Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 3, 62–67
DOI: 10.7546/nntdm.2018.24.3.62-67

Conditions equivalent to the Descartes–Frenicle–Sorli Conjecture on odd perfect numbers – Part II

Jose Arnaldo Bebita Dris[1] and Doli-Jane Uvales Tejada[2]

[1] Institute of Mathematics, University of the Philippines
Carlos P. Garcia Avenue, Diliman, Quezon City, Philippines
e-mails: jbdris@upd.edu.ph, josearnaldobdris@gmail.com

[2] Mathematics Department, College of Natural Sciences and Mathematics
Mindanao State University, General Santos City, Philippines
e-mail: dolijanetejada@gmail.com

Received: 20 January 2018 **Revised:** 9 July 2018 Accepted: 17 August 2018

Abstract: The Descartes–Frenicle–Sorli conjecture predicts that $k = 1$ if $q^k n^2$ is an odd perfect number with Euler prime q. In this note, we present some further conditions equivalent to this conjecture.

Keywords: Odd perfect number, Abundancy index, Deficiency.
AMS Classification: 11A25.

1 Introduction

Let x be a positive integer. Recall that we denote

$$\sum_{d|x} d = \sigma_1(x) = \sigma(x)$$

as the sum of divisors of x. Denote the abundancy index of x by $I(x) = \sigma(x)/x$, and the deficiency of x by $D(x) = 2x - \sigma(x)$. Note that we have the identity

$$\frac{D(x)}{x} + \frac{\sigma(x)}{x} = \frac{D(x)}{x} + I(x) = 2.$$

Note further that, if $y = \prod_{i=1}^{w} {z_i}^{s_i}$ is the prime factorization of y, then we have the formula

$$\sigma(y) = \sigma\left(\prod_{i=1}^{w} {z_i}^{s_i}\right) = \prod_{i=1}^{w}\left(\sigma({z_i}^{s_i})\right) = \prod_{i=1}^{w} \frac{{z_i}^{s_i+1} - 1}{z_i - 1},$$

where $w = \omega(y)$ is the number of distinct prime factors of y. This means that σ as a function satisfies $\sigma(ab) = \sigma(a)\sigma(b)$ if and only if $\gcd(a, b) = 1$, which means that σ is multiplicative.

Therefore, if $\gcd(a, b) = 1$, it follows from the above formula for σ that

$$I(ab) = \frac{\sigma(ab)}{ab} = \frac{\sigma(a)\sigma(b)}{ab} = \left(\frac{\sigma(a)}{a}\right) \cdot \left(\frac{\sigma(b)}{b}\right) = I(a)I(b)$$

which shows that the abundancy index I as a function is also multiplicative. Lastly, note that the deficiency D as a function is in general not multiplicative [6].

We will repeatedly use the multiplicativity of the divisor sum σ and the abundancy index I to derive results in this paper.

We say that a number N is perfect if $\sigma(N) = 2N$. The following result (due to Euclid and Euler) gives a necessary and sufficient condition for an even integer E to be perfect.

Theorem 1.1. *An even integer E is perfect if and only if $E = (2^p - 1)2^{p-1}$ for some integer p for which $2^p - 1$ is prime.*

Refer to Dickson [2] to see different proofs of Theorem 1.1. Prime numbers of the form $2^p - 1$ are called Mersenne primes, and if $2^p - 1$ is prime then p must be prime. (The converse of this last statement does not hold.) Observe that 6, 28, 496, and 8128 are examples of even perfect numbers, and these correspond to the Mersenne primes $2^p - 1$ with p given by 2, 3, 5, and 7, respectively. We still do not know if there are infinitely many even perfect numbers. Also, it is not known if there are odd perfect numbers. It is widely believed that no odd perfect numbers exist.

If there is an odd perfect number O, then Euler proved that it must have the form $O = q^k n^2$, where q is a prime satisfying $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$. We call q the special or Euler prime of O, q^k is the Euler factor, and n^2 is the non-Euler factor. (Note that both E and O have the forms $N = Q^K M^2$, where Q is prime, $K \equiv 1 \pmod 4$, and $\gcd(Q, M) = 1$.) Descartes, Frenicle, and subsequently Sorli [7] predicted that $k = 1$ always holds. Sorli conjectured $k = 1$ after testing large odd numbers N' with $\omega(N') = 8$ for perfection. More recently, Beasley [1] reports that "Dickson has documented Descartes' conjecture as occurring in a letter to Marin Mersenne [on November 15,] 1638, with Frenicle's subsequent observation occurring in 1657".

Holdener [4] presented some conditions equivalent to the existence of odd perfect numbers. In [3], Dris gives some conditions equivalent to the Descartes–Frenicle–Sorli Conjecture.

In this paper we reprove the following result from our previous paper [3] on this topic.

Lemma 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$

We shall use Lemma 1.1 to show the veracity of the following statements:

Theorem 1.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ if and only if*

$$\sigma(n^2) - n^2 = \left(\frac{q-1}{2}\right) \cdot D(n^2).$$

Theorem 1.3. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ if and only if*

$$N = \left(\frac{q(q+1)}{2}\right) \cdot D(n^2).$$

We also use Lemma 1.1 to give a new proof of the following result from Lustig [5]:

Theorem 1.4. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ if and only if*

$$N = \frac{n^2 \sigma(n^2)}{D(n^2)}.$$

All of the proofs given in this note are elementary.

2 The proof of Lemma 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.
Since N is perfect, we have

$$\sigma(q^k)\sigma(n^2) = \sigma(N) = 2N = 2q^k n^2$$

where we have used the divisibility constraint $\gcd(q, n) = 1$ and the fact that σ is multiplicative. It follows that $q^k \mid \sigma(n^2)$ (because $\gcd\left(q^k, \sigma(q^k)\right) = 1$). Hence,

$$\frac{\sigma(n^2)}{q^k} = \frac{\sigma(N/q^k)}{q^k} = \frac{2n^2}{\sigma(q^k)}$$

is an integer.
Consequently, by setting

$$A = \sigma(n^2), \quad B = q^k, \quad C = 2n^2, \quad D = \sigma(q^k),$$

we can use the algebraic identity

$$\frac{A}{B} = \frac{C}{D} = \frac{C-A}{D-B}$$

to show that

$$\frac{\sigma(N/q^k)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})},$$

since $D - B = \sigma(q^k) - q^k = 1 + q + \ldots + q^{k-1} = \sigma(q^{k-1})$. The remaining part is to show that

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})}$$

and this follows easily from

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} = \frac{D(n^2)}{\sigma(q^{k-1})}$$

and the fact that $\gcd\left(q^k, \sigma(q^k)/2\right) = 1$.
This finishes the proof of Lemma 1.1. □

3 The proof of Theorem 1.2

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.
By Lemma 1.1, we have

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} = \frac{D(n^2)}{\sigma(q^{k-1})}.$$

Suppose that $k = 1$. Then we obtain

$$\frac{\sigma(n^2)}{q} = \frac{n^2}{\frac{q+1}{2}} = D(n^2).$$

Consequently, by setting

$$A' = \sigma(n^2), \quad B' = q, \quad C' = n^2, \quad D' = \frac{q+1}{2},$$

we can use the algebraic identity

$$\frac{A'}{B'} = \frac{C'}{D'} = \frac{A' - C'}{B' - D'}$$

to show that

$$D(n^2) = \frac{\sigma(n^2) - n^2}{\frac{q-1}{2}},$$

since $B' - D' = q - (q+1)/2 = (q-1)/2$.
It follows that

$$\sigma(n^2) - n^2 = \left(\frac{q-1}{2}\right) \cdot D(n^2),$$

and this establishes one direction of Theorem 1.2.
Next, suppose that

$$\sigma(n^2) - n^2 = \left(\frac{q-1}{2}\right) \cdot D(n^2).$$

We get that

$$2\left(\sigma(n^2) - n^2\right) = \left(q - 1\right) \cdot \left(2n^2 - \sigma(n^2)\right),$$

which gives

$$2\sigma(n^2) - 2n^2 = 2qn^2 - 2n^2 - q\sigma(n^2) + \sigma(n^2).$$

Collecting like terms and simplifying, we obtain

$$(q+1)\sigma(n^2) = 2qn^2.$$

This implies that

$$I(n^2) = \frac{\sigma(n^2)}{n^2} = \frac{2q}{q+1},$$

from which it follows that

$$I(q^k) = \frac{2}{I(n^2)} = \frac{q+1}{q}.$$

Since q is the Euler prime, we conclude that $k = 1$.
This completes the proof of Theorem 1.2. (Note that $\sigma(n^2) - n^2$ is called the *sum of the aliquot parts* of the non-Euler factor n^2.) □

4 The proof of Theorem 1.3

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.
The proof of the biconditional

$$k = 1 \iff N = \left(\frac{q(q+1)}{2}\right) \cdot D(n^2)$$

follows from the fact that every odd perfect number $N = q^k n^2$ can be written in the form

$$N = \left(\frac{q^k \sigma(q^k)}{2}\right) \cdot \frac{D(n^2)}{\sigma(q^{k-1})},$$

and a proof of this statement follows directly from Lemma 1.1. (Note that one also needs to use the fact that $\gcd\left(q^k, \sigma(q^{k-1})\right) = \gcd\left(\sigma(q^k), \sigma(q^{k-1})\right) = 1$.) □

5 The proof of Theorem 1.4

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.
First, assume that $k = 1$. Then, from Lemma 1.1, we have

$$\frac{\sigma(n^2)}{q} = D(n^2)$$

which implies that

$$\frac{n^2 \sigma(n^2)}{D(n^2)} = qn^2 = N.$$

This establishes one direction of Theorem 1.4.
For the other direction, suppose that

$$q^k n^2 = N = \frac{n^2 \sigma(n^2)}{D(n^2)}.$$

This implies that

$$\frac{\sigma(n^2)}{q^k} = D(n^2).$$

However, by Lemma 1.1 we know that

$$\frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})},$$

from which it follows that

$$\sigma(q^{k-1}) = 1.$$

This means that $q^{k-1} = 1$, which implies that $k - 1 = 0$, whence we finally derive $k = 1$. This concludes the proof of Theorem 1.4. □

6 Further research

Following the proof of Lemma 1.1, one can use the same algebraic trick to derive a plethora of identities linking the various quantities associated with the divisors q^k and n^2 of an odd perfect number $q^k n^2$ with Euler prime q. As the ultimate goal of this research style is to derive a contradiction from among the stringent conditions that an odd perfect number must satisfy, we leave this as a research thrust for other researchers to pursue.

Acknowledgements

The authors are indebted to the anonymous referees whose valuable feedback improved the overall presentation and style of this manuscript.

References

[1] Beasley, B. D. (2013) Euler and the ongoing search for odd perfect numbers, *Proc. of ACMS 19-th Biennial Conference Proceedings*, Bethel University, 29 May–1 June, 2013, pp. 21–31. Available online: http://godandmath.files.wordpress.com/2013/07/acms-2013-proceedings.pdf.

[2] Dickson, L. E. (1971) *History of the Theory of Numbers*, Vol. 1, Chelsea Pub. Co., New York, pp. 3–33.

[3] Dris, J. A. B. (2017) Conditions equivalent to the Descartes–Frenicle–Sorli conjecture on odd perfect numbers, *Notes on Number Theory and Discrete Mathematics*, 23 (2), 12–20.

[4] Holdener, J. A. (2006) Conditions equivalent to the existence of odd perfect numbers, *Math. Mag.*, 79, 389–391.

[5] Lustig, D. (2010) The algebraic independence of the sum of divisors functions, *Journal of Number Theory*, 130, 2628–2633.

[6] Sloane, N. J. A., OEIS sequence A033879 - Deficiency of n, or $2n - \sigma(n)$. Available online: http://oeis.org/A033879.

[7] Sorli, R. M. (2003) *Algorithms in the study of multiperfect and odd perfect numbers*, Ph. D. Thesis, University of Technology, Sydney.