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On the Iyengar–Madhava Rao–Nanjundiah inequality and its hyperbolic version

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Abstract: We provide a new proof of the trigonometric inequality obtained by K. S. K. Iyengar, B. S. Madhava Rao and T. S. Nanjundiah in 1945, and offer also the hyperbolic version of this result. Certain related results are pointed out, too.
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1 Introduction

Inequalities for trigonometric and hyperbolic functions have been studied extensively in the last 20 years. Among such inequalities we mention the famous inequalities by Adamović–Mitrinović, which states that

$$\frac{\sin x}{x} > \sqrt[3]{\cos x} \tag{1.1}$$

for any $x \in \left(0, \frac{\pi}{2}\right)$; and the Lazarović inequality

$$\frac{\sinh x}{x} > \sqrt[3]{\cosh x} \tag{1.2}$$

for any $x > 0$. For a basic reference to (1.1) and (1.2), see [2]. See also [4, 6].

In a little known paper [1] Iyengar, Madhava Rao and Nanjundiah have proved that

$$\frac{\sin x}{x} > \cos\left(\frac{x}{\sqrt{3}}\right) > \sqrt[3]{\cos x} \tag{1.3}$$

for $x \in \left(0, \frac{\pi}{2}\right)$.

In a recent paper [7], we have discovered (1.3) by an application of the famous Hadamard integral inequality. By using the same method, we have proved also the hyperbolic analogue of (1.3), namely

$$\frac{\sinh x}{x} > \cosh\left(\frac{x}{\sqrt{3}}\right) > \sqrt[3]{\cosh x} \tag{1.4}$$

for any $x > 0$. We note that, the second inequality of (1.4) follows by a comparison of certain series inequalities.

Iyengar, Madhava Rao and Nanjundiah have considered by other arguments, the signs of the functions

$$\frac{\sin x}{\cos \mu x} - x = a(x)$$

and showed that for $\mu_1 = \frac{1}{\sqrt{3}} = 0.577\ldots$ one has $a(x) > 0$, and for $\mu_2 = \frac{2}{\pi} \arccos \frac{2}{\pi} = 0.560\ldots$ we have $a(x) < 0$, therefore,

$$\frac{\sin x}{x} < \cos \mu_2 x, \tag{1.5}$$

$x \in \left(0, \frac{\pi}{2}\right)$ is also true. In fact, μ_1 and μ_2 above are the best constants such that these inequalities are valid.

The aim of this paper is to give a new proof of the above results, and also a new proof of the first inequality of (1.4).

2 Main results

Our method (which we discovered in fact in 2010 [5]) is based on the auxiliary functions

$$a_1(x) = \arccos\left(\frac{\sin x}{x}\right), x \in \left(0, \frac{\pi}{2}\right)$$

and

$$a_2(x) = \operatorname{arccosh}\left(\frac{\sinh x}{x}\right), x > 0.$$

In the proof, the well-known "L'Hospital rule for monotonicity" lemma will be used, as follows (see e.g. [3]).

Lemma 2.1. *Suppose that $f, g : (a, b) \to \mathbb{R}$ are differentiable on (a, b), and $g'(x) \neq 0$ for $x \in (a, b)$. Suppose that $f(a+) = g(a+) = 0$. Then, if $\frac{f'}{g'}$ is monotone on (a, b), then $\frac{f}{g}$ is also monotone, having the same type of monotonicity.*

Theorem 2.1. *The function* $f_1(x) = \dfrac{1}{x}\arccos\left(\dfrac{\sin x}{x}\right)$ *is strictly decreasing for* $x \in \left(0, \frac{\pi}{2}\right)$.

Proof. We will use successively Lemma 2.1. Put $f(x) = \arccos\left(\dfrac{\sin x}{x}\right)$, $g(x) = x$. Then $\dfrac{f'(x)}{g'(x)} = \dfrac{\sin x - x\cos x}{x\sqrt{x^2 - \sin^2 x}}$, and $f(0+) = g(0+) = 0$.

To avoid the radicals, we will consider $\left(\dfrac{f'(x)}{g'(x)}\right)^2 = \dfrac{(\sin x - x\cos x)^2}{x^2(x^2 - \sin^2 x)} = \dfrac{u(x)}{v(x)}$. Here $u(0+) = v(0+) = 0$ and $\dfrac{u'(x)}{v'(x)} = \dfrac{\sin^2 x - x\sin x\cos x}{2x^2 - \sin^2 x - x\sin x\cos x}$ clearly $v'(x) \neq 0$ as $x > \sin x$ and $x > \sin x\cos x$. As $u'(x) = v'(x) = 0$, consider $\dfrac{u''(x)}{v''(x)} = \dfrac{\sin x\cos x - x + 2x\sin^2 x}{3x - 3\sin x\cos x + 2x\sin^2 x.}$ By continuing, we obtain finally

$$\frac{u'''(x)}{v'''(x)} = \frac{\cos^2 x - \sin^2 x - 1 + 2\sin^2 x + 4x\sin x\cos x}{3 - 3\cos^2 x + 3\sin^2 x + 2\sin^2 x + 4x\sin x\cos x}$$

$$= \frac{4x\sin x\cos x}{8\sin^2 x + 4x\sin x\cos x} = \frac{x\cos x}{2\sin x + x\cos x} = \frac{A(x)}{B(x)}.$$

Now, remark that $\dfrac{B(x)}{A(x)} = \dfrac{2\tan x}{x} + 1$, and it is well-known that $\dfrac{\tan x}{x}$ is strictly increasing on $\left(0, \frac{\pi}{2}\right)$.

This means that $\dfrac{A(x)}{B(x)}$ is strictly decreasing, and the proof of Theorem 2.1. is finished. □

Theorem 2.2. *The function* $f_2(x) = \dfrac{1}{x}\operatorname{arccosh}\left(\dfrac{\sinh x}{x}\right)$, $x > 0$ *is strictly increasing on* $(0, \infty)$.

Proof. Apply again Lemma 2.1 for $f(x) = \operatorname{arccosh}\left(\dfrac{\sinh x}{x}\right)$ and $g(x) = x$. As $f(0+) = g(0+) = 0$ and $\dfrac{f'(x)}{g'(x)} = \dfrac{x\cosh x - \sinh x}{x\sqrt{\sinh^2 x - x^2}}$, we will consider again

$$\left(\frac{f'(x)}{g'(x)}\right)^2 = \frac{(x\cosh x - \sinh x)^2}{x^2(\sinh^2 x - x^2)} = \frac{u(x)}{v(x)}$$

and all can be repeated as in the proof of Theorem 2.1. One arrives finally to the function $\dfrac{A(x)}{B(x)} = \dfrac{x\cosh x}{2\sinh x + x\cosh x}$ and remark that in this case $\dfrac{B(x)}{A(x)} = \dfrac{2\tanh x}{x} + 1$, which is strictly decreasing, as the function $x \to \dfrac{\tanh x}{x}$ is known to be strictly decreasing for $x > 0$. Thus $\dfrac{A(x)}{B(x)}$ will be strictly increasing, so $\left(\dfrac{f'(x)}{g'(x)}\right)^2$ and $\left(\dfrac{f(x)}{g(x)}\right)^2$ will be strictly increasing, too. This last assertion implies also that $\dfrac{f(x)}{g(x)}$ is strictly increasing, too. □

Theorem 2.3. *The function* $f_3(x) = \arccos\left(\dfrac{\sin x}{x}\right) - x$ *is strictly decreasing on* $\left(0, \frac{\pi}{2}\right)$.

Proof. Remark that $f_3(x) = x(f_1(x) - 1)$, where $f_1(x)$ is the decreasing function of Theorem 2.1. On the other hand, $f_1(x) - 1 = h(x)$ is also decreasing, but $h(x) < 0$. This follows by $\frac{\sin x}{x} > \cos x$ so the $\arccos$ function being decreasing, we get $f_1(x) < 1$. Now $f_3(x) = xh(x)$, so $f_3'(x) = h(x) + xh'(x) < 0$ as $h(x) < 0$ and $h'(x) < 0, x > 0$. □

Theorem 2.4. *The function* $f_4(x) = \operatorname{arccosh}\left(\frac{\sinh x}{x}\right) - x$ *is strictly decreasing.*

Proof. The method of proof of Theorem 2.3 cannot be applied here, since $f_4(x) = x(f_2(x) - 1)$, where $f_2(x)$ is the strictly increasing function of Theorem 2.2. Now, $f_2(x) - 1 < 0$ as $\frac{\sinh x}{x} < \cosh x$ and the function $\operatorname{arccosh}$ is strictly increasing. By letting $k(x) = f_2(x) - 1 < 0$, strictly increasing and $f_4'(x) = k(x) + xk'(x)$, where $k(x) < 0$, but $k'(x) > 0$.

We will compute

$$f_4'(x) = \frac{x \cosh x - \sinh x - x\sqrt{\sinh^2 x - x^2}}{x\sqrt{\sinh^2 x - x^2}}.$$

Now $b(x) = x \cosh x - \sinh x - x\sqrt{\sinh^2 x - x^2} < 0$ is equivalent to $(x \cosh x - \sinh x)^2 < x^2(\sinh^2 x - x^2)$, or $c(x) = \sinh^2 x - 2x \sinh x \cosh x + x^2 + x^4 < 0$. One has $c'(x) = 4x(x^2 - \sinh^2) < 0$, after some elementary computations. Thus it follows $c(x) < c(0+) = 0$, implying $b(x) < 0$, i.e. $f_4'(x) < 0$. This finishes the proof of the theorem. □

Corollary 2.1. *The best constants* $a, b > 0$ *such that*

$$\cos bx < \frac{\sin x}{x} < \cos ax, \tag{2.1}$$

for $x \in \left(0, \frac{\pi}{2}\right)$, *are* $b = \frac{1}{\sqrt{3}}$, $a = \frac{2}{\pi} \arccos \frac{2}{\pi}$.

Proof. By Theorem 2.1 one has $f_1\left(\frac{\pi}{2}\right) < f_1(x) < f_1(0+)$. As $f_1\left(\frac{\pi}{2}\right) = \frac{2}{\pi} \arccos \frac{2}{\pi}$, and $f_1(0+) = \frac{1}{\sqrt{3}}$ (we will prove below), and the function $\cos x$ being strictly decreasing, inequality (2.1) follows with best constants $a = f_1\left(\frac{\pi}{2}\right), b = f_1(0+)$. Now, to compute $b = \lim_{x\to 0} f_1(x)$, put $y = \frac{\sin x}{x}$ and as $y \to 1$ as $x \to 0+$, remark that $\frac{\arccos y}{\sin(\arccos y)} \to 1$ as $z = \arccos y \to 0$ and $\frac{z}{\sin z} \to 1$. Put $\sin(\arccos y) = \sqrt{1 - y^2} = \sqrt{1 - \frac{\sin^2 x}{x^2}}$. Therefore, as $\frac{\arccos y}{x} = \frac{\arccos y}{\sin(\arccos y)} \cdot \frac{\sin(\arccos y)}{x}$, it is sufficient to compute the limit of $\frac{1}{x}\sqrt{1 - \frac{\sin^2 x}{x^2}}$, or avoiding the radicals, the limit of $\sqrt{\frac{x^2 - \sin^2 x}{x^4}}$.

Now it is immediate by L'Hospital's rule that $\frac{x^2 - \sin^2 x}{x^4} \to \frac{1}{3}$, as $x^2 - \sin^2 x = (x - \sin x)(x + \sin x)$, and $\frac{x + \sin x}{x} \to 2$, $\frac{x - \sin x}{x^3} \to \frac{1}{6}$, so the limit follows. □

Corollary 2.2. *The best constants* c, d *such that*

$$\cos(x+c) < \frac{\sin x}{x} < \cos(x+d) \tag{2.2}$$

are for $x \in \left(0, \frac{\pi}{2}\right)$, $c = 0, d = \arccos\frac{2}{\pi} - \frac{\pi}{2} = -0.689\ldots$

Proof. Applying Theorem 2.3, we get $f_3\left(\frac{\pi}{2}\right) < f_3(x) < f_3(0+)$. Now, $f_3\left(\frac{\pi}{2}\right) = \arccos\frac{2}{\pi} - \frac{\pi}{2}$, while $f_3(0+) = 0$. Thus we get $d < \arccos\left(\frac{\sin x}{x}\right) - x < c$, and applying the decreasing function $\cos x$, we get (2.2). □

Corollary 2.3. *The best constant* $k > 0$ *such that*

$$\frac{\sinh x}{x} > \cosh kx, \tag{2.3}$$

for $x > 0$ *is* $k = \frac{1}{\sqrt{3}}$.

Proof. By Theorem 2.2 we get $f_2(x) > f_2(0+)$. Now, it follows by the same lines as in the proof of Corollary 2.1 that $f_2(0+) = \lim_{x\to 0+} \frac{1}{x} \operatorname{arccosh}\left(\frac{\sinh x}{x}\right) = \frac{1}{\sqrt{3}}$. As the function $\cosh x$ is strictly increasing, relation (2.3) follows. □

Remark 2.1. *If* $x \in (0, x_0)$, *the the best constant* $l > 0$ *such that*

$$\frac{\sinh x}{x} < \cosh lx, \tag{2.4}$$

$x > 0$, *will be*

$$l = \operatorname{arccosh}\left(\frac{\sinh x_0}{x_0}\right)$$

Corollary 2.4. *The best constant* s *such that*

$$\frac{\sinh x}{x} < \cosh(x+s) \tag{2.5}$$

for $x > 0$, *is* $s = 0$.

Proof. By Theorem 2.4 one has $f_4(x) < f_4(0+) = s$, so $\operatorname{arccosh}\left(\frac{\sinh x}{x}\right) < x + s$, implying $\frac{\sinh x}{x} < \cosh(x+s)$, with best $s = 0$. □

Remark 2.2. *If* $x \in (0, x_0)$, *the best constant* p *such that*

$$\frac{\sinh x}{x} > \cosh(x+p) \tag{2.6}$$

for $x > 0$, *is* $p = \operatorname{arccosh}\frac{\sinh x_0}{x_0} - x_0$.

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