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On products of quartic polynomials over consecutive indices which are perfect squares

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Abstract: Let a be a positive integer. We study the Diophantine equation

$$\prod_{k=1}^{n}(a^2k^4 + (2a - a^2)k^2 + 1) = y^2.$$

This Diophantine equation generalizes a result of Gürel [5] for $a = 2$. We also prove that the product $(2^2 - 1)(3^2 - 1)\dots(n^2 - 1)$ is a perfect square only for the values n for which the triangular number T_n is a perfect square.
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1 Introduction

The study of sequences containing infinitely many squares is a common topic in number theory. Let $\Omega_\mu(n) = (1^\mu + 1)(2^\mu + 1)\dots(n^\mu + 1)$ where $\mu \geq 2$ is an integer. Amdeberhan et al. [1] conjectured that $\Omega_2(n)$ is not a square for any integer $n > 3$. Cilleruelo [3] confirmed this conjecture. Gürel and Kisisel [6] proved that $\Omega_3(n)$ is not a square. Later, an idea due to Zudilin was applied to $\Omega_p(n)$ by Zhang and Wang [9] and to $\Omega_{p^t}(n)$ by Chen et al. [2] for any odd prime p. Fang [4] confirmed another similar conjecture posed by Amdeberhan et al. [1] to the products of quadratic polynomials $\prod_{k=1}^{n}(4k^2+1)$ and $\prod_{k=1}^{n}(2k^2-2k+1)$ are not perfect squares. Yang et al. [8] studied the Diophantine equation $\prod_{k=1}^{n}(ak^2+bk+c)=dy^l$. Gürel [5] proved that the product $\prod_{k=1}^{n}(4k^4+1)$ is a perfect square for infinitely many n.

In this manuscript, we will extend the result of Gürel [5] on the polynomial $4k^4+1$ to the polynomial $P_a(k)=a^2k^4+(2a-a^2)k^2+1$, where a is a positive integer. Next, we prove the product $(2^2-1)(3^2-1)\dots(n^2-1)$ is a perfect square only for the values n for which the triangular number T_n is a perfect square.

2 Main results

Let a be a positive integer and $P_a(x)=a^2x^4+(2a-a^2)x^2+1$. Denote $\mathcal{X}_a(n)$ is the product of first n consecutive values of the $P_a(n)$, i.e.,

$$\mathcal{X}_a(n)=P_a(1)P_a(2)\dots P_a(n).$$

Lemma 1. *$\mathcal{X}_a(n)$ is a square if and only if an^2+an+1 is a square.*

Proof. Let $f(x)=ax^2-ax+1$. Then $f(x+1)=ax^2+ax+1$ and

$$P_a(x)=a^2x^4+(2a-a^2)x^2+1=(ax^2-ax+1)(ax^2+ax+1)=f(x)f(x+1).$$

We have

$$\mathcal{X}_a(n)=\prod_{k=1}^{n}P_a(k)=\prod_{k=1}^{n}f(k)f(k+1)=\left(\prod_{k=2}^{n}f(k)\right)^2 f(1)f(n+1).$$

Since $f(1)=1$, it follows that $\mathcal{X}_a(n)$ is a square if and only if $f(n+1)$ is a square. □

Consider $4n^2+4n+1=(2n+1)^2$, we obtain that

$$\prod_{k=1}^{n}(16k^4-8k^2+1) \text{ is a perfect square for all } n.$$

Theorem 1. *Let a,d,n be positive integers with $a=d^2\neq 4$. Suppose $p=\left\lfloor\frac{d+1}{2}\right\rfloor$. Then $\mathcal{X}_a(n)$ is not a perfect square for $n>\frac{p^2-2p}{d^2-2dp+2d}$.*

Proof. By Lemma 1, the problem is reduced to finding square values of $f(n+1)$, i.e., finding integer solutions to the following equation,

$$an^2+an+1=m^2. \tag{1}$$

We see that

$$an^2+an+1=d^2n^2+d^2n+1<(dn+p)^2.$$

Assume $n>(p^2-2p)/(d^2-2dp+2d)$.

If d is even, we get that $p = \dfrac{d}{2}$ and $n > \dfrac{d-4}{8}$, so

$$d^2n^2 + d^2n + 1 > (dn + p - 1)^2.$$

And if d is odd, we have that $p = (d+1)/2$ and $n > (d^2 - 2d - 3)/4d$, so

$$d^2n^2 + d^2n + 1 > (dn + p - 1)^2.$$

We obtain that

$$(dn + p - 1)^2 < d^2n^2 + d^2n + 1 < (dn + p)^2.$$

Since there is no perfect square between two consecutive perfect squares, $\mathcal{X}_a(n)$ is not a perfect square for $n > \dfrac{p^2 - 2p}{d^2 - 2dp + 2d}$ and $a = d^2 \neq 4$. □

For a not a perfect square, we conjecture that the Diophantine equation (1) has infinitely many solutions. The case $a = 2$ has been shown in [5]. In the next theorem, we will only show the case $3 \leq a \leq 13$.

Theorem 2. *Let $3 \leq a \leq 13$ be not a perfect square. Then $\mathcal{X}_a(n)$ is a perfect square for infinitely many n.*

Proof. It suffices to find the integer solutions of (1).

- Case $a = 3$, we consider

$$3n^2 + 3n + 1 = m^2. \tag{2}$$

  We see that $(7, 13)$ is a solution of (2). For each solution (x, y) of (2), the map sends (x, y) to $(7x + 4y + 3, 12x + 7y + 6)$, which gives another solution of (2). It can be verified that

$$\begin{aligned} 3(7x + 4y + 3)^2 + 3(7x + 4y + 3) + 1 &= 147x^2 + 48y^2 + 168xy + 147x + 84y + 37 \\ &= (3x^2 + 3x + 1) - y^2 + (12x + 7y + 6)^2 \\ &= (12x + 7y + 6)^2. \end{aligned}$$

  Therefore, the equation (2) has infinitely many distinct solutions.
  (*Note: If (n_i, m_i) are all solutions of* (2)*, the sequence $\{n_i\}$ satisfies $n_i = 14n_{i-1} - n_{i-2} + 6$, where $n_0 = 0$, $n_1 = 7$, see* A001921 *in* [7]*, and the sequence $\{m_i\}$ is* A001570 *in* [7].)

- Case $a = 5$, we consider

$$5n^2 + 5n + 1 = m^2. \tag{3}$$

  Clearly, $(8, 19)$ is a solution of (3). For each solution (x, y) of (3), the map sends (x, y) to $(9x + 4y + 4, 20x + 9y + 10)$, which gives another solution of (3).
  (*Note: If (n_i, m_i) are all solutions of* (3)*, then $n_i = (F_{6i+3} - 2)/4$ and $m_i = (F_{6n+4} + F_{6n+2})/4$, where F_i is i^{th} Fibonacci number, see* A053606 *and* A049629 *in* [7].)

- Case $a = 6$, we consider
$$6n^2 + 6n + 1 = m^2. \tag{4}$$
Clearly, $(4, 11)$ is a solution of (4). For each solution (x, y) of (4), the map sends (x, y) to $(5x + 2y + 2, 12x + 5y + 6)$, which gives another solution of (4).
(*Note: If* (n_i, m_i) *are all solutions of* (4)*, then* $n_i = 11n_{i-1} - 11n_{i-2} + n_{i-3}$*, where* $n_0 = 0$, $n_1 = 4$ *and* $n_2 = 44$*, see* $A105038$ *in* [7] *and* m_i *is* $A054320$ *in* [7].)

- Case $a = 7$, we consider
$$7n^2 + 7n + 1 = m^2. \tag{5}$$
Clearly, $(15, 41)$ is a solution of (5). For each solution (x, y) of (5), the map sends (x, y) to $(127x + 48y + 63, 336x + 127y + 168)$, which gives another solution of (5).
(*Note: If* (n_i, m_i) *are all solutions of* (5)*, then* $n_i = 254n_{i-2} - n_{i-4} + 126$*, where* $n_0 = 0$, $n_1 = 15, n_2 = 111$ *and* $n_3 = 3936$*, see* $A105051$ *or* $A105040$ *in* [7].)

- Case $a = 8$, we consider
$$8n^2 + 8n + 1 = m^2. \tag{6}$$
Clearly, $(2, 7)$ is a solution of (6). For each solution (x, y) of (6), the map sends (x, y) to $(3x + y + 1, 8x + 3y + 4)$, which gives another solution of (6).
(*Note: If* (n_i, m_i) *are all solutions of* (6)*, the sequences* $\{n_i\}$ *and* $\{m_i\}$ *are respectively* $A053141$ *and* $A002315$ *in* [7].)

- Case $a = 10$, we consider
$$10n^2 + 10n + 1 = m^2. \tag{7}$$
Clearly, $(3, 11)$ is a solution of (7). For each solution (x, y) of (7), the map sends (x, y) to $(19x + 6y + 9, 60x + 19y + 30)$, which gives another solution of (7).
(*Note: If* (n_i, m_i) *are all solutions of* (7)*, the sequence* $\{n_i\}$ *is* $A222390$ *in* [7].)

- Case $a = 11$, we consider
$$11n^2 + 11n + 1 = m^2. \tag{8}$$
Clearly, $(39, 131)$ is a solution of (8). For each solution (x, y) of (8), the map sends (x, y) to $(199x + 60y + 99, 660x + 199y + 330)$, which gives another solution of (8).
(*Note: If* (n_i, m_i) *are all solutions of* (8)*, the sequences* $\{n_i\}$ *and* $\{m_i\}$ *are respectively* $A105838$ *and* $A105837$ *in* [7].)

- Case $a = 12$, we consider
$$12n^2 + 12n + 1 = m^2. \tag{9}$$
Clearly, $(1, 5)$ is a solution of (9). For each solution (x, y) of (9), the map sends (x, y) to $(7x + 2y + 3, 24x + 7y + 12)$, which gives another solution of (9).
(*Note: If* (n_i, m_i) *are all solutions of* (9)*, the sequences* $\{n_i\}$ *and* $\{m_i\}$ *are respectively* $A061278$ *and* $A001834$ *in* [7].)

- Case $a = 13$, we consider
$$13n^2 + 13n + 1 = m^2. \tag{10}$$
Clearly, $(7, 27)$ is a solution of (10). For each solution (x, y) of (10), the map sends (x, y) to $(649x + 180y + 324, 234x + 649y + 1170)$, which gives another solution of (10). (*Note: If* (n_i, m_i) *are all solutions of* (10)*, the sequence* $\{n_i\}$ *is* $A104240$ *in* [7].)

Therefore, $\mathcal{X}_a(n)$ is a perfect square for infinitely many n, where a is not a perfect square. □

For $a > 13$ not a perfect square, the authors consider that $\mathcal{X}_a(n)$ is a perfect square for infinitely many n. We can find the linear map for each solution (n_0, m_0) that gives another solution of (1). But the authors will leave this problem to the interested reader.

From Theorems 1 and 2, we give the following examples for $a = 1, 3, 5$.

(1) $\prod_{k=1}^{n}(k^4 + k^2 + 1)$ is not a square.

(2) $\prod_{k=1}^{n}(9k^4 - 3k^2 + 1)$ and $\prod_{k=1}^{n}(25k^4 - 15k^2 + 1)$ are perfect squares for infinitely many n.

Next, we give analogue of $\Omega_2(n)$ for the product $(2^2 - 1)(3^2 - 1)\dots(n^2 - 1)$.

Theorem 3. *The product* $\prod_{k=2}^{n}(k^2 - 1)$ *is a perfect square if and only if the triangular number* T_n *is a perfect square for* $n > 1$.

Proof. The triangular number T_n is a number obtained by adding all positive integers less than or equal to a given positive integer n, i.e., $T_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. We have

$$\begin{aligned}\prod_{k=2}^{n}(k^2 - 1) &= \prod_{k=2}^{n}(k - 1)(k + 1)\\ &= \left(\prod_{k=3}^{n-1} k\right)^2 2n(n + 1)\\ &= \left(\prod_{k=3}^{n-1} k\right)^2 4T_n.\end{aligned}$$

Thus, this product is a square if and only if T_n is a square. □

The triangular number T_n is a square (see $A001108$ in [7]) when the value of n is $1, 8, 49, 288, 1681, 9800, 57121, 332928, 1940449, 11309768, \dots$.

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