - a_{n-1}(i-1) \\
&\quad = \sum_{j=2}^{i-2} a^-_{n-1}(j) - \sum_{j=1}^{i-3} a_{n-2}(j) \\
&\quad + a^-_{n-1}(i-1) - a_{n-2}(i-2) + a^-_{n-1}(i) - a_{n-1}(i,1) - a_{n-2}(i-1).
\end{aligned}$$

Note that $a^-_n(1) = 0$, $a^-_n(n) = a_{n-1}$ and $a^-_n(n-1) = a_{n-1} - a_{n-2}$. Also, $a_n(i,1) = 2^{i-2}$ since $|S_n(123,132)| = 2^{n-1}$. Hence,

$$a^-_n(i) = \sum_{j=1}^{i} a^-_{n-1}(j) - \sum_{j=1}^{i-1} a_{n-2}(j) + a_{n-1}(i-1)$$

with $a^-_n(n) = a_{n-1}$ and $a^-_n(n-1) = a_{n-1} - a_{n-2}$.

Multiplying the last recurrence by v^{i-1} and summing over $i = 1,2,\ldots,n-2$, we obtain

$$\begin{aligned}
A^-_n(v) &= \frac{1}{1-v}(A^-_{n-1}(v) - v^{n-2}A^-_{n-1}(1)) - \frac{v}{1-v}(A_{n-2}(v) - v^{n-1}A_{n-2}(1)) \\
&+ vA_{n-1}(v) + v^{n-2}(1+v)A_{n-1}(1) - v^{n-2}A_{n-2}(1),
\end{aligned}$$

with $A^-_2(v) = v$. Hence, multiplying by x^n and summing over $n \geq 3$, we get

$$\begin{aligned}
A^-(x,v) &= \frac{x}{1-v}(A^-(x,v) - \frac{1}{v}A^-(vx,1)) - \frac{vx^2}{1-v}(A(x,v) - vA(vx,1)) \\
&+ vx(A(x,v)-1) + x(A(vx,1)-1) + \frac{x}{v}(A(vx,1)-1) - x^2A(vx,1). \qquad (1)
\end{aligned}$$

By Lemma 3, we have

$$A(x,v) = 1 + x + A^+(x,v) + A^-(x,v) = 1 + x + \frac{x^3}{1-2x} + x(A(x,v)-1) + A^-(x,v),$$

which leads to

$$A(x,v) = \frac{1-x-x^2}{1-2x} + \frac{1}{1-x}A^-(x,v).$$

Lemma 3 also gives

$$A^+(vx,1) = \frac{v^3x^3}{1-2vx} + vx(A(vx,1)-1),$$

Hence, by plugging the expressions for $A(x,v)$ and $A^+(vx,1)$ into (1), we obtain

$$\begin{aligned}\frac{(v^2-v+x)(vx-v+x)}{v(1-v)(v-x)}&A^-(x/v,v)\\ &= \frac{x}{v^2(1-v)}A^-(x,1) - \frac{x(v^2x-v^2+vx-x+1)}{v^2(1-v)}A(x,1)\\ &+ \frac{x(v^3x-v^2x^2-v^3+2v^2x-x^3+v-2x)}{v^2(v-2x)/(1-v)}.\end{aligned}$$

By taking $v = 1/C(x)$, we complete the proof. □

Since $A(x,1) = 1 + x + A^-(x,1) + A^+(x,1)$, Lemmas 3 and 4 imply the following result.

Theorem 5. *Let* $T = \{2341, 2314, 1243, 1234\}$. *Then*

$$F_T(x) = C(x) + \frac{x^3}{1-2x}C(x)^4.$$

2.2 Case 890

2.2.1 The symmetry class of $\{2143, 1243, 1423, 1432\}$

Let $a_n = |S_n(T)|$ with $T = \{2143, 1243, 1423, 1432\}$. Define $a_n(i_1, i_2, \ldots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$, and set $b_n(i) = a_n(i, i+1)$ and $c_n(i) = a_n(i, i+2)$.

Lemma 6. *For* $1 \le i \le n-2$, $a_n(i,j) = 0$ *whenever* $j > i+3$, $c_n(i) = a_{n-1}(i)$, $b_n(i) = a_{n-1}(i) - c_{n-1}(i)$, *and*

$$a_n(i) = c_n(1) + c_n(2) + \cdots + c_n(i-2) + b_n(i-1) + b_n(i) + c_n(i),$$

while $a_n(n) = a_n(n-1) = a_{n-1}$.

Proof. For $\pi = ij\pi' \in S_n$ with $i+3 \le j$, π has an occurrence of either 1423 or 1432. Thus $a_n(i,j) = 0$ whenever $j > i+3$. From now we assume that $1 \le i \le n-2$. First, suppose $\pi = i(i+2)j\pi' \in S_n(T)$. Then either $j \le i-1$ or $j = i+1$ or $j = i+3$. Thus

$$c_n(i) = \sum_{j=1}^{i+2} a_{n-1}(i,j) = a_{n-1}(i).$$

Next, let $\pi = i(i+1)j\pi' \in S_n(T)$. Then either $j \le i-1$ or $j = i+2$. Thus

$$b_n(i) = \sum_{j=1}^{i+1} a_{n-1}(i,j) = a_{n-1}(i) - c_{n-1}(i).$$

Now, let $\pi = ij\pi' \in S_n(T)$. If $j = i-1$, then by exchanging the position of the letters i and $i-1$, we see that $a_n(i,i-1) = a_n(i-1,i) = b_n(i-1)$ for all $i = 2,3,\ldots,n$. If $j = i-2$, then

$$a_n(i,i-2) = \sum_{k=1}^{i-3} a_n(i,i-2,k) + a_n(i,i-2,i-1) + a_n(i,i-2,i+1),$$

which leads to

$$a_n(i,i-2) = \sum_{k=1}^{i-3} a_{n-1}(i-2,k) + a_{n-1}(i-2,i-1) + a_{n-1}(i-2,i) = a_{n-1}(i-2) = c_n(i-2)\,.$$

So, let us assume that $1 \le j \le i-3$. Then

$$a_n(i,j) = \sum_{k=1}^{j-1} a_n(i,j,k) + \sum_{k=j+1}^{j+2} a_n(i,j,k),$$

which gives

$$a_n(i,j) = \sum_{k=1}^{j-1} a_{n-1}(j,k) + \sum_{k=j+1}^{j+2} a_{n-1}(j,k) = a_{n-1}(j) = c_n(j).$$

Hence,

$$a_n(i) = c_n(1) + \cdots + c_n(i-2) + b_n(i-1) + b_n(i) + c_n(i),$$

as required. □

By Lemma 6 we have for $n \ge 3$,

$$a_n(i) = a_{n-1}(1) + \cdots + a_{n-1}(i) + a_{n-1}(i) - a_{n-2}(i-1) - a_{n-2}(i)\delta_{i<n-2},$$

and $a_n(n) = a_n(n-1) = a_{n-1}$. Define $A_n(v) = \sum_{i=1}^{n} a_n(i)v^{i-1}$. Then

$$\begin{aligned} A_n(v) &= \frac{1}{1-v}\big(A_{n-1}(v) - v^n A_{n-1}(1)\big) \\ &\quad + A_{n-1}(v) - v^{n-2}A_{n-2}(1) - (1+v)\big(A_{n-2}(v) - v^{n-3}A_{n-3}(1)\big). \end{aligned}$$

with $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1+v$. Define $A(x,v) = \sum_{n\ge 0} A_n(v)x^n$, then by multiplying the last recurrence by x^n and summing over $n \ge 3$, we obtain

$$\begin{aligned} A(x,v) &= 1 - x + (1+v)x^2 + \frac{x}{1-v}\big(A(x,v) - vA(vx,1)\big) \\ &\quad + x\big(1-(1+v)x\big)A(x,v) - x^2\big(1-(1+v)x\big)A(vx,1), \end{aligned}$$

which is equivalent to

$$\begin{aligned} &\left(1 - \frac{x}{v(1-v)} - \frac{x}{v^2}(v-(1+v)x)\right) A\Big(\frac{x}{v},v\Big) \\ &= 1 - \frac{x}{v} + (1+v)\frac{x^2}{v^2} - x\left(\frac{1}{1-v} + \frac{x}{v^3}(v-(1+v)x)\right) A(x,1). \end{aligned}$$

By taking v as the root of $v = 1 + x - x^2 - 2x/v + x^2/v^2$, we obtain the following result.

Theorem 7. *Let $T = \{2143, 1243, 1423, 1432\}$. Then*

$$F_T(x) = \frac{v\big(v(v-x) + (1+v)x^2\big)(1-v)}{x(v-x)},$$

where $v = 1 - x - 2x^2 - 4x^3 - 11x^3 - \cdots$ is the root of $v = 1 + x - x^2 - 2x/v + x^2/v^2$.

2.2.2 The symmetry class of $\{2413, 1423, 1432, 1342\}$

Let $a_n = |S_n(T)|$ with $T = \{2413, 1423, 1432, 1342\}$. Define $a_n(i_1, i_2, \ldots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$. By similar techniques as in the proof of Lemma 6, we obtain

Lemma 8. *Define $b_n(i) = a_n(i, i+1)$ and $c_n(i) = a_n(i, i+2)$. For all $1 \leq i \leq n-2$, $a_n(i,j) = 0$ whenever $j > i+3$, $c_n(i) = a_{n-2}(i)$ with $c_n(n-2) = a_{n-3}$, $b_n(i) = a_{n-1}(i)$ with $b_n(n-1) = a_{n-2}$, and*

$$a_n(i) = b_n(1) + \cdots + b_n(i) + c_n(i),$$

and $a_n(n) = a_n(n-1) = a_{n-1}$.

Lemma 8 shows

$$a_n(i) = a_{n-1}(1) + \cdots + a_{n-1}(i) + a_{n-2}(i),$$

with $a_n(n) = a_n(n-1) = a_{n-1}$ and $a_n(n-2) = a_{n-1} - a_{n-2} + a_{n-3}$. Defining $A_n(v) = \sum_{i=1}^{n} a_n(i) v^{i-1}$ and $A(x,v) = \sum_{n \geq 0} A_n(v) x^n$, we obtain

$$A_n(v) = \frac{1}{1-v}(A_{n-1}(v) - v^n A_{n-1}(1)) + A_{n-2}(v)$$

with $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1 + v$. So

$$A(x/v, v) = 1 - \frac{x^2}{v^2} + \frac{x}{v(1-v)}(A(x/v,v) - vA(x,1)) + \frac{x^2}{v^2} A(x/v, v).$$

Hence, we can state the following result.

Theorem 9. *Let $T = \{2413, 1423, 1432, 1342\}$. Then*

$$F_T(x) = \frac{v\big(v(v-x) + (1+v)x^2\big)(1-v)}{x(v-x)},$$

where $v = 1 - x - 2x^2 - 4x^3 - 11x^3 - \cdots$ is the root of $v = 1 + x - x^2 - 2x/v + x^2/v^2$.

2.2.3 The symmetry class of $\{2314, 1324, 1243, 1234\}$

Let $a_n = |S_n(T)|$ with $T = \{2314, 1324, 1243, 1234\}$. Define $a_n(i_1, i_2, \ldots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$. By similar techniques as in the proof of Lemma 6, we obtain

Lemma 10. *Define $b_n(i) = a_n(i, n)$ and $c_n(i) = a_n(i, n-1)$. For all $1 \leq i \leq n-2$, $a_n(i,j) = 0$ whenever $i+1 \leq j \leq n-1$, $c_n(i) = a_{n-2}(i)$ with $c_n(n-2) = a_{n-3}$, $b_n(i) = a_{n-1}(i)$ with $b_n(n-1) = a_{n-2}$, and*

$$a_n(i) = b_n(1) + \cdots + b_n(i) + c_n(i),$$

and $a_n(n) = a_n(n-1) = a_{n-1}$.

Lemma 10 shows

$$a_n(i) = a_{n-1}(1) + \cdots + a_{n-1}(i) + a_{n-2}(i),$$

with $a_n(n) = a_n(n-1) = a_{n-1}$ and $a_n(n-2) = a_{n-1} - a_{n-2} + a_{n-3}$. Since this is the same recurrence as in the previous case, we obtain the following result.

Theorem 11. *Let* $T = \{2314, 1324, 1243, 1234\}$. *Then*

$$F_T(x) = \frac{v(v(v-x) + (1+v)x^2)(1-v)}{x(v-x)},$$

where v *is the root of* $v = 1 + x - x^2 - 2x/v + x^2/v^2$.

2.3 Case 1054

2.3.1 The symmetry class of $\{2314, 2431, 2341, 1342\}$

Let $G_m(x)$ be the generating function for T-avoiders with m left-right maxima. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$.

Let us write an equation for $G_2(x)$. Let $\pi = i\pi' n\pi'' \in S_n(T)$ with exactly 2 left-right maxima. If $i = n-1$, then we have a contribution of $x(F_T(x) - 1)$. So, we can assume that $i < n-1$, and then π can be written as $\pi = i\pi' n\alpha\beta$, where $\pi'\alpha < i < \beta < n$ and $\beta \neq \emptyset$ and the contribution is $H(x)$. The contributions of the cases $\pi'\alpha = \emptyset$, $\pi' = (i-1)\pi'''$, $\pi' = \pi''(i-1)\pi'''$ with $\pi'' \neq \emptyset$, and α contains $i-1$ are given by $x^2(C(x)-1)$, $xH(x)$, $x^3(C(x)-1)^2C(x)$ and $xC(x)H(x)$ respectively. Thus, $G_2(x) = x\big(F_T(x) - 1\big) + H(x)$, where

$$H(x) = x^2\big(C(x) - 1\big) + xH(x) + x^3\big(C(x) - 1\big)^2 C(x) + xC(x)H(x).$$

Hence,

$$G_2(x) = x(F_T(x) - 1) + \frac{x^2(C(x)-1) + x^3(C(x)-1)^2C(x)}{1 - x - xC(x)}.$$

Now let us write an equation for $G_m(x)$ with $m \geq 3$. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 3$ left-right maxima. Since π avoids T, we have $\pi^{(j)} > i_{j-1}$ for all j (where $i_0 := 0$). Thus, π avoids T if and only if $\pi^{(j)}$ avoids 231 for all j. Hence, $G_m(x) = x^m C^m(x)$ [3].

Summing over $m \geq 0$, we have

$$F_T(x) = 1 + xF_T(x) + x(F_T(x)-1) + \frac{x^2(C(x)-1) + x^3(C(x)-1)^2C(x)}{1 - x - xC(x)} + \frac{x^3C(x)^3}{1 - xC(x)},$$

and solving for $F_T(x)$ implies the following result.

Theorem 12. *Let* $T = \{2314, 2431, 2341, 1342\}$. *Then*

$$F_T(x) = C(x) + x^3C(x)^5 + \frac{x^4C(x)^5}{1-2x}.$$

2.3.2 The symmetry class of $\{2314, 2341, 1342, 1243\}$

Let $a_n = |S_n(T)|$ and define $a_n(i_1, i_2, \ldots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$.

Lemma 13. *Let* $n \geq 3$. *Then* $a_n(i,j) = 0$ *for* $2 \leq i < j \leq n-1$, $a_n(1,2) = 1$ *and* $a_n(1,i) = 2^{i-3}$ *for* $i = 3, 4, \ldots, n-3$, $a_n(i,n) = a_{n-1}(i)$ *for* $i = 1, 2, \ldots, n-1$, $a_n(i, i-1) = a_{n-1}(i-1)$ *for* $i = 2, 3, \ldots, n$, $a_n(n) = a_n(n-1) = a_{n-1}$, $a_n(2,1) = 2^{n-3}$, $a_n(i,1) = a_{n-1}(i,1) + 2^{i-3}$ *for* $i = 3, 4, \ldots, n-2$, *and*

$$a_n(i,j) = a_{n-1}(i,j) + \sum_{k=1}^{j-1} a_{n-1}(j,k) - a_{n-2}(j-1) + a_{n-1}(i-1, j-1),$$

for $2 \leq j < i-1 \leq n-3$.

Proof. We consider the case $2 \leq j < i-1 \leq n-3$. By the definitions, we have

$$a_n(i,j) = \sum_{k=1}^{j-2} a_n(i,j,k) + a_n(i,j,j-1) + \sum_{k=j+1}^{i-1} a_n(i,j,k) + \sum_{k=i+1}^{n-1} a_n(i,j,k) + a_n(i,j,n).$$

Clearly, $a_n(i,j,k) = a_{n-1}(j,k)$ and $a_n(i,j,j-1) = a_{n-1}(i-1,j-1)$ and $a_n(i,j,n) = a_{n-1}(i,j)$. Moreover, $a_n(i,j,k) = 0$ for all $j+1 \leq k \leq n-1$ because we avoid 2314 and 2341. Hence,

$$a_n(i,j) = a_{n-1}(i,j) + \sum_{k=1}^{j-2} a_{n-1}(j,k) + a_{n-1}(i-1,j-1)\,.$$

Since $a_n(j, j-1) = a_{n-1}(j-1)$ for $2 \leq j \leq n$, this completes the proof. $\square$

Define $a_n^-(i) = \sum_{j=1}^{i-1} a_n(i,j)$. By Lemma 13, we have

$$\begin{aligned}
&a_n^-(i) - a_n(i,1) - a_n(i,i-1) \\
&= a_{n-1}^-(i) - a_{n-1}(i,1) - a_{n-1}(i,i-1) + \big(a_{n-1}^-(2) + \cdots + a_{n-1}^-(i-2)\big) \\
&- \big(a_{n-2}(1) + \cdots + a_{n-2}(i-3)\big) + a_{n-1}^-(i-1) - a_{n-2}(i-2),
\end{aligned}$$

which is equivalent to

$$a_n^-(i) = a_{n-1}^-(1) + \cdots + a_{n-1}^-(i) - (a_{n-2}(1) + \cdots + a_{n-2}(i-1)) + a_{n-1}(i-1) + 2^{i-3},$$

where $a_n^-(1) = 0$, $a_n^-(2) = 2^{n-3}$ and $a_n^-(n) = a_{n-1}$. Also, $a_n(i) = a_n^-(i) + a_{n-1}(i) + 2^{n-3}\delta_{i=1}$ for all $i = 1, 2, \ldots, n$.

Define $A_n(v) = \sum_{i=1}^n a_n(i) v^{i-1}$ and $A_n^-(v) = \sum_{i=1}^n a_n^-(i) v^{i-1}$. Then, for all $n \geq 3$,

$$A_n(v) = A_n^-(v) + A_{n-1}(v) + 2^{n-3},$$

and

$$\begin{aligned}
A_n^-(v) &= vA_{n-1}(v) + \big(A_{n-1}(1) - A_{n-2}(1)\big)v^{n-1} \\
&+ \frac{1}{1-v}\big(A_{n-1}^-(v) - v^{n-1}A_{n-1}^-(1)\big) - \frac{1}{1-v}\big(vA_{n-2}(v) - v^{n-1}A_{n-2}(1)\big) + v^2\sum_{i=0}^{n-4}(2v)^i.
\end{aligned}$$

with $A_1(v) = 1$, $A_2(v) = 1 + v$ and $A_2^-(v) = v$.

Define $A(x, v) = \sum_{n\geq 0} A_n(v)x^n$ and $A^-(x, v) = \sum_{n\geq 0} A_n^-(v)x^n$. Then

$$A(x, v) = \frac{1}{1-x}\left(1 + \frac{x^3}{1-2x} + A^-(x, v)\right),$$

and

$$\begin{aligned}A^-(x, v) &= vx\big(A(x, v) - 1\big) + x\big(A(vx, 1) - 1\big) - vx^2 A(vx, 1)\\ &\quad + \frac{x}{1-v}\big(A^-(x, v) - A^-(vx, 1)\big) - \frac{vx^2}{1-v}\big(A(x, v) - A(vx, 1)\big) + \frac{v^2x^4}{1 - x - 2vx + 2vx^2}.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{(v^2 - v + x)(vx - v + x)}{v(1-v)(v-x)}A^-(x, v) &= \frac{x(vx - v + x)}{v(1-v)(1-x)}A^-(x, 1)\\ &\quad + \frac{x^2}{v} + \frac{x^4(v^3 - 3v^2x + 2vx^2 - 3vx + 2x^2 + v - x)}{v^2(1-2x)(v-x)(v-2x)}.\end{aligned}$$

By taking $v = 1/C(x)$, we obtain

$$A^-(x, 1) = (1 - x)\left(C(x) + x^3C(x)^5 + \frac{x^4C(x)^5}{1-2x}\right) - 1 - \frac{x^3}{1-2x},$$

which implies the following result.

Theorem 14. *Let $T = \{2314, 2341, 1342, 1243\}$. Then*

$$F_T(x) = C(x) + x^3C(x)^5 + \frac{x^4C(x)^5}{1-2x}.$$

2.3.3 The symmetry class of $\{1324, 2341, 1342, 1234\}$

Let $a_n = |S_n(T)|$ and define $a_n(i_1, i_2, \dots, i_s)$ to be the number of permutations $\pi = i_1 i_2 \cdots i_s \pi' \in S_n(T)$.

Lemma 15. *We have $a_n(i, j) = 0$ for $2 \leq i+1 < j \leq n-1$, $a_n(1, 2) = 1$ and $a_n(i, i+1) = 2^{i-2}$ for $i = 2, 3, \dots, n-2$, $a_n(i, n) = a_{n-1}(i)$ for $i = 1, 2, \dots, n-1$, and $a_n(i, j) = a_{n-1}(j)$ for $1 \leq j < i \leq n$.*

Proof. We prove only $a_n(i, j) = a_{n-1}(j)$ for $1 \leq j < i \leq n$, all the rest are done in the same fashion. For $\pi = ij\pi' \in S_n(T)$, if there is an occurrence of 1324, 1342 or 1234 that starts with i, then there is an occurrence of the same pattern that starts with j. Moreover, if there is an occurrence $ixyz$ of 2341 in π, then $jxyz$ is order isomorphic to 2341 or 1342. Thus, π avoids T if and only if $j\pi'$ avoids T, which implies that $a_n(i, j) = a_{n-1}(j)$ for $1 \leq j < i \leq n$. □

By Lemma 15, we have

$$a_n(i) = a_{n-1}(1) + a_{n-1}(2) + \cdots + a_{n-1}(i) + 2^{i-2}$$

with $a_n(n) = a_n(n-1) = a_{n-1}$ and $a_n(1) = 1 + a_{n-1}(1)$. Define $A_n(v) = \sum_{i=1}^{n-1} a_n(i)v^{i-1}$. Then,

$$A_n(v) = \frac{1}{1-v}\big(A_{n-1}(v) - v^n A_{n-1}(1)\big) + 1 + v\frac{1-(2v)^{n-3}}{1-2v}.$$

with $A_0(v) = A_1(v) = 1$ and $A_2(v) = 1 + v$.

Define $A(x,v) = \sum_{n\geq 0} A_n(v)x^n$. Multiplying the recurrence for $A_n(v)$ by x^n/v^n and summing over $n \geq 3$, we obtain

$$A(x/v,v) = 1 + \frac{x}{v(1-v)}\big(A(x/v,v) - vA(x,1)\big) + \frac{x^3}{v^2(v-x)} + \frac{x^4}{v^2(v-x)(1-2x)}.$$

Taking $v = 1/C(x)$ and using the identity $A(x,1) = F_T(x)$, we obtain the following result.

Theorem 16. *Let* $T = \{2314, 2431, 2341, 1342\}$. *Then*

$$F_T(x) = C(x) + x^3C(x)^5 + \frac{x^4C(x)^5}{1-2x}.$$

3 Proof of Theorem 2

We treat the following selection of cases. Proofs for the others are similar and are omitted.

3.1 Case 623

3.1.1 The symmetry class of $\{2413, 3142, 3241, 1342\}$

Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Let us write an equation for $G_m(x)$ with $m \geq 0$. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has m left-right maxima. If $\pi^{(m)} > i_{m-1}$, then π avoids T if and only if $i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_{m-1}\pi^{(m-1)}$ avoids T and $\pi^{(m)}$ avoids 231, which implies the contribution of $xC(x)G_{m-1}(x)$ [3]. Otherwise, $\pi^{(m)}$ has a letter smaller than i_1, which implies that $\pi^{(j)} = \emptyset$ for all $j = 1, 2, \ldots, m$ and $\pi^{(m)} = \alpha\beta$ where $\alpha > i_{m-1} > i_1 > \beta$, β is not empty, and α avoids $\{213, 231\}$ and β avoids T. Thus, by [6], we have a contribution of $x^m\frac{1-x}{1-2x}\big(F_T(x) - 1\big)$. Hence,

$$G_m(x) = xC(x)G_{m-1}(x) + x^m\frac{1-x}{1-2x}(F_T(x) - 1).$$

Summing over $m \geq 2$, we have

$$F_T(x) = 1 + xF_T(x) + xC(x)(F_T(x) - 1) + \frac{x^2}{1-2x}(F_T(x) - 1),$$

and solving for $F_T(x)$ implies the following result.

Theorem 17. *Let* $T = \{2413, 3142, 3241, 1342\}$. *Then*

$$F_T(x) = \frac{x(2x-1)C(x) - x^2 - 2x + 1}{x(2x-1)C(x) + x^2 - 3x + 1}.$$

3.1.2 The symmetry class of $\{2431, 1432, 1324, 1423\}$

Note that each pattern in T contains 132. We recall the cell decomposition of $\pi \in S_n(T)$ as described in [4] (also, see [5]). If π contains 132, then π can be written as

$$\pi = \alpha^{(1)} \cdots \alpha^{(j+2)} a \beta^{(1)} (a+1) \cdots \beta^{(i)} (a+i) \gamma (a+i+1) k_1 \cdots k_j,$$

where $\alpha^{(1)} > k_{j+1} > \alpha^{(2)} > k_j > \cdots > \alpha^{(j+2)} > a+i+1$, k_{j+1} is the maximal letter of γ, $ak_{j+1}(a+i+1)$ is order isomorphic to 132, $\beta^{(1)} > \cdots > \beta^{(i)} > \gamma'$ where γ' is obtained from γ by removing the letter k_{j+1} and each of $\alpha^{(s)}$, $\beta^{(s)}$ and γ avoids 132. Hence, by [3], we have

$$F_T(x) = C(x) + \frac{x^2 C(x)^2 \big(C(x) - 1\big)}{\big(1 - xC(x)\big)^2},$$

which leads to the following result.

Theorem 18. *Let* $T = \{2431, 1432, 1324, 1423\}$. *Then*

$$F_T(x) = \frac{x(2x-1)C(x) - x^2 - 2x + 1}{x(2x-1)C(x) + x^2 - 3x + 1}.$$

3.2 Case 651

3.2.1 The symmetry class of $\{3142, 2431, 1423, 1324\}$

Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Let us write an equation for $G_m(x)$ with $m \geq 2$. Suppose $\pi = i_1 \pi^{(1)} i_2 \pi^{(2)} \cdots i_m \pi^{(m)} \in S_n(T)$ has $m \geq 2$ left-right maxima. Since π avoids 1324, we have that $\pi^{(j)} < i_1$ for all $j = 1, 2, \ldots, m-1$. Since π avoids 3142 and 2431, $\pi^{(m)}$ can be written as $\alpha^{(1)} \alpha^{(2)} \cdots \alpha^{(m)}$ where $i_{j-1} < \alpha^{(j)} < i_j$ with $i_0 = 0$. Thus, π avoids T if and only if either

- there exists j, $2 \leq j \leq m$, such that $\pi^{(1)} > \cdots > \pi^{(j-1)} > \alpha^{(1)}$, $\alpha^{(j)}$ is decreasing and $\alpha^{(s)} = \emptyset$ for all $s = 2, \ldots, j-1, j+1, \ldots, m$, and $\pi^{(s)}$ avoids 132 for all $s = 1, 2, \ldots, j-1$, and $\alpha^{(1)}$ avoids 132, or
- $\pi^{(1)} > \cdots > \pi^{(m-1)} > \alpha^{(1)}$, $\alpha^{(s)} = \emptyset$ for all $s = 2, 3, \ldots, m$, and $\pi^{(s)}$ avoids 132 for all $s = 1, 2, \ldots, m-1$, and $\alpha^{(1)}$ avoids T.

Thus, by [3], we have

$$G_m(x) = \sum_{j=2}^{m} x^m C(x)^j \frac{x}{1-x} + x^m C(x)^{m-1} F_T(x).$$

Hence, since $F_T(x) = 1 + xF_T(x) + \sum_{m \geq 2} G_m(x)$, we obtain the following result.

Theorem 19. *Let* $T = \{3142, 2431, 1423, 1324\}$. *Then*

$$F_T(x) = \frac{\big(1 - 2x - x(x^2 - 3x + 1)C(x)\big) C(x)^2}{(1-x)^2}.$$

3.3 Case 729

3.3.1 The symmetry class of $\{3142, 1324, 1423, 1243\}$

Let $T = \{3142, 1243, 1423, 1324\}$. Clearly, $G_0(x) = 1$ and $G_1(x) = xF_T(x)$. Let us write an equation for $G_m(x)$ with $m \geq 2$. Suppose $\pi = i_1\pi^{(1)}i_2\pi^{(2)}\cdots i_m\pi^{(m)} \in S_n(T)$ has $m \geq 2$ left-right maxima. Since π avoids 1324 and 1243, we have that $\pi^{(j)} < i_1$ for all $j = 1, 2, \ldots, m-1$ and $\pi^{(m)} < i_2$. Since π avoids 1423, $\pi^{(m)}$ contains the subsequence $(i_2-1)(i_2-2)\cdots(i_1+1)$. So $\pi^{(m)}$ has the form $\alpha^{(1)}(i_2-1)\cdots\alpha^{(i_2-i_1-1)}(i_1+1)\alpha^{(i_2-i_1)}$, and π avoids T if and only if either

- $\pi^{(1)} > \cdots > \pi^{(m)}$, $\pi^{(s)}$ avoids 132 for all $s = 1, 2, \ldots, m-1$, and $\pi^{(m)}$ avoids T (case $i_2 = i_1 + 1$).
- $\pi^{(1)}\cdots\pi^{(m-1)}\alpha^{(1)}\cdots\alpha^{(i_2-i_1-2)}$ is decreasing, $\pi^{(2)}\cdots\pi^{(m-1)} = \emptyset$, $\alpha^{(i_2-i_1-1)}$ avoids 132, and $\alpha^{(i_2-i_1)}$ avoids T (case $i_2 > i_1 + 1$).

Thus, by [3], we have

$$G_m(x) = x^m C(x)^{m-1} F_T(x) + \frac{x^{m+1}C(x)F_T(x)}{1-2x}.$$

Since $F_T(x) = 1 + xF_T(x) + \sum_{m\geq 2} G_m(x)$, we obtain the following result.

Theorem 20. *Let $T = \{3142, 1243, 1423, 1324\}$. Then*

$$F_T(x) = \frac{(1-x)(1-2x)}{(1-x)(1-2x) - x(1-3x+3x^2)C(x)}.$$

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Appendix

Table 1. 3-Wilf classes of four 4-letter patterns

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
75	$\{4231, 2341, 4312, 4123\}$		INSENC
	$\{2314, 3124, 1432, 1324\}$	$\frac{(1-x)^4(1-3x+2x^2-x^3)}{x^8-5x^7+17x^6-36x^5+52x^4-47x^3+26x^2-8x+1}$	INSENC
	$\{2413, 3142, 1423, 1234\}$		INSENC
141	$\{4231, 3412, 1423, 1234\}$		INSENC
	$\{2314, 4312, 1342, 1324\}$	$\frac{9x^4-11x^3+11x^2-5x+1}{(1-x)^6}$	INSENC
	$\{2134, 4312, 3124, 1342\}$		INSENC
212	$\{2431, 4213, 1432, 1324\}$		INSENC
	$\{3412, 3124, 1342, 4123\}$	$\frac{1-6x-15x^3-2x^5+9x^4+14x^2}{(x^2-3x+1)(1-x)^4}$	EX
	$\{3412, 3124, 1342, 1423\}$		EX
217	$\{2431, 4213, 1324, 1423\}$		INSENC
	$\{2413, 4132, 1243, 1234\}$	$\frac{3x^4-6x^3+9x^2-5x+1}{(x^2-3x+1)(1-x)^3}$	INSENC
	$\{2143, 3412, 1342, 1423\}$		EX
231	$\{2431, 2134, 4312, 1423\}$		INSENC
	$\{2413, 4312, 1342, 1234\}$	$\frac{x^7-2x^6-x^5+9x^4-11x^3+11x^2-5x+1}{(1-x)^6}$	INSENC
	$\{3412, 4132, 1243, 1234\}$		INSENC
301	$\{2341, 2143, 4132, 4123\}$		INSENC
	$\{2314, 4213, 3412, 1432\}$	$\frac{x^6-7x^5+13x^4-18x^3+15x^2-6x+1}{(1-2x)(1-x)^5}$	INSENC
	$\{3412, 1432, 1324, 4123\}$		INSENC
354	$\{2341, 3412, 4123, 1243\}$		EX
	$\{3412, 2341, 1324, 1234\}$	$\frac{1-7x+21x^2-33x^3+31x^4-16x^5+3x^6-2x^7}{(1-2x)(1-x)^6}$	EX
	$\{3412, 1324, 1243, 1234\}$		EX
482	$\{4213, 2143, 3142, 1342\}$		INSENC
	$\{3142, 1342, 4123, 1243\}$	$\frac{(1-2x)(3x^4-7x^3+9x^2-5x+1}{(1-x)^5(1-3x)}$	INSENC
	$\{4132, 1432, 4123, 1243\}$		INSENC
538	$\{4213, 3142, 1342, 1243\}$		INSENC
	$\{2143, 3142, 1342, 4123\}$	$\frac{-(4x^4-7x^3+9x^2-5x+1)}{(2x^3-4x^2+4x-1)(1-x)^2}$	INSENC
	$\{3412, 1432, 4123, 1423\}$		INSENC
548	$\{4213, 3124, 1342, 1423\}$		INSENC
	$\{2143, 3142, 1324, 4123\}$	$\frac{x^5-3x^4+4x^3-8x^2+5x-1}{(x^3-2x^2+3x-1)(x^2-3x+1)}$	EX
	$\{3142, 3124, 1432, 4123\}$		INSENC
602	$\{2413, 4312, 3412, 1342\}$		INSENC
	$\{2413, 4312, 3142, 1342\}$	$\frac{x^5-5x^4+16x^3-17x^2+7x-1}{(3x-1)(x^2-3x+1)(1-x)^2}$	INSENC
	$\{2413, 3412, 4132, 1342\}$		EX
609	$\{2413, 4312, 1432, 1324\}$		INSENC
	$\{2413, 4312, 1342, 1324\}$	$\frac{x^6-9x^5+21x^4-28x^3+20x^2-7x+1}{(1-2x)^2(1-x)^4}$	INSENC
	$\{2143, 4312, 3142, 1324\}$		EX
614	$\{2413, 3412, 3142, 1234\}$		INSENC
	$\{2134, 4312, 4132, 1432\}$	$\frac{x^6-8x^5+20x^4-22x^3+16x^2-6x+1}{(1-x)^7}$	INSENC
	$\{2134, 4312, 1342, 1324\}$		INSENC

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
619	$\{2413, 3412, 4132, 1234\}$ $\{2134, 4312, 3412, 1342\}$ $\{2134, 3412, 4132, 1342\}$	$\frac{2x^6-10x^5+20x^4-22x^3+16x^2-6x+1}{(1-x)^7}$	INSENC INSENC INSENC
643	$\{2413, 3124, 1432, 1324\}$ $\{2143, 3142, 3124, 1432\}$ $\{3142, 3124, 1423, 1234\}$	$\frac{(1-x)^3(2x-1)}{x^5-6x^4+14x^3-13x^2+6x-1}$	INSENC INSENC INSENC
656	$\{2431, 3142, 1324, 1423\}$ $\{2431, 2413, 1324, 1423\}$ $\{2341, 2314, 1243, 1234\}$	$C(x) + \frac{x^3}{1-2x}C(x)^4$	[4] [4] Thm. 5
686	$\{2143, 4312, 3124, 1243\}$ $\{4312, 3124, 1432, 1243\}$ $\{4312, 1432, 1324, 4123\}$	$\frac{x^7-10x^5+21x^4-28x^3+20x^2-7x+1}{(x-1)^4(2x-1)^2}$	INSENC INSENC INSENC
702	$\{2143, 3412, 3142, 1342\}$ $\{3412, 3142, 1342, 1324\}$ $\{3412, 3142, 1342, 1243\}$	$\frac{1-6x+11x^2-5x^3}{(1-x)(1-3x)(1-3x+x^2)}$	EX EX EX
721	$\{2143, 3412, 1423, 1243\}$ $\{4312, 3142, 3124, 1342\}$ $\{3124, 4132, 1342, 1423\}$	$\frac{x^6-4x^5+10x^4-20x^3+18x^2-7x+1}{(x^2-3x+1)^2(x-1)^2}$	EX INSENC INSENC
726	$\{2143, 3142, 1432, 1423\}$ $\{2143, 3142, 1342, 1423\}$ $\{3142, 1432, 1423, 1243\}$	$\frac{(1-x-\sqrt{1-6x+9x^2-8x^3})(1-x)}{2x(1-2x+2x^2)}$	[4] [4] [4]
733	$\{2143, 3142, 1423, 1234\}$ $\{3142, 1432, 1243, 1234\}$ $\{3124, 1432, 1342, 1234\}$	$\frac{x^3-2x^2+3x-1}{x^4+3x^3-4x^2+4x-1}$	INSENC INSENC INSENC
824	$\{2134, 3142, 1432, 1342\}$ $\{2134, 3142, 1432, 1423\}$ $\{3142, 1432, 1342, 1234\}$	$\frac{(x^3-2x^2+3x-1)^2}{3x^6-9x^5+20x^4-24x^3+18x^2-7x+1}$	INSENC INSENC INSENC
845	$\{2134, 4132, 1432, 1324\}$ $\{3124, 4132, 1423, 1243\}$ $\{4132, 1432, 1324, 4123\}$	$-\frac{(x^5-7x^4+19x^3-18x^2+7x-1)}{(1-x)(x^2-3x+1)(1-2x)^2}$	INSENC INSENC INSENC
882	$\{4312, 3412, 3142, 4123\}$ $\{2143, 3142, 1342, 1432\}$ $\{3142, 1432, 1342, 1243\}$	$f = 1 - x + x^2 f + x(x^2 - 2x + 2)f^2 - x^2(1-x)f^3$	[4] [4] [4]
890	$\{2143, 1243, 1423, 1432\}$ $\{2413, 1432, 1423, 1342\}$ $\{2314, 1324, 1243, 1234\}$	$\frac{v(v(v-x)+(1+v)x^2)(1-v)}{x(v-x)}$ where $v = 1 + x - x^2 - 2x/v + x^2/v^2$	Thm. 7 Thm. 9 Thm. 11
1034	$\{3142, 3124, 1432, 1342\}$ $\{3142, 1324, 1243, 1234\}$ $\{3124, 1432, 1342, 1423\}$	$\frac{(1-2x)(x^2+2x-1)}{(2x^3+2x^2-4x+1)(x-1)}$	INSENC INSENC INSENC
1054	$\{2314, 2431, 2341, 1342\}$ $\{2314, 2341, 1342, 1243\}$ $\{1324, 2341, 1342, 1234\}$	$C(x) + x^3C(x)^5 + \frac{x^4C(x)^5}{1-2x}$	Thm. 12 Thm. 14 Thm. 16

Table 2. 4-Wilf classes of four 4-letter patterns

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
74	$\{4231, 2341, 2143, 4123\}$	$\frac{x^6-2x^5-5x^4+4x^3-7x^2+4x-1}{(x-1)^5}$	INSENC
	$\{4231, 3412, 1243, 1234\}$		INSENC
	$\{2143, 3412, 1324, 1234\}$		INSENC
	$\{2143, 3412, 4123, 1234\}$		INSENC
132	$\{4231, 3412, 3142, 1234\}$	$\frac{-(x^5-9x^4+11x^3-11x^2+5x-1)}{(x-1)^6}$	INSENC
	$\{2143, 3412, 3142, 1234\}$		INSENC
	$\{2143, 3412, 1243, 1234\}$		INSENC
	$\{2134, 4312, 3142, 1432\}$		INSENC
156	$\{4231, 3124, 1342, 1324\}$	$\frac{-(6x^5-21x^4+28x^3-20x^2+7x-1)}{(2x-1)^2(x-1)^4}$	INSENC
	$\{4231, 1342, 1324, 1423\}$		EX
	$\{3412, 3124, 1324, 1423\}$		EX
	$\{3412, 1324, 4123, 1423\}$		EX
163	$\{4231, 4132, 1342, 1324\}$	$\frac{-(13x^5-35x^4+42x^3-26x^2+8x-1)}{(2x-1)^3(x-1)^3}$	EX
	$\{4312, 3124, 4132, 1324\}$		EX
	$\{3412, 3142, 3124, 1243\}$		INSENC
	$\{3412, 3142, 1342, 1234\}$		INSENC
361	$\{2341, 3142, 4123, 1423\}$	$\frac{x^5-6x^4+12x^3-13x^2+6x-1}{(3x^3-5x^2+4x-1)(x^2-3x+1)}$	INSENC
	$\{4213, 2413, 3124, 1432\}$		INSENC
	$\{4213, 3142, 3124, 1432\}$		INSENC
	$\{4213, 3142, 1432, 1324\}$		INSENC
375	$\{2341, 4132, 1432, 1324\}$	$\frac{-(2x^4+x^3+4x^2-4x+1)}{(2x-1)(x^2-3x+1)}$	INSENC
	$\{4213, 3124, 4132, 1342\}$		INSENC
	$\{2143, 3412, 1432, 1243\}$		EX
	$\{2143, 1324, 4123, 1234\}$		INSENC
534	$\{4213, 3142, 4132, 1324\}$	$\frac{-(3x^4-15x^3+17x^2-7x+1)}{(2x-1)(x^2-3x+1)^2}$	EX
	$\{3142, 3124, 4132, 1324\}$		EX
	$\{3142, 3124, 1324, 4123\}$		EX
	$\{1432, 1324, 4123, 1423\}$		INSENC
623	$\{2413, 3142, 3241, 1342\}$	$\frac{x(2x-1)C(x)-x^2-2x+1)}{x(2x-1)C(x)+x^2-3x+1}$	Theorem 17
	$\{2431, 1432, 1324, 1423\}$		Theorem 18
	$\{4132, 1432, 1324, 1243\}$		EX
	$\{1342, 4123, 1423, 1234\}$		EX
647	$\{2413, 3124, 1342, 1324\}$	$\frac{-(2x-1)^3}{5x^4-17x^3+17x^2-7x+1}$	EX
	$\{2134, 3142, 3124, 1423\}$		EX
	$\{3142, 3124, 1324, 1423\}$		EX
	$\{3124, 1432, 1324, 1423\}$		INSENC
651	$\{2413, 4132, 1432, 1324\}$	$\frac{(1-2x-x(x^2-3x+1)C(x))C^2(x)}{(1-x)^2}$	EX
	$\{3142, 2431, 1423, 1324\}$		Theorem 19
	$\{2143, 4132, 1432, 1324\}$		EX

No.	Pattern set T	Generating function $F_T(x)$	Thm./[Ref]
	$\{3142, 4132, 1432, 1243\}$		EX
659	$\{2413, 4132, 4123, 1234\}$	$\frac{-(7x^4-14x^3+14x^2-6x+1)}{(x-1)^3(2x-1)^2}$	INSENC
	$\{2134, 3412, 3142, 1423\}$		EX
	$\{3412, 1342, 4123, 1243\}$		EX
	$\{3412, 1342, 4123, 1234\}$		EX
670	$\{2143, 2134, 4132, 1342\}$	$\frac{2x^5-4x^4+11x^3-13x^2+6x-1}{(2x-1)(x^2-3x+1)(x-1)^2}$	INSENC
	$\{2143, 3412, 1432, 1423\}$		EX
	$\{3412, 3142, 1432, 1324\}$		EX
	$\{3412, 3142, 1432, 1243\}$		EX
729	$\{2143, 3142, 1324, 1423\}$	$\frac{(1-x)(1-2x)}{(1-x)(1-2x)-x(1-3x+3x^2)C(x)}$	EX
	$\{3412, 3142, 4123, 1423\}$		EX
	$\{3142, 1324, 1423, 1243\}$		Theorem 20
	$\{3124, 1342, 1423, 1243\}$		EX
769	$\{2143, 1342, 4123, 1423\}$	$\frac{x^4-x^3+4x^2-4x+1}{(x-1)(x^3-3x^2+4x-1)}$	INSENC
	$\{2143, 1342, 4123, 1243\}$		INSENC
	$\{3412, 4132, 1432, 4123\}$		INSENC
	$\{1432, 1342, 4123, 1243\}$		INSENC
778	$\{2134, 4312, 3142, 1342\}$	$\frac{-(2x^6-9x^5+20x^4-22x^3+16x^2-6x+1)}{(x-1)^7}$	INSENC
	$\{4312, 3412, 1342, 1234\}$		INSENC
	$\{3412, 3124, 4132, 1234\}$		INSENC
	$\{3412, 4132, 1342, 1234\}$		INSENC