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An inequality involving a ratio of zeta functions

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Abstract: We prove an inequality for a ratio of zeta functions. This extends a classical result (see [2]). The method is based on Dirichlet series, combined with real analysis.
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Let $\omega(n)$ denote the number of distinct prime divisors of n. Then $\omega(1) = 0$ and $\omega(n)$ is an additive function, i.e.

$$\omega(mn) = \omega(m) + \omega(n) \text{ for all } (m, n) = 1.$$

This implies immediately that the function

$$f(n) = k^{\omega(n)}$$

(where $k \geq 2$ is fixed) is a multiplicative function, i.e. satisfies the functional equation

$$f(mn) = f(m) \cdot f(n) \text{ for all } (m, n) = 1, \tag{1}$$

where $f(1) = 1$.

A general Dirichlet series is an infinite series of type $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$, where $s \in \mathbb{C}$ is such that the series is convergent. For $a_n = 1$, we get the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is convergent e.g. for all $\operatorname{Re} s > 1$. Another Dirichlet series is obtained when

$$a_n = f(n) = k^{\omega(n)}.$$

We will prove in what follows the following result:

Theorem. *Let $s > 1$ a fixed positive integer. Then one has the inequality*

$$\sum_{k=1}^{\infty} \frac{k^{\omega(n)}}{n^s} \leq \frac{\zeta^k(s)}{\zeta(ks)}, \tag{2}$$

with equality only for $k = 2$.

For the proof, the following well-known result will be applied (see e.g. [1]).

Lemma 1. *Let f be a multiplicative arithmetical function, and let the series $\sum_{n=1}^{\infty} f(n)$ be absolutely convergent. Then we have the identity:*

$$\sum_{n=1}^{\infty} f(n) = \prod_{p \text{ prime}} (1 + f(p) + f(p^2) + \ldots). \tag{3}$$

We shall need also the following auxiliary result:

Lemma 2. *Let $0 < x \leq \dfrac{1}{2}$ and $k \geq 2$. Then*

$$1 - x^k \geq (1-x)^{k-1}[1 + x(k-1)]. \tag{4}$$

Inequality (4) may be written also as

$$1 - x^k \geq (1-x)^{k-1}[(1 - x + kx)] = (1-x)^k + kx(1-x)^{k-1}.$$

Let us define

$$g(x) = x^k = (1-x)^k + kx(1-x)^{k-1}, \; g : [0,1] \to \mathbb{R}.$$

We have to prove that $g(x) \leq 1$. One has

$$g(1) = g(0) = 1 \quad \text{and} \quad g'(x) = kx[x^{k-2} - (k-1)(1-x)^{k-2}].$$

Remark that, as $0 < x \leq \dfrac{1}{2}$, we have $0 < x \leq 1 - x$, so

$$x^{k-2} \leq (1-x)^{k-2} \leq (k-1)(1-x)^{k-2},$$

with equality only for $k = 2$. Thus we get $g'(x) \leq 0$, implying

$$g(x) \leq g(0) = 1.$$

Remark. The above proof shows that there is equality in (4) only for $k = 2$.

Proof of Theorem. Letting

$$f(n) = \frac{k^{\omega(n)}}{n^s}$$

in Lemma 1, we get

$$\sum_{n=1}^{\infty} \frac{k^{\omega(n)}}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{k}{p^s} + \frac{k}{p^{2s}} + \ldots\right) \tag{5}$$

For $f(n) = \dfrac{1}{n^s}$ in the same Lemma 1, we get Euler's identity

$$\sum_{k=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 + \frac{k}{p^s} + \frac{k}{p^{2s}} + \ldots\right) = \prod_{p \text{ prime}} \frac{1}{1 - \dfrac{1}{p^s}}.$$

Thus, by using Euler's identity, we get

$$\zeta(ks) = \prod_{p \text{ prime}} \frac{1}{1 - \dfrac{1}{p^{ks}}},$$

i.e.,

$$\frac{\zeta^k(s)}{\zeta(ks)} = \prod_{p \text{ prime}} \frac{1 - \dfrac{1}{p^{ks}}}{\left(1 - \dfrac{1}{p^s}\right)^k} \tag{6}$$

Put now $x = \dfrac{1}{p^s}$. As $s > 1$ and $p \geq 2$, clearly $x < \dfrac{1}{2}$. So we can apply Lemma 2, which implies

$$\frac{1 - x^k}{(1-x)^k} \geq \frac{1 + x(k-1)}{1 - x} \tag{7}$$

In relation (5) one has

$$\begin{aligned} 1 + \frac{k}{p^s} + \frac{k}{p^{2s}} + \ldots &= 1 + kx + kx^2 + \ldots \\ &= 1 + kx(1 + x + x^2 + \ldots) \\ &= 1 + \frac{kx}{x-1} = \frac{1 + x(k-1)}{x-1}. \end{aligned}$$

By relations (6) and (7), this implies inequality (2), finishing the proof of Theorem. □

Remark. For $k = 2$ we get the known identity (see [2])

$$\sum_{k=1}^{\infty} \frac{2^{\omega(n)}}{n^s} = \frac{\zeta^2(s)}{\zeta(2s)}.$$

References

[1] Hardy, G. H., & Wright, E. M. (1964) *An Introduction to the Theory of Numbers*, Oxford Univ. Press.

[2] Titchmarsh, E. C. (1951) *The Theory of the Riemann Zeta Function*, Oxford.