Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 2, 6–20
DOI: 10.7546/nntdm.2018.24.2.6-20

The ternary Goldbach problem with prime numbers of a mixed type

S. I. Dimitrov

Faculty of Applied Mathematics and Informatics, Technical University of Sofia
8 St. Kliment Ohridski Blvd., 1756 Sofia, Bulgaria
e-mail: sdimitrov@tu-sofia.bg

Received: 21 December 2017 **Revised:** 21 May 2018 **Accepted:** 25 May 2018

Abstract: In the present paper we prove that every sufficiently large odd integer N can be represented in the form

$$N = p_1 + p_2 + p_3\,,$$

where p_1, p_2, p_3 are primes, such that $p_1 = x^2 + y^2 + 1,\ p_2 = [n^c]$.

Keywords: Goldbach problem, Prime numbers, Circle method.
2010 Mathematics Subject Classification: 11N36, 11P32.

1 Notations

Let N be a sufficiently large odd integer. The letter p, with or without subscript, will always denote prime numbers. Let $A > 100$ be a constant. By ε we denote an arbitrary small positive number, not the same in all appearances. The relation $f(x) \ll g(x)$ means that $f(x) = \mathcal{O}(g(x))$. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n\,(k)$. As usual $e(t)$=exp($2\pi it$). We denote by (d, q), $[d, q]$ the greatest common divisor and the least common multiple of d and q respectively. As usual $\varphi(d)$ is Euler's function; $\mu(d)$ is Möbius' function; $r(d)$ is the number of solutions of the equation $d = m_1^2 + m_2^2$ in integers m_j; $\chi(d)$ is the non-principal character modulo 4 and $L(s, \chi)$ is the corresponding Dirichlet's L-function. By c_0 we denote some positive number, not necessarily the same in different occurrences. Let c be a real constant such that $1 < c < 73/64$.

Denote

$$\gamma = 1/c\,; \tag{1}$$

$$D = \frac{N^{1/2}}{(\log N)^A}\,; \tag{2}$$

$$\psi(t) = \{t\} - 1/2\,; \tag{3}$$

$$\theta_0 = \frac{1}{2} - \frac{1}{4}e\log 2 = 0.0289...\,; \tag{4}$$

$$\begin{aligned}\mathfrak{S}_{d,l}(N) = &\prod_{\substack{p\nmid d\\ p\mid N}}\left(1-\frac{1}{(p-1)^2}\right)\prod_{\substack{p\mid d\\ p\nmid N-l}}\left(1-\frac{1}{(p-1)^2}\right)\\ &\times\prod_{p\nmid dN}\left(1+\frac{1}{(p-1)^3}\right)\prod_{\substack{p\mid d\\ p\mid N-l}}\left(1+\frac{1}{p-1}\right)\,;\end{aligned} \tag{5}$$

$$\mathfrak{S}(N) = \prod_{p\mid N}\left(1-\frac{1}{(p-1)^2}\right)\prod_{p\nmid N}\left(1+\frac{1}{(p-1)^3}\right)\,; \tag{6}$$

$$\begin{aligned}\mathfrak{S}_\Gamma(N) = \pi\mathfrak{S}(N)&\prod_{p\nmid N(N-1)}\left(1+\chi(p)\frac{p-3}{p(p^2-3p+3)}\right)\prod_{p\mid N}\left(1+\chi(p)\frac{1}{p(p-1)}\right)\\ &\times\prod_{p\mid N-1}\left(1+\chi(p)\frac{2p-3}{p(p^2-3p+3)}\right)\,;\end{aligned} \tag{7}$$

$$\Delta(t,h) = \max_{y\le t}\max_{(l,h)=1}\left|\sum_{\substack{p\le y\\ p\equiv l\,(h)}}\log p - \frac{y}{\varphi(h)}\right|\,. \tag{8}$$

$$\tag{9}$$

2 Introduction and statement of the result

In 1937, I. M. Vinogradov [15] solved the ternary Goldbach problem. He proved that for a sufficiently large odd integer N

$$\sum_{p_1+p_2+p_3=N}\log p_1\log p_2\log p_3 = \frac{1}{2}\mathfrak{S}(N)N^2 + \mathcal{O}\left(\frac{N^2}{\log^A N}\right)\,,$$

where $\mathfrak{S}(N)$ is defined by (6) and $A > 0$ is an arbitrarily large constant.

In 1953, Piatetski-Shapiro [9] proved that for any fixed $c \in (1, 12/11)$ the sequence

$$([n^c])_{n\in\mathbb{N}}$$

contains infinitely many prime numbers. Such prime numbers are named in honor of Piatetski-Shapiro. The interval for c was subsequently improved many times and the best result up to now belongs to Rivat and Wu [10] for $c \in (1, 243/205)$.

In 1992, A. Balog and J. P. Friedlander [1] considered the ternary Goldbach problem with variables restricted to Piatetski-Shapiro primes. They proved that, for any fixed $1 < c < 21/20$, every sufficiently large odd integer N can be represented in the form

$$N = p_1 + p_2 + p_3 \, ,$$

where p_1, p_2, p_3 are primes, such that $p_k = [n_k^c]$, k=1,2,3. Rivat [10] extended the range to $1 < c < 199/188$; Kumchev [7] extended the range to $1 < c < 53/50$. Jia [5] used a sieve method to enlarge the range to $1 < c < 16/15$.

Furthermore, Kumchev [7] proved that for any fixed $1 < c < 73/64$ every sufficiently large odd integer may be written as the sum of two primes and prime number of type $p = [n^c]$.

On the other hand, in 1960, Linnik [8] showed that there exist infinitely many prime numbers of the form $p = x^2 + y^2 + 1$, where x and y are integers. In 2010 Tolev [14] proved that every sufficiently large odd integer N can be represented in the form

$$N = p_1 + p_2 + p_3 \, ,$$

where p_1, p_2, p_3 are primes, such that $p_k = x_k^2 + y_k^2 + 1$, k=1,2. In 2017 Teräväinen [12] improved Tolev's result for primes p_1, p_2, p_3, such that $p_k = x_k^2 + y_k^2 + 1, k = 1, 2, 3$.

Recently the author [2] proved that there exist infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = 2p_2 - p_1$ such that $p_1 = x^2 + y^2 + 1, \ p_3 = [n^c]$.

Define

$$\Gamma(N) = \sum_{\substack{p_1+p_2+p_3=N \\ p_2=[n^c]}} r(p_1 - 1) p_2^{1-\gamma} \log p_1 \log p_2 \log p_3 \, . \tag{10}$$

Motivated by these results we shall prove the following theorem.

Theorem 1. *Assume that* $1 < c < 73/64$. *Then the asymptotic formula*

$$\Gamma(N) = \frac{\gamma}{2} \mathfrak{S}_\Gamma(N) N^2 + \mathcal{O}\big(N^2 (\log N)^{-\theta_0} (\log\log N)^6\big) \, ,$$

holds. Here γ, θ_0 *and* $\mathfrak{S}_\Gamma(N)$ *are defined by* (1), (4) *and* (7).

Bearing in mind that $\mathfrak{S}_\Gamma(N) \gg 1$ for N odd, from Theorem 1 it follows that for any fixed $1 < c < 73/64$ every sufficiently large odd integer N can be written in the form

$$N = p_1 + p_2 + p_3 \, ,$$

where p_1, p_2, p_3 are primes, such that $p_1 = x^2 + y^2 + 1, \ p_2 = [n^c]$.

The asymptotic formula obtained for $\Gamma(N)$ is the product of the individual asymptotic formulas

$$\sum_{p_1+p_2+p_3=N} r(p_1 - 1) \log p_1 \log p_2 \log p_3 \sim \frac{1}{2} \mathfrak{S}_\Gamma(N) N^2$$

and

$$\frac{1}{N} \sum_{\substack{p \le N \\ p=[n^c]}} p^{1-\gamma} \log p \sim \gamma \, .$$

The proof of Theorem 1 follows the same ideas as the proof in [2].

3 Outline of the proof

Using (10) and well-known identity $r(n) = 4\sum_{d|n}\chi(d)$ we find

$$\Gamma(N) = 4\big(\Gamma_1(N) + \Gamma_2(N) + \Gamma_3(N)\big), \tag{11}$$

where

$$\Gamma_1(N) = \sum_{\substack{p_1+p_2+p_3=N\\ p_2=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ d\le D}} \chi(d)\right) p_2^{1-\gamma}\log p_1 \log p_2 \log p_3\,, \tag{12}$$

$$\Gamma_2(N) = \sum_{\substack{p_1+p_2+p_3=N\\ p_2=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ D<d<N/D}} \chi(d)\right) p_2^{1-\gamma}\log p_1 \log p_2 \log p_3\,, \tag{13}$$

$$\Gamma_3(N) = \sum_{\substack{p_1+p_2+p_3=N\\ p_2=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ d\ge N/D}} \chi(d)\right) p_2^{1-\gamma}\log p_1 \log p_2 \log p_3\,. \tag{14}$$

In order to estimate $\Gamma_1(N)$ and $\Gamma_3(N)$ we have to consider the sum

$$I_{d,l;J}(N) = \sum_{\substack{p_1+p_2+p_3=N\\ p_1\equiv l\ (d)\\ p_1\in J\\ p_2=[n^c]}} p_2^{1-\gamma}\log p_1 \log p_2 \log p_3\,, \tag{15}$$

where d and l are coprime natural numbers, and $J \subset [1, N]$. The left and the right side of the interval J, we shall denote with J_1 and J_2, i.e. $J = (J_1, J_2]$. If $J = [1, N]$ then we write for simplicity $I_{d,l}(N)$. We apply the circle method. Clearly

$$I_{d,l;J}(N) = \int\limits_0^1 S_{d,l;J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha)d\alpha\,, \tag{16}$$

where

$$S_{d,l;J}(\alpha) = \sum_{\substack{p\in J\\ p\equiv l\ (d)}} e(\alpha p)\log p\,, \tag{17}$$

$$S(\alpha) = S_{1,1;[1,N]}(\alpha)\,, \tag{18}$$

$$S_c(\alpha) = \sum_{\substack{p\le N\\ p=[n^c]}} p^{1-\gamma}e(\alpha p)\log p\,. \tag{19}$$

We define major and minor arcs by

$$E_1 = \bigcup_{q\le Q}\ \bigcup_{\substack{a=0\\ (a,q)=1}}^{q-1} \left[\frac{a}{q} - \frac{1}{q\tau}, \frac{a}{q} + \frac{1}{q\tau}\right],\ \ E_2 = \left[\frac{1}{\tau}, 1 + \frac{1}{\tau}\right] \setminus E_1\,, \tag{20}$$

where

$$Q = (\log N)^B\,, \quad \tau = NQ^{-1}\,, \quad A > 4B + 3\,, \quad B > 14\,. \tag{21}$$

Then we have the decomposition

$$I_{d,l;J}(N) = I^{(1)}_{d,l;J}(N) + I^{(2)}_{d,l;J}(N)\,, \tag{22}$$

where

$$I^{(i)}_{d,l;J}(N) = \int\limits_{E_i} S_{d,l;J}(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha)d\alpha\,, \quad i = 1, 2. \tag{23}$$

We shall estimate $I^{(1)}_{d,l;J}(N)$, $\Gamma_3(N)$, $\Gamma_2(N)$ and $\Gamma_1(N)$ respectively in the sections 4, 5, 6 and 7. In section 8 we shall complete the proof of the Theorem.

4 Asymptotic formula for $\mathbf{I^{(1)}_{d,l;J}(N)}$

We have

$$I^{(1)}_{d,l;J}(N) = \sum_{q\le Q} \sum_{\substack{a=0\\(a,q)=1}}^{q-1} H(a, q)\,, \tag{24}$$

where

$$H(a, q) = \int\limits_{-1/q\tau}^{1/q\tau} S_{d,l;J}\left(\frac{a}{q} + \alpha\right) S\left(\frac{a}{q} + \alpha\right) S_c\left(\frac{a}{q} + \alpha\right) e\left(-N\left(\frac{a}{q} + \alpha\right)\right) d\alpha\,. \tag{25}$$

On the other hand,

$$S_{d,l;J}\left(\frac{a}{q} + \alpha\right) = \sum_{\substack{1\le m\le q\\(m,q)=1\\m\equiv l\ ((d,q))}} e\left(\frac{am}{q}\right) T(\alpha) + \mathcal{O}\big(q\log N\big)\,, \tag{26}$$

where

$$T(\alpha) = \sum_{\substack{p\in J\\p\equiv l\ (d)\\p\equiv m\ (q)}} e(\alpha p)\log p\,.$$

According to the Chinese remainder theorem there exists integer $f = f(l, m, d, q)$ such that $(f, [d, q]) = 1$ and

$$T(\alpha) = \sum_{\substack{p\in J\\p\equiv f\ ([d,q])}} e(\alpha p)\log p\,.$$

Applying Abel's transformation we obtain

$$T(\alpha) = -\int\limits_{J_1}^{J_2} \left(\sum_{\substack{J_1 < p < t \\ p \equiv f\ ([d,q])}} \log p \right) \frac{d}{dt}(e(\alpha t))dt + \left(\sum_{\substack{p \in J \\ p \equiv f\ ([d,q])}} \log p \right) e(\alpha J_2)$$

$$= -\int\limits_{J_1}^{J_2} \left(\frac{t - J_1}{\varphi([d,q])} + \mathcal{O}\big(\Delta(J_2, [d,q])\big) \right) \frac{d}{dt}(e(\alpha t))dt$$

$$+ \left(\frac{J_2 - J_1}{\varphi([d,q])} + \mathcal{O}\big(\Delta(J_2, [d,q])\big) \right) e(\alpha J_2)$$

$$= \frac{1}{\varphi([d,q])} \int\limits_{J_1}^{J_2} e(\alpha t)dt + \mathcal{O}\big((1 + |\alpha|(J_2 - J_1))\Delta(J_2, [d,q])\big) . \tag{27}$$

We use the well known formula

$$\int\limits_{J_1}^{J_2} e(\alpha t)dt = M_J(\alpha) + \mathcal{O}(1) , \tag{28}$$

where

$$M_J(\alpha) = \sum_{m \in J} e(\alpha m) .$$

Bearing in mind that $|\alpha| \leq 1/q\tau$ and $J \subset (1\,, N]$, from (21), (27) and (28) we get

$$T(\alpha) = \frac{M_J(\alpha)}{\varphi([d,q])} + \mathcal{O}\left(\left(1 + \frac{Q}{q} \right) \Delta(N, [d,q]) \right) . \tag{29}$$

From (26) and (29) it follows

$$S_{d,l;J}\left(\frac{a}{q} + \alpha \right) = \frac{c_d(a,q,l)}{\varphi([d,q])} M_J(\alpha) + \mathcal{O}\big(Q(\log N)\Delta(N, [d,q])\big) , \tag{30}$$

where

$$c_d(a,q,l) = \sum_{\substack{1 \leq m \leq q \\ (m,q)=1 \\ m \equiv l\ ((d,q))}} e\left(\frac{am}{q} \right) .$$

We shall find asymptotic formula for $S_c\left(\frac{a}{q} + \alpha \right)$. From (19) we have

$$S_c(\alpha) = \sum_{p \leq N} p^{1-\gamma}\big([-p^\gamma] - [-(p+1)^\gamma]\big) e(\alpha p) \log p$$

$$= \Omega(\alpha) + \Sigma(\alpha) , \tag{31}$$

where

$$\Omega(\alpha) = \sum_{p \leq N} p^{1-\gamma}\big((p+1)^\gamma - p^\gamma\big) e(\alpha p) \log p , \tag{32}$$

$$\Sigma(\alpha) = \sum_{p \leq N} p^{1-\gamma}\big(\psi(-(p+1)^\gamma) - \psi(-p^\gamma)\big) e(\alpha p) \log p . \tag{33}$$

According to Kumchev ([7], Theorem 2) for $64/73 < \gamma < 1$ uniformly in α we have that

$$\Sigma\left(\frac{a}{q}+\alpha\right) \ll N^{1-\varepsilon}\,. \tag{34}$$

On the other hand,

$$(p+1)^\gamma - p^\gamma = \gamma p^{\gamma-1} + \mathcal{O}\left(p^{\gamma-2}\right)\,. \tag{35}$$

The formulas (32) and (35) give us

$$\Omega(\alpha) = \gamma S(\alpha) + \mathcal{O}(N^\varepsilon)\,, \tag{36}$$

where $S(\alpha)$ is defined by (18).

According to ([6], Lemma 3, §10) we have

$$S\left(\frac{a}{q}+\alpha\right) = \frac{\mu(q)}{\varphi(q)}M(\alpha) + \mathcal{O}\left(Ne^{-c_0\sqrt{\log N}}\right)\,, \tag{37}$$

where

$$M(\alpha) = \sum_{m\leq N} e(\alpha m)\,.$$

Bearing in mind (31), (34), (36) and (37) we obtain

$$S_c\left(\frac{a}{q}+\alpha\right) = \gamma\frac{\mu(q)}{\varphi(q)}M(\alpha) + \mathcal{O}\left(Ne^{-c_0\sqrt{\log N}}\right)\,. \tag{38}$$

Furthermore, we need the trivial estimates

$$\left|S_{d,l;J}\left(\frac{a}{q}+\alpha\right)\right| \ll \frac{N\log N}{d}\,, \quad \left|S\left(\frac{a}{q}+\alpha\right)\right| \ll N\,, \quad |M(\alpha)| \ll N\,, \quad |\mu(q)| \ll 1\,. \tag{39}$$

By (30), (37) – (39) and the well-known inequality $\varphi(n) \gg n(\log\log n)^{-1}$ we find

$$\begin{aligned}
&S_{d,l;J}\left(\frac{a}{q}+\alpha\right) S\left(\frac{a}{q}+\alpha\right) S_c\left(\frac{a}{q}+\alpha\right) e\left(-N\left(\frac{a}{q}+\alpha\right)\right) \\
&= \gamma\frac{c_d(a,q,l)\mu^2(q)}{\varphi([d,q])\varphi^2(q)}M_J(\alpha)M^2(\alpha)e\left(-N\left(\frac{a}{q}+\alpha\right)\right) + \mathcal{O}\left(\frac{N^3}{d}e^{-c_0\sqrt{\log N}}\right) \\
&+ \mathcal{O}\left(\frac{N^2Q\log^2 N}{q^2}\Delta(N,[d,q])\right)\,.
\end{aligned} \tag{40}$$

Having in mind (21), (25) and (40) we get

$$\begin{aligned}
H(a,q) &= \gamma\frac{c_d(a,q,l)\mu^2(q)}{\varphi([d,q])\varphi^2(q)}e\left(-N\frac{a}{q}\right)\int\limits_{-1/q\tau}^{1/q\tau} M_J(\alpha)M^2(\alpha)e(-N\alpha)d\alpha \\
&\quad + \mathcal{O}\left(\frac{N^2}{qd}e^{-c_0\sqrt{\log N}}\right) + \mathcal{O}\left(\frac{NQ^2\log^2 N}{q^3}\Delta(N,[d,q])\right)\,.
\end{aligned} \tag{41}$$

Taking into account (24), (41) and following the method in [13] we obtain

$$I_{d,l;J}^{(1)}(N) = \gamma \frac{\mathfrak{S}_{d,l}(N)}{\varphi(d)} \sum_{\substack{m_1+m_2+m_3=N \\ m_1\in J}} 1 + \mathcal{O}\left(\frac{N^2}{d}(\log N)\sum_{q>Q}\frac{(d,q)\log q}{q^2}\right)$$
$$+ \mathcal{O}\left(\tau^2(\log N)\sum_{q\le Q}\frac{q}{[d,q]}\right) + \mathcal{O}\left(NQ^2(\log N)^2\sum_{q\le Q}\frac{\Delta(N,[d,q])}{q^2}\right)$$
$$+ \mathcal{O}\left(\frac{N^2}{d}e^{-c_0\sqrt{\log N}}\right), \tag{42}$$

where $\mathfrak{S}_{d,l}(N)$ is defined by (5).

5 Upper bound for $\Gamma_3(N)$

Consider the sum $\Gamma_3(N)$.

Since

$$\sum_{\substack{d|p_1-1 \\ d\ge N/D}} \chi(d) = \sum_{\substack{m|p_1-1 \\ m\le (p_1-1)D/N}} \chi\left(\frac{p_1-1}{m}\right) = \sum_{j=\pm1}\chi(j) \sum_{\substack{m|p_1-1 \\ m\le (p_1-1)D/N \\ \frac{p_1-1}{m}\equiv j\,(4)}} 1$$

then from (14) and (15) it follows

$$\Gamma_3(N) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm1} \chi(j) I_{4m,1+jm;J_m}(N)\,,$$

where $J_m = [1+mN/D, N]$.

Therefore from (22) we get

$$\Gamma_3(N) = \Gamma_3^{(1)}(N) + \Gamma_3^{(2)}(N)\,, \tag{43}$$

where

$$\Gamma_3^{(\nu)}(N) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm1} \chi(j) I^{(\nu)}_{4m,1+jm;J_m}(N)\,, \quad \nu=1,2. \tag{44}$$

Let us consider first $\Gamma_3^{(2)}(N)$. Bearing in mind (23) for $i=2$ and (44) for $\nu=2$ we have

$$\Gamma_3^{(2)}(N) = \int\limits_{E_2} K(\alpha)S(\alpha)S_c(\alpha)e(-N\alpha)d\alpha\,,$$

where

$$K(\alpha) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm1} \chi(j) S_{4m,1+jm;J_m}(\alpha)\,. \tag{45}$$

Using Cauchy's inequality we obtain

$$\Gamma_3^{(2)}(N) \ll \sup_{\alpha\in E_2\setminus\{1\}} |S_c(\alpha)| \int\limits_{E_2} |K(\alpha)S(\alpha)|d\alpha + \mathcal{O}(N^\varepsilon)$$
$$\ll \sup_{\alpha\in E_2\setminus\{1\}} |S_c(\alpha)| \left(\int\limits_0^1 |K(\alpha)|^2 d\alpha\right)^{1/2} \left(\int\limits_0^1 |S(\alpha)|^2 d\alpha\right)^{1/2} + \mathcal{O}(N^\varepsilon)\,. \tag{46}$$

From (31) and (36) we have

$$S_c(\alpha) = \gamma S(\alpha) + \Sigma(\alpha) + \mathcal{O}(N^{\varepsilon})\,, \tag{47}$$

where $S(\alpha)$ and $\Sigma(\alpha)$ are defined by (18) and (33).

Using (20) and (21) we can prove in the same way as in ([6], Ch.10, Th.3) that

$$\sup_{\alpha\in E_2\backslash\{1\}} |S(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}\,. \tag{48}$$

According to Kumchev ([7], Theorem 2) we have that

$$\sup_{\alpha\in E_2\backslash\{1\}} |\Sigma(\alpha)| \ll N^{1-\varepsilon}\,. \tag{49}$$

Bearing in mind (47)–(49) we get

$$\sup_{\alpha\in E_2\backslash\{1\}} |S_c(\alpha)| \ll \frac{N}{(\log N)^{B/2-4}}\,. \tag{50}$$

From (18) after straightforward computations we find

$$\int\limits_0^1 |S(\alpha)|^2 d\alpha \ll N\log N\,. \tag{51}$$

On the other hand, from (17) and (45) we obtain

$$\begin{aligned}
\int\limits_0^1 |K(\alpha)|^2\, d\alpha &= \sum_{\substack{m_1,m_2<D\\ 2|m_1,2|m_2}} \sum_{\substack{j_1=\pm1\\ j_2=\pm1}} \chi(j_1)\chi(j_2)\\
&\times \int\limits_0^1 S_{4m_1,1+j_1m_1;J_{m_1}}(\alpha) S_{4m_2,1+j_2m_2;J_{m_2}}(-\alpha)d\alpha\\
&= \sum_{\substack{m_1,m_2<D\\ 2|m_1,2|m_2}} \sum_{\substack{j_1=\pm1\\ j_2=\pm1}} \chi(j_1)\chi(j_2)\\
&\times \sum_{\substack{p_i\in J_{m_i},i=1,2\\ p_i\equiv 1+j_im_i\,(4m_i),i=1,2}} \log p_1\log p_2 \int\limits_0^1 e(\alpha(p_1-p_2))d\alpha\\
&= \sum_{\substack{m<D\\ 2|m}} \sum_{j=\pm1} \chi(j) \sum_{\substack{p\in J_m\\ p\equiv 1+jm\,(4m)}} (\log p)^2\\
&\ll (\log N)^2 \sum_{\substack{m<D\\ 2|m}} \sum_{\substack{p\in J_m\\ p\equiv 1+jm\,(4m)}} 1\\
&\ll N(\log N)^2 \sum_{m<D} \frac{1}{m}\\
&\ll N\log^3 N\,.
\end{aligned} \tag{52}$$

Thus from (46), (50) – (52) it follows

$$\Gamma_3^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2-6}} \,. \tag{53}$$

Now let us consider $\Gamma_3^{(1)}(N)$. From (42) and (44) for $\nu = 1$ we get

$$\begin{aligned}\Gamma_3^{(1)}(N) = \Gamma^* &+ \mathcal{O}\big(N^2(\log N)\Sigma_1\big) + \mathcal{O}\big(\tau^2(\log N)\Sigma_2\big) \\ &+ \mathcal{O}\big(NQ^2(\log N)^2\Sigma_3\big) + \mathcal{O}\big(N^2 e^{-c_0\sqrt{\log N}}\Sigma_4\big) \,, \end{aligned} \tag{54}$$

where

$$\begin{aligned}
\Gamma^* &= \gamma\Bigg(\sum_{\substack{m_1+m_2+m_3=N\\ m_1\in J_m}} 1\Bigg) \sum_{\substack{m<D\\ 2|m}} \frac{1}{\varphi(4m)} \sum_{j=\pm1} \chi(j)\mathfrak{S}_{4m,1+jm}(N)\,,\\
\Sigma_1 &= \sum_{m<D}\sum_{q>Q} \frac{(4m,q)\log q}{mq^2}\,,\\
\Sigma_2 &= \sum_{m<D}\sum_{q\leq Q} \frac{q}{[4m,q]}\,,\\
\Sigma_3 &= \sum_{m<D}\sum_{q\leq Q} \frac{\Delta(N,[4m,q])}{q^2}\,,\\
\Sigma_4 &= \sum_{m<D} \frac{1}{m}\,.
\end{aligned}$$

From the definition (5) it follows that $\mathfrak{S}_{4m,1+jm}(N)$ does not depend on j. Then we have $\sum\limits_{j=\pm1} \chi(j)\mathfrak{S}_{4m,1+jm}(N) = 0$ and that leads to

$$\Gamma^* = 0\,. \tag{55}$$

Arguing as in [13] and using Bombieri–Vinogradov's theorem we find the following estimates

$$\Sigma_1 \ll \frac{\log^3 N}{Q}\,, \quad \Sigma_2 \ll Q\log^2 N\,, \tag{56}$$

$$\Sigma_3 \ll \frac{N}{(\log N)^{A-B-5}}\,, \quad \Sigma_4 \ll \log N\,. \tag{57}$$

Bearing in mind (21), (54) – (57) we obtain

$$\Gamma_3^{(1)}(N) \ll \frac{N^2}{(\log N)^{B-4}}\,. \tag{58}$$

Now from (43), (53) and (58) we find

$$\Gamma_3(N) \ll \frac{N^2}{(\log N)^{B/2-6}}\,. \tag{59}$$

6 Upper bound for $\Gamma_2(N)$

Consider the sum $\Gamma_2(N)$ defined by (13). We denote by $\mathcal{F}$ the set of all primes $p \leq N$ such that $p-1$ has a divisor belongs to the interval $(D, N/D)$. Using the inequality $uv \leq u^2 + v^2$ and taking into account the symmetry with respect to d and t we get

$$\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \sum_{\substack{p_1+p_2+p_3=N \\ p_4+p_5+p_6=N \\ p_2=[n_1^c],\, p_5=[n_2^c]}} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<N/D}} \chi(d) \Bigg| \Bigg| \sum_{\substack{t|p_4-1 \\ D<t<N/D}} \chi(t) \Bigg|$$

$$\ll (\log N)^6 N^{2-2\gamma} \sum_{\substack{p_1+p_2+p_3=N \\ p_4+p_5+p_6=N \\ p_2=[n_1^c],\, p_5=[n_2^c] \\ p_4\in\mathcal{F}}} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<N/D}} \chi(d) \Bigg|^2 . \tag{60}$$

Further, we use that if n is a natural such that $n \leq N$, then the number of solutions of the equation $p_1 + p_2 = n$ in primes $p_1, p_2 \leq N$ such that $p_1 = [m^{1/\gamma}]$ is $\mathcal{O}\big(N^\gamma (\log N)^{-2} \log\log N\big)$, i.e.

$$\#\{p_1 \, : p_1 + p_2 = n, \;\; p_1 = [m^{1/\gamma}], \;\; n \leq N\} \ll \frac{N^\gamma \log\log N}{\log^2 N} . \tag{61}$$

This follows for example from ([3], Ch.2, Th.2.4).
Thus the summands in the sum (60) for which $p_1 = p_4$ can be estimated with $\mathcal{O}(N^{3+\varepsilon})$.

Therefore

$$\Gamma_2(N)^2 \ll (\log N)^6 N^{2-2\gamma} \Sigma_1 + N^{3+\varepsilon} , \tag{62}$$

where

$$\Sigma_1 = \sum_{p_1 \leq N} \Bigg| \sum_{\substack{d|p_1-1 \\ D<d<N/D}} \chi(d) \Bigg|^2 \sum_{\substack{p_4 \leq N \\ p_4 \in \mathcal{F} \\ p_4 \neq p_1}} \sum_{\substack{p_2+p_3=N-p_1 \\ p_5+p_6=N-p_4 \\ p_2=[n_1^c],\, p_5=[n_2^c]}} 1 .$$

We use again (61) and find

$$\Sigma_1 \ll \frac{N^{2\gamma}}{\log^4 N} (\log\log N)^2 \Sigma_2 \Sigma_3 , \tag{63}$$

where

$$\Sigma_2 = \sum_{p \leq N} \Bigg| \sum_{\substack{d|p-1 \\ D<d<N/D}} \chi(d) \Bigg|^2 , \qquad \Sigma_3 = \sum_{\substack{p \leq N \\ p \in \mathcal{F}}} 1 .$$

Arguing as in ([4], Ch.5) we find

$$\Sigma_2 \ll \frac{N (\log\log N)^7}{\log N} , \qquad \Sigma_3 \ll \frac{N (\log\log N)^3}{(\log N)^{1+2\theta_0}} . \tag{64}$$

where θ_0 is denoted by (4).

From (62) – (64) it follows

$$\Gamma_2(N) \ll N^2 (\log N)^{-\theta_0} (\log\log N)^6 . \tag{65}$$

7 Asymptotic formula for $\Gamma_1(N)$

In this section our argument is a modification of Tolev's [14] argument.

Consider the sum $\Gamma_1(N)$. From (12), (15) and (22) we get

$$\Gamma_1(N) = \Gamma_1^{(1)}(N) + \Gamma_1^{(2)}(N)\,, \tag{66}$$

where

$$\Gamma_1^{(1)}(N) = \sum_{d\leq D} \chi(d) I_{d,1}^{(1)}(N)\,,$$
$$\Gamma_1^{(2)}(N) = \sum_{d\leq D} \chi(d) I_{d,1}^{(2)}(N)\,.$$

We estimate the sum $\Gamma_1^{(2)}(N)$ by the same way as the sum $\Gamma_3^{(2)}(N)$ and obtain

$$\Gamma_1^{(2)}(N) \ll \frac{N^2}{(\log N)^{B/2-6}}\,. \tag{67}$$

Now we consider $\Gamma_1^{(1)}(N)$. We use the formula (42) for $J = [1, N]$. The error term is estimated by the same way as for $\Gamma_3^{(1)}(N)$. We have

$$\Gamma_1^{(1)}(N) = \frac{\gamma}{2}\mathfrak{S}(N) N^2 \sum_{d\leq D} \frac{\chi(d)\mathfrak{S}_{d,1}^*(N)}{\varphi(d)} + \mathcal{O}\bigg(\frac{N^2}{(\log X)^{B-4}}\bigg)\,, \tag{68}$$

where $\mathfrak{S}(N)$ is defined by (6) and

$$\begin{aligned}\mathfrak{S}_{d,1}^*(N) = &\prod_{\substack{p\mid d\\ p\mid N}}\bigg(1-\frac{1}{(p-1)^2}\bigg)^{-1} \prod_{\substack{p\mid d\\ p\nmid N-1}}\bigg(1-\frac{1}{(p-1)^2}\bigg)\\ &\times\prod_{\substack{p\mid d\\ p\nmid N}}\bigg(1+\frac{1}{(p-1)^3}\bigg)^{-1} \prod_{\substack{p\mid d\\ p\mid N-1}}\bigg(1+\frac{1}{p-1}\bigg)\,;\end{aligned} \tag{69}$$

Denote

$$\Sigma = \sum_{d\leq D} f(d)\,, \qquad f(d) = \frac{\chi(d)\mathfrak{S}_{d,1}^*(N)}{\varphi(d)}\,. \tag{70}$$

We have

$$f(d) \ll d^{-1}(\log\log(10d))^2 \tag{71}$$

with absolute constant in the Vinogradov's symbol. Hence the corresponding Dirichlet series

$$F(s) = \sum_{d=1}^{\infty} \frac{f(d)}{d^s}$$

is absolutely convergent in $Re(s) > 0$. On the other hand, $f(d)$ is multiplicative with respect to d and applying Euler's identity we find

$$F(s) = \prod_p T(p,s)\,, \quad T(p,s) = 1 + \sum_{l=1}^{\infty} f(p^l)p^{-ls}\,. \tag{72}$$

From (69), (70) and (72) we establish that

$$T(p,s)=\left(1-\frac{\chi(p)}{p^{s+1}}\right)^{-1}\left(1+\frac{\chi(p)}{p^{s+1}}E_d(p)\right)\,,$$

where

$$E_d(p)=\begin{cases}(p-3)(p^2-3p+3)^{-1} & \text{if } \ p\nmid N(N-1)\,,\\ (p-1)^{-1} & \text{if } \ p\mid N\,,\\ (2p-3)(p^2-3p+3)^{-1} & \text{if } \ p\mid N-1\,.\end{cases}$$

Hence we find

$$F(s)=L(s+1,\chi)\mathcal{N}(s)\,, \tag{73}$$

where $L(s+1,\chi)$ is Dirichlet series corresponding to the character χ and

$$\mathcal{N}(s)=\prod_{p\nmid N(N-1)}\left(1+\chi(p)\frac{p-3}{p^{s+1}(p^2-3p+3)}\right)\prod_{p\mid N}\left(1+\chi(p)\frac{1}{p^{s+1}(p-1)}\right)$$
$$\times\prod_{p\mid N-1}\left(1+\chi(p)\frac{2p-3}{p^{s+1}(p^2-3p+3)}\right)\,. \tag{74}$$

From the properties of the L-functions it follows that $F(s)$ has an analytic continuation to $Re(s)>-1$. It is well known that

$$L(s+1,\chi)\ll 1+|Im(s)|^{1/6}\quad\text{for }\ Re(s)\geq-\frac{1}{2}\,. \tag{75}$$

Moreover,

$$\mathcal{N}(s)\ll 1\,. \tag{76}$$

Using (73), (75) and (76) we get

$$F(s)\ll N^{1/6}\quad\text{for }\ Re(s)\geq-\frac{1}{2}\,,\ \ |Im(s)|\leq N\,. \tag{77}$$

We apply Perron's formula given at Tenenbaum ([11], Chapter II.2) and also (71) to obtain

$$\Sigma=\frac{1}{2\pi\imath}\int\limits_{\varkappa-\imath N}^{\varkappa+\imath N}F(s)\frac{D^s}{s}ds+\mathcal{O}\left(\sum_{t=1}^{\infty}\frac{D^{\varkappa}\log\log(10t)}{t^{1+\varkappa}\left(1+N\left|\log\frac{D}{t}\right|\right)}\right)\,, \tag{78}$$

where $\varkappa=1/10$. It is easy to see that the error term above is $\mathcal{O}\left(N^{-1/20}\right)$.

Applying the residue theorem we see that the main term in (78) is equal to

$$F(0)+\frac{1}{2\pi\imath}\left(\int\limits_{1/10-\imath N}^{-1/2-\imath N}+\int\limits_{-1/2-\imath N}^{-1/2+\imath N}+\int\limits_{-1/2+\imath N}^{1/10+\imath N}\right)F(s)\frac{D^s}{s}ds\,.$$

From (77) it follows that the contribution from the above integrals is $\mathcal{O}\left(N^{-1/20}\right)$.

Hence

$$\Sigma=F(0)+\mathcal{O}\left(N^{-1/20}\right)\,. \tag{79}$$

Using (73) we get

$$F(0) = \frac{\pi}{4}\mathcal{N}(0)\,. \tag{80}$$

Bearing in mind (68), (70), (74), (79) and (80) we find a new expression for $\Gamma_1^{(1)}(N)$

$$\Gamma_1^{(1)}(N) = \frac{\gamma}{8}\mathfrak{S}_\Gamma(N)N^2 + \mathcal{O}\bigg(\frac{N^2}{(\log N)^{B-4}}\bigg)\,, \tag{81}$$

where $\mathfrak{S}_\Gamma$ is defined by (7).

From (66), (67) and (81) we obtain

$$\Gamma_1(N) = \frac{\gamma}{8}\mathfrak{S}_\Gamma(N)N^2 + \mathcal{O}\bigg(\frac{N^2}{(\log N)^{B/2-6}}\bigg)\,. \tag{82}$$

8 Proof of the Theorem

Therefore using (11), (59), (65) and (82) we find

$$\Gamma(N) = \frac{\gamma}{2}\mathfrak{S}_\Gamma(N)N^2 + \mathcal{O}\big(N^2(\log N)^{-\theta_0}(\log\log N)^6\big)\,.$$

This implies that $\Gamma(N) \to \infty$ as $N \to \infty$.

The Theorem is proved. □

References

[1] Balog, A. & Friedlander, J. P.(1992) A hybrid of theorems of Vinogradov and Piatetski-Shapiro, *Pacific J. Math.*, 156, 45–62.

[2] Dimitrov, S. I. (2017) Prime triples p_1, p_2, p_3 in arithmetic progressions such that $p_1 = x^2 + y^2 + 1$, $p_3 = [n^c]$, *Notes on Number Theory and Discrete Mathematics*, 23(4), 22–33.

[3] Halberstam, H. & Richert , H.-E. (1974) *Sieve Methods*, Academic Press.

[4] Hooley, C. (1976) *Applications of sieve methods to the theory of numbers*, Cambridge Univ. Press.

[5] Jia, C.-H. (1995) On the Piatetski-Shapiro-Vinogradov theorem, *Acta Arith.*, 73, 1–28.

[6] Karatsuba, A. (1983) *Principles of the Analytic Number Theory*, Nauka, Moscow, (in Russian).

[7] Kumchev, A. (1997) On the Piatetski-Shapiro-Vinogradov Theorem, *Journal de Théorie des Nombres de Bordeaux*, 9, 11–23.

[8] Linnik, Ju. (1960) An asymptotic formula in an additive problem of Hardy and Littlewood, *Izv. Akad. Nauk SSSR, Ser.Mat.*, 24, 629–706 (in Russian).

[9] Piatetski-Shapiro, I. I. (1953) On the distribution of prime numbers in sequences of the form $[f(n)]$, *Mat. Sb.*, 33, 559–566.

[10] Rivat, J. & Wu, J. (2001) Prime numbers of the form $[n^c]$, *Glasg. Math. J*, 43(2), 237–254.

[11] Tenenbaum, G. (1995) *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Univ. Press.

[12] Teräväinen, J. (2018) The Goldbach problem for primes that are sums of two squares plus one, Mathematika, 64(1), 20–70.

[13] Tolev, D. (1997) On the number of representations of an odd integer as a sum of three primes, one of which belongs to an arithmetic progression, *Proc. Steklov Math. Inst.*, 218, 415–432.

[14] Tolev, D. (2010) The ternary Goldbach problem with arithmetic weights attached to one of the variables, *J. Number Theory*, 130, 439–457.

[15] Vinogradov, I. M. (1937) Representation of an odd number as the sum of three primes, *Dokl. Akad. Nauk. SSSR*, 15, 291–294 (in Russian).