

# Relationships between Fibonacci-type sequences and Golden-type ratios

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**Abstract:** The classical Fibonacci sequence is defined so that the first two terms are each equal to 1, and each term after this is the sum of the two terms immediately before it. The golden ratio is the ratio of the longer to shorter side of a rectangle with the property that if we remove a square from the rectangle such that the remainder is also a rectangle, that the old and new rectangles are proportional. Johannes Kepler showed that if we take the sequence of ratios of consecutive Fibonacci numbers, the limit of this sequence is the golden ratio [5]. In this paper, we give a higher dimension extension of Fibonacci sequences and golden ratios and provide a connection between the two.

**Keywords:** Fibonacci sequence, Golden ratio.

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## 1 Fibonacci-type sequences

Let  $m \in \mathbb{Z}^+$  such that  $m \geq 2$ , and let  $k \in \mathbb{Z}^+$  such that  $1 \leq k < m$ . Let  $a_1 = \cdots = a_m = 1$ , and for each  $i \in \mathbb{Z}^+$  such that  $i > m$ , let  $a_i = a_{i-k} + a_{i-m}$ . We will refer to this sequence as an  **$m$ -dimensional Fibonacci-type sequence with index  $k$** .

**Lemma 1.1.** *Every Fibonacci-type sequence is non-decreasing.*

*Proof.* If  $i \leq m$ , then  $a_i = a_{i-1} = 1$ , so the lemma is true for the first  $m$  terms. Now let  $i > m$ , and suppose  $a_j \geq a_{j-1}$  for all  $j \leq i$ . Then

$$\begin{aligned}
a_{i+1} &= a_{i-k+1} + a_{i-m+1} \\
&\geq a_{i-k} + a_{i-m} \\
&= a_i.
\end{aligned}$$

Hence the proof.  $\square$

Let  $\{a_i\}$  be a Fibonacci-type sequence. For each  $i \in \mathbb{Z}^+$ , let  $R_i = \frac{a_{i+1}}{a_i}$ . From a result by Szczyrba in [4], the sequence  $\{R_i\}$  will have a limit if the greatest common divisor of  $k$  and  $m$  is 1. If the limit exists, we will call it the ratio limit of the sequence and denote it  $R$ .

**Theorem 1.2.** *If the ratio limit  $R$  of a Fibonacci-type sequence exists, then it is the unique solution to the equation  $x^m - x^{m-k} - 1 = 0$  on the interval  $(1, \infty)$ .*

*Proof.* Suppose  $R$  exists. Then the following is true.

$$\begin{aligned}
a_{i+1} &= a_{i-m+1} + a_{i-k+1} \\
\frac{a_{i+1}}{a_i} &= \frac{a_{i-m+1}}{a_i} + \frac{a_{i-k+1}}{a_i} \\
\frac{a_{i+1}}{a_i} &= \frac{a_{i-m+1}}{a_{i-m+2}} \cdots \frac{a_{i-1}}{a_i} + \frac{a_{i-k+1}}{a_{i-k+2}} \cdots \frac{a_{i-1}}{a_i} \\
R_i &= \frac{1}{R_{i-m+1}} \cdots \frac{1}{R_{i-1}} + \frac{1}{R_{i-k+1}} \cdots \frac{1}{R_{i-1}} \\
R &= \frac{1}{R^{m-1}} + \frac{1}{R^{k-1}} \\
R^m &= 1 + R^{m-k} \\
R^m - R^{m-k} - 1 &= 0.
\end{aligned}$$

Next, we wish to show that there is only one solution to the above equation that could be the ratio limit. Since  $a_i \geq a_{i-1}$  for all  $i$ , we know that  $R \geq 1$ . Let  $f(x) = x^m - x^{m-k} - 1$ . Note that  $f(1) = -1 \neq 0$ , so  $R > 1$ . Next, consider the derivative of  $f$ :

$$\begin{aligned}
f'(x) &= mx^{m-1} - (m-k)x^{m-k-1} \\
&= x^{m-k-1}(mx^k - (m-k)).
\end{aligned}$$

Then  $f'(x) = 0$  only when  $x = 0 < 1$  or  $x^k = \frac{m-k}{m} < 1$ . So  $f$  is one-to-one on the interval  $(1, \infty)$ , and thus the value of  $R$  is unique.  $\square$

If  $m = 2$  and  $k = 1$ , then this Fibonacci-type sequence is the standard Fibonacci sequence. There are also two Fibonacci-type sequences in dimension 3, which are known as the Narayana's cows sequence and the Padovan sequence. The Padovan sequence has a ratio limit that is referred to as the plastic number [3]. For a given dimension  $m$ , if  $k = m - 1$ , we obtain ratio limits that Marohnić and Strmečki have defined as harmonious numbers [2], and that Krčadinac refers to as the  $k$ -th lower Fibonacci sequence [1].

$m$	$k$	Sequence	Ratio limit
2	1	1, 1, 2, 3, 5, 8, 13, ... (Fibonacci sequence)	1.1680...
3	1	1, 1, 1, 2, 3, 4, 6, 9, 13, 19, ... (Narayana's cows sequence)	1.4655...
3	2	1, 1, 1, 2, 2, 3, 4, 5, 7, 9, ... (Padovan sequence)	1.3247...
4	1	1, 1, 1, 1, 2, 3, 4, 5, 7, 10, 14, 19, ...	1.3802...
4	2	1, 1, 1, 1, 2, 2, 3, 3, 5, 5, 8, 8, ...	none
4	3	1, 1, 1, 1, 2, 2, 2, 3, 4, 4, 5, 7, 8, ...	1.2207...

## 2 Square base extensions and Fibonacci-type sequences

Define a box  $A$  in  $\mathbb{R}^m$  as a product  $\prod_{i=1}^m [0, s_i]$ , where  $s_i$  is a positive real number for each  $i$ . Each  $s_i$  will be called a side length of  $A$ .

Define a square base extension  $B$  of  $A$  in dimension  $j$  and index  $k$ , denoted  $E_{j,k}(A)$ , as the product  $\prod_{i=1}^m [c_i, d_i]$ , where  $[c_j, d_j] = [s_j, s_j + s_k]$  and  $[c_i, d_i] = [0, s_i]$  if  $i \neq j$ . Notice that the projection of  $B$  onto its  $j$ -th and  $k$ -th coordinates gives a square of side length  $s_k$ . Moreover,  $A \cup B$  is a box in  $\mathbb{R}^m$ .

Let  $A_1 = [0, 1]^m$  and let  $a_1 = \dots = a_m = 1$  denote the side lengths of  $A_1$ . For each  $i \in \mathbb{Z}^+$ , let  $B_i$  be the square base extension of  $A_i$  in dimension  $((i-1) \bmod m) + 1$  with index  $((m-k+i-1) \bmod m) + 1$ . Let  $A_{i+1} = A_i \cup B_i$ , and let  $a_i + m$  be the new side length of  $A_{i+1}$  (the side length in the  $((i-1) \bmod m) + 1$ -th dimension). Then  $a_n = a_{n-m} + a_{n-k}$ , with the first  $m$  terms equal to 1, so this is an  $m$ -dimensional Fibonacci-type sequence of index  $k$ .

The following corollary is a result of our choices of index in each step and the result of Lemma 1.1.

**Corollary 2.1.** *In each step of the process that defines the sequence of boxes  $\{A_i\}$ , we have:*

1. *The side being extended is the shortest side,*
2. *The side is being extended by the length of the  $k$ -th longest side, and*
3. *The newly extended side is the longest side.*

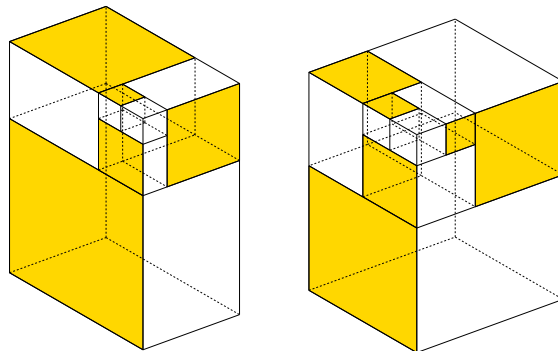


Figure 1. The Narayana (left) and Padovan (right) sequences, represented using square base extensions

### 3 Square base reductions and Golden-type ratios

Let  $A$  be a box in  $\mathbb{R}^m$ . If  $s_j > s_k$ , define a square base reduction  $B$  of  $A$  in dimension  $j$  with index  $k$ , denoted  $R_{j,k}(A)$ , as the product  $\prod_{i=1}^m [c_i, d_i]$ , where  $[c_j, d_j] = [s_j - s_k, s_j]$  and  $[c_i, d_i] = [0, s_i]$  if  $i \neq j$ . Note that the projection of  $B$  onto its  $j$ -th and  $k$ -th coordinates is a square, and also note that  $\overline{A - B}$  is a box in  $\mathbb{R}^n$ .

Let  $A = \prod_{i=1}^m [0, b_i]$  be a box in  $\mathbb{R}^m$  such that  $b_i < b_{i+1}$  for each  $i$  in  $1, \dots, m-1$ , and  $b_1 = 1$ . Let  $B$  be the square base reduction with dimension  $m$  and index  $m-k$ , and let  $A' = \overline{A - B}$ . Then  $A' = \prod_{i=1}^m [0, r_i]$  for some  $r_1, \dots, r_m > 0$ . Suppose  $A'$  is proportional to  $B$ . That is, there exists a bijection  $h$  on  $\{1, \dots, m\}$  and  $b > 0$  such that  $b_i = b \cdot r_{h(i)}$  for each  $i$  in  $1, \dots, m$ . Define the  $m$ -dimensional golden-type ratio with index  $k$  to be this proportion  $b$ . Note that  $A$  and  $A'$  have all sides lengths equal with one exception:  $b_m$  is present only in  $A$ , and  $b_m - b_{m-k}$  is only a length of  $A'$ . Moreover,  $b_m - b_{m-k}$  must be the shortest side of  $A'$ ; otherwise 1 would be the shortest side of both boxes and so  $c$  must equal 1, a contradiction. Listing both sets of sides in decreasing order, we have the following:

$b_m$	$b_{m-1}$	$\cdots$	$b_3$	$b_2$	$b_1 (= 1)$
$b_{m-1}$	$b_{m-2}$	$\cdots$	$b_2$	$b_1 (= 1)$	$b_m - b_{m-k}$

Then  $\frac{b_{i+1}}{b_i} = b$  for any  $i$  in  $1, \dots, m-1$ . In particular,  $b_2 = b$  since  $b_1 = 1$ .

**Lemma 3.1.** *For the terms defined above,  $b_i = b^{i-1}$  for each  $i$  in  $1, \dots, m-1$ .*

*Proof.* Note that  $b_1 = 1 = b^0$ . Now let  $i > 1$  and suppose that for each  $j \leq i$  we have  $b_j = b^{j-1}$ . Then

$$\begin{aligned} \frac{b_{i+1}}{b_i} &= \frac{b_i}{b_{i-1}} \\ \frac{b_{i+1}}{b^{i-1}} &= \frac{b^{i-1}}{b^{i-2}} \\ b_{i+1} &= b^i. \end{aligned}$$

So each dimension of  $A$  is a power of  $b$ . □

**Theorem 3.2.** *The  $m$ -dimensional golden-type ratio with index  $k$  is equal to the ratio limit of the  $m$ -dimensional Fibonacci-type sequence with index  $k$ , when the ratio limit exists.*

*Proof.* To find the value of  $b$ , we use the last two columns of the table above.

$$\begin{aligned} \frac{b_2}{b_1} &= \frac{b_1}{b_m - b_{m-k}} \\ \frac{b}{1} &= \frac{1}{b^{m-1} - b^{m-k-1}} \\ b^m - b^{m-k} &= 1 \\ b^m - b^{m-k} - 1 &= 0. \end{aligned}$$

Since the terms were listed in descending order, we know that  $b > 1$ . This is the same equation we used to solve for  $R$  previously, so the solution is the same as that of  $R$  (and is thus unique). □

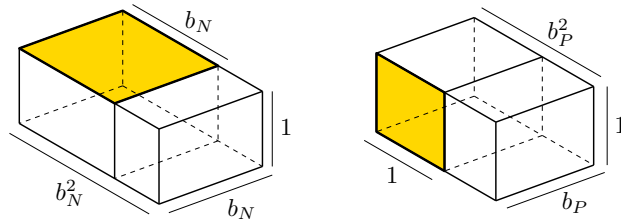


Figure 2. The ratio limits of the Narayana (left) and Padovan (right) sequences, represented using square base reduction

## References

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