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On generalizations of the Jacobsthal sequence

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Abstract: In this paper, the generalized Jacobsthal and generalized complex Jacobsthal and generalized dual Jacobsthal sequences using the Jacobsthal numbers are investigated. Also, special cases of these sequences are investigated. Furthermore, recurrence relations, vectors, the golden ratio and Binet's formula for the generalized Jacobsthal sequences and generalized dual Jacobsthal sequences are given.
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1 Introduction

In 1973, the first use of these numbers appears "A Handbook of Integer Sequences" in a paper by Sloane by the title applications of Jacobsthal sequences to curves [17].

Further, in 1988, Horadam [12] introduced the Jacobsthal and Jacobsthal–Lucas sequences recurrence relation $\{J_n\}$ and $\{j_n\}$ are defined by the recurrence relations

$$J_0 = 0, \;\; J_1 = 1, \;\; J_n = J_{n-1} + 2J_{n-2}, \;\; for\; n \geq 2,$$

$$j_0 = 2, \;\; j_1 = 1, \;\; j_n = j_{n-1} + 2j_{n-2}, \;\; for\; n \geq 2,$$

respectively.

The first eleven terms of the Jacobsthal sequence J_n are $0,\ 1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43,\ 85,$ $171,$ and 341. This sequence is given by the formula

$$J_n = \frac{2^n - (-1)^n}{3}\,.$$

The first eleven terms of the Jacobsthal–Lucas sequence $\{j_n\}$ are $2, 1, 5, 7, 17, 31, 65, 127,$ $257, 511$ and 1025. This sequence is given by the formula

$$j_n = 2^n + (-1)^n.$$

There are many articles in the literature that study on the Jacobsthal sequences [2, 3, 11, 13, 14]. Generalization of the Jacobsthal sequences is given in [4, 5, 8, 9, 10]. Also, we can see the matrix representations of Jacobsthal and Jacobsthal–Lucas numbers in [6, 7, 15, 16]. Several authors worked on the Jacobsthal quaternions in [1, 18, 19].

For the Jacobsthal and Jacobsthal–Lucas numbers, the following properties

$$J_n^2 + 2\,J_{n-1}^2 = J_{2n-1} \tag{1}$$

$$J_{n+1}^2 + 2\,J_n^2 = J_{2n+1} \tag{2}$$

$$J_{n+1}^2 - 4\,J_{n-1}^2 = J_{2n} \tag{3}$$

$$J_n^2 - 4\,J_{n-1}^2 = (-1)^{n+1} J_{n+1} \tag{4}$$

$$J_{n-1}J_{n+1} - J_n^2 = (-1)^n\, 2^{n-1} \tag{5}$$

$$J_{n+1} - 4\,J_{n-1} = (-1)^{n+1} \tag{6}$$

$$J_{n+1} + 4\,J_n = j_{n+1} \tag{7}$$

$$j_n J_n = J_{2n} \tag{8}$$

$$J_m J_{n+1} + 2\,J_{m-1}J_n = J_{m+n} \tag{9}$$

$$J_m J_{n-1} - J_{m-1}J_n = (-1)^n\, 2^{n-1} J_{m-n} \tag{10}$$

hold [6, 7, 13, 14, 15, 16, 18, 19].

2 The generalized Jacobsthal sequence

In this section, the generalized Jacobsthal sequence denoted by $\mathbb{J}_n$ will be defined. The generalized Jacobsthal sequence is defined by

$$\mathbb{J}_n = \mathbb{J}_{n-1} + 2\,\mathbb{J}_{n-2}\,, \quad (n \geq 3) \tag{11}$$

with $\mathbb{J}_0 = q\,,\ \mathbb{J}_1 = p + q,\ \mathbb{J}_2 = p + 3\,q,$ where p, q are arbitrary integers. That is, the generalized Jacobsthal sequence is

$$q\,,\ p + q\,,\ p + 3\,q\,,\ 3\,p + 5\,q\,,\ 5\,p + 11\,q\,,\ 11\,p + 21\,q\,,\ \ldots\,, (p + q)J_n + 2\,qJ_{n-1}, \ldots \tag{12}$$

Using the equations (11) and (12) , we get

$$\begin{aligned} \mathbb{J}_n &= p\,J_n + \,q\,J_{n+1} \\ \mathbb{J}_{n+1} &= (p+q)\,J_{n+1} + 2\,q\,J_n \\ \mathbb{J}_{n+2} &= (p+3\,q)\,J_{n+1} + 2\,(p+q)\,J_n. \end{aligned} \tag{13}$$

Putting $n = r$ in (13) and using (11), we find

$$\begin{aligned} \mathbb{J}_{r+3} &= (3\,p+5\,q\,)J_{r+1} + 2\,(p+3\,q\,)\,J_r = \mathbb{J}_3\,J_{r+1} + 2\,\mathbb{J}_2\,J_r \\ \mathbb{J}_{r+4} &= (5\,p+11\,q)\,J_{r+1} + 2\,(3\,p+5\,q)\,J_r = \mathbb{J}_4\,J_{r+1} + 2\,\mathbb{J}_3\,J_r. \end{aligned} \tag{14}$$

So, in general, we obtain relation between generalized Jacobsthal sequence and Jacobsthal sequence as follows:

$$\mathbb{J}_{n+r} = \mathbb{J}_n\,J_{r+1} + 2\,\mathbb{J}_{n-1}\,J_r. \tag{15}$$

Also, certain results follow almost immediately from (11)

$$\mathbb{J}_{n+2} - 3\,\mathbb{J}_n - 2\,\mathbb{J}_{n-1} = 0. \tag{16}$$

For the generalized Jacobsthal sequence, we have the following properties:

$$(\mathbb{J}_n)^2 + 2\,(\mathbb{J}_{n-1})^2 = (2p+q)\mathbb{J}_{2n-1} - e_J\,J_{2n-1}\,, \tag{17}$$

$$(\mathbb{J}_{n+1})^2 - 4\,(\mathbb{J}_{n-1})^2 = (2p+q)\mathbb{J}_{2n} - e_J\,J_{2n}\,, \tag{18}$$

$$\mathbb{J}_{n-1}\,\mathbb{J}_{n+1} - (\mathbb{J}_n)^2 = (-1)^n\,2^{n-1}\,e_J\,, \tag{19}$$

$$\mathbb{J}_{n-r}\,\mathbb{J}_{n+r} - (\mathbb{J}_n)^2 = (-1)^{n-r+1}\,2^{n-r}\,J_r^2\,e_J\,, \tag{20}$$

$$4\,(\mathbb{J}_n)^2 + 2\,e_J\,J_{n+1}^2 = 2\,p\,\mathbb{J}_{2n+1}\,, \tag{21}$$

$$\mathbb{J}_m\,\mathbb{J}_{n+1} + 2\,\mathbb{J}_{m-1}\,\mathbb{J}_n = (2p+q)\mathbb{J}_{m+n} - e_J\,J_{m+n}\,, \tag{22}$$

$$\mathbb{J}_m\,\mathbb{J}_{n-1} - \,\mathbb{J}_{m-1}\,\mathbb{J}_n = (-1)^n\,2^{n-1}\,e_J\,J_{m-n}, \tag{23}$$

where $e_J = p^2 + p\,q - 2\,q^2$.

Also, for $p = 1,\ q = 0$, we get the well-known results in $(1-10)$.

Theorem 1. If $\mathbb{J}_n$ is the generalized Jacobsthal number, then

$$\lim_{n\to\infty} \frac{\mathbb{J}_{n+1}}{\mathbb{J}_n} = \frac{(p+q)\,\alpha + 2\,q}{q\alpha + p},$$

where $\alpha = 2$.

Proof. We have for the Jacobsthal number J_n,

$$\lim_{n\to\infty} \frac{J_{n+1}}{J_n} = \alpha,$$

where $\alpha = 2$ [17].

Then for the generalized Jacobsthal number $\mathbb{J}_n$, we obtain

$$\lim_{n\to\infty} \frac{\mathbb{J}_{n+1}}{\mathbb{J}_n} = \lim_{n\to\infty} \frac{(p+q)\,J_{n+1} \,+\, 2\,qJ_n}{p\,J_n + q\,J_{n+1}} = \frac{(p+q)\,\alpha + 2\,q}{q\,\alpha + p}. \tag{24}$$

□

Theorem 2. The Binet's formula[1] for the generalized Jacobsthal sequence is as follows;

$$\mathbb{J}_n = \frac{(\bar{\alpha}\ \alpha^n - \bar{\beta}\ \beta^n)}{\alpha - \beta}. \tag{25}$$

Proof. The characteristic equation of recurrence relation $\mathbb{J}_{n+2} = \mathbb{J}_{n+1} + 2\mathbb{J}_n$ is

$$t^2 - t - 2 = 0. \tag{26}$$

The roots of this equation are

$$\alpha = -1 \text{ and } \beta = 2, \tag{27}$$

where $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$. Using recurrence relation and initial values $\mathbb{J}_0 = q$, $\mathbb{J}_1 = p + q$, for the Binet's formula $\mathbb{J}_n$, we get

$$\mathbb{J}_n = A\,\alpha^n + B\,\beta^n = \frac{(\bar{\alpha}\ \alpha^n - \bar{\beta}\ \beta^n)}{\alpha - \beta}, \tag{28}$$

where $A = \frac{\mathbb{J}_1 - \beta\,\mathbb{J}_0}{\alpha - \beta}$, $B = \frac{\alpha\,\mathbb{J}_0 - \mathbb{J}_1}{\alpha - \beta}$ and $\bar{\alpha} = p + q\,(1-\beta)$, $\bar{\beta} = q\,(\alpha - 1) - p$. □

2.1 The generalized Jacobsthal vectors

A generalized Jacobsthal vector is defined by $\overrightarrow{\mathbb{J}_{\mathrm{n}}} = (\mathbb{J}_n, \mathbb{J}_{n+1}, \mathbb{J}_{n+2})$. Also, from equation (12) it can be expressed as

$$\overrightarrow{\mathbb{J}_{\mathrm{n}}} = (p+q)\,\overrightarrow{\mathrm{J}}_n + 2\,q\,\overrightarrow{\mathrm{J}}_{\mathrm{n}-1}, \tag{29}$$

where $\overrightarrow{\mathrm{J}}_{\mathrm{n}} = (J_n, J_{n+1}, J_{n+2})$ and $\overrightarrow{\mathrm{J}}_{\mathrm{n}+1} = (J_{n+1}, J_{n+2}, J_{n+3})$ are the Jacobsthal vectors. The product of $\overrightarrow{\mathbb{J}_{\mathrm{n}}}$ and $\lambda \in \mathbb{R}$ is given by

$$\lambda\,\overrightarrow{\mathbb{J}_{\mathrm{n}}} = (\lambda\,\overrightarrow{\mathbb{J}}_{\mathrm{n}}, \lambda\,\overrightarrow{\mathbb{J}}_{\mathrm{n}+1}, \lambda\,\overrightarrow{\mathbb{J}}_{\mathrm{n}+2})$$

and $\overrightarrow{\mathbb{J}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{J}_{\mathrm{m}}}$ are equal if and only if $\mathbb{J}_n = \mathbb{J}_m$, $\mathbb{J}_{n+1} = \mathbb{J}_{m+1}$, $\mathbb{J}_{n+2} = \mathbb{J}_{m+2}$.

Theorem 3. Let $\overrightarrow{\mathbb{J}}_{\mathrm{n}}$ and $\overrightarrow{\mathbb{J}}_{\mathrm{m}}$ be two generalized Jacobsthal vectors. The dot product of $\overrightarrow{\mathbb{J}}_{\mathrm{n}}$ and $\overrightarrow{\mathbb{J}}_{\mathrm{m}}$ is given by

$$\begin{aligned}\langle \overrightarrow{\mathbb{J}}_{\mathrm{n}}, \overrightarrow{\mathbb{J}}_{\mathrm{m}} \rangle &= p^2[\frac{1}{3}(J_{n+m} + J_{n+m+2} + J_{n+m+4}) \\ &+ (-1)^{n+1} J_m + (-1)^{m+1} J_n] \\ &+ 2\,p\,q\,[\frac{1}{3}(J_{n+m+1} + J_{n+m+3} + J_{n+m+5}) \\ &+ (-1)^{n+1} J_{m-1} + (-1)^{m+1} J_{n-1}] \\ &+ q^2[\frac{1}{3}(J_{n+m+2} + J_{n+m+4} + J_{n+m+6}) \\ &+ (-1)^{n+2} J_{m+1} + (-1)^{m+2} J_{n+1}]\,.\end{aligned} \tag{30}$$

[1]Binet's formula is the explicit formula to obtain the n-th Jacobsthal and Jacobsthal–Lucas numbers. It is well known that for the Jacobsthal and Jacobsthal–Lucas numbers, Binet's formulas are $J_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $j_n = \alpha^n + \beta^n$ respectively, where $\alpha + \beta = 1$, $\alpha - \beta = 3$, $\alpha\beta = -2$ and $\alpha = 2$, $\beta = -1$, [15, 16].

Proof. The dot product of $\overrightarrow{\mathbb{J}}_{\mathrm{n}} = (\mathbb{J}_n,\, \mathbb{J}_{n+1},\, \mathbb{J}_{n+2})$ and $\overrightarrow{\mathbb{J}}_{\mathrm{m}} = (\mathbb{J}_m\,,\, \mathbb{J}_{m+1},\, \mathbb{J}_{m+2})$ are defined by $\left\langle \overrightarrow{\mathbb{J}}_{\mathrm{n}}, \overrightarrow{\mathbb{J}}_{\mathrm{m}} \right\rangle = \mathbb{J}_n\mathbb{J}_m + \mathbb{J}_{n+1}\mathbb{J}_{m+1} + \mathbb{J}_{n+2}\mathbb{J}_{m+2}$. Also, using the equations (11), (12) and (13), we obtain

$$\mathbb{J}_n\, \mathbb{J}_m = p^2\,(J_n\, J_m) + p\, q\, [J_n J_{m+1} + J_{n+1}\, J_m] + q^2\,(J_{n+1}\, J_{m+1}), \tag{31}$$

$$\begin{aligned} \mathbb{J}_{n+1}\, \mathbb{J}_{m+1} &= p^2\,(J_{n+1}\, J_{m+1}) \\ &\quad + p\, q\, [J_{n+1} J_{m+2} + J_{n+2}\, J_{m+1}] + q^2\,(J_{n+2}\, J_{m+2}), \end{aligned} \tag{32}$$

$$\begin{aligned} \mathbb{J}_{n+2}\, \mathbb{J}_{m+2} &= p^2\,(J_{n+2} J_{m+2}) \\ &\quad + p\, q\, [J_{n+2} J_{m+3} + J_{n+3} J_{m+2}] + q^2\,(J_{n+3} J_{m+3}). \end{aligned} \tag{33}$$

Then, from the equations (31), (32) and (33), we have

$$\begin{aligned} \left\langle \overrightarrow{\mathbb{J}}_{\mathrm{n}}, \overrightarrow{\mathbb{J}}_{\mathrm{m}} \right\rangle &= p^2(J_n J_m + J_{n+1} J_{m+1} + J_{n+2} J_{m+2}) \\ &\quad + p\, q\, [J_n J_{m+1} + J_{n+1} J_m + J_{n+1} J_{m+2} \\ &\quad + J_{n+2} J_{m+1} + J_{n+2} J_{m+3} + J_{n+3} J_{m+2}] \\ &\quad + q^2\,(J_{n+1}\, J_{m+1} + J_{n+2}\, J_{m+2} + J_{n+3}\, J_{m+3}) \\ &= p^2[\frac{1}{3}\,(J_{n+m} + J_{n+m+2} + J_{n+m+4}) \\ &\quad + (-1)^{n+1} J_m + (-1)^{m+1} J_n] \\ &\quad + 2\, p\, q\, [\frac{1}{3}\,(J_{n+m+1} + J_{n+m+3} + J_{n+m+5}) \\ &\quad + (-1)^{n+1} J_{m-1} + (-1)^{m+1} J_{n-1}] \\ &\quad + q^2[\frac{1}{3}\,(J_{n+m+2} + J_{n+m+4} + J_{n+m+6}) \\ &\quad + (-1)^{n+2} J_{m+1} + (-1)^{m+2} J_{n+1}]\,. \end{aligned} \tag{34}$$

Then for the norm of the generalized Jacobsthal vector, using identities as follows:

$$J_{n+1}^2 + 2\, J_n^2 = J_{2n+1},$$

$$J_{n+1}^2 - J_n^2 = 2^{n+1} J_{n-1},$$

$$J_m J_{n+1} + 2\, J_{m-1} J_n = J_{m+n},$$

we have

$$\begin{aligned} \left\| \overrightarrow{\mathbb{J}_{\mathrm{n}}} \right\|^2 &= \left\langle \overrightarrow{\mathbb{J}_{\mathrm{n}}}, \overrightarrow{\mathbb{J}_{\mathrm{n}}} \right\rangle = {\mathbb{J}_n}^2 + {\mathbb{J}_{n+1}}^2 + {\mathbb{J}_{n+2}}^2 \\ &= p^2[\,\frac{1}{3}(J_{2n} + J_{2n+2} + J_{2n+4}) + 2\,(-1)^{n+1}\, J_n\,] \\ &\quad +\ 2\, p\, q\, [\,\frac{1}{3}(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}\, J_{n-1}\,] \\ &\quad +\ q^2[\,\frac{1}{3}(J_{2n+2} + J_{2n+4} + J_{2n+6}) + 2\,(-1)^{n+2}\, J_{n+1}\,] \\ &\qquad \textit{or} \\ &= p^2[J_{2n+3} - 2^{n+1} J_{n-1}\,] \\ &\quad +\ 2\, p\, q\, [\,\frac{1}{3}(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}\, J_{n-1}\,] \\ &\quad +\ q^2[\, J_{2n+5} - 2^{n+2} J_n\,]. \end{aligned} \tag{35}$$

- **Special case 1:** For the dot product of the generalized Jacobsthal vectors $\overrightarrow{\mathbb{J}}_{\mathrm{n}}$ and $\overrightarrow{\mathbb{J}}_{\mathrm{n+1}}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{J}}_{\mathrm{n}}, \overrightarrow{\mathbb{J}}_{\mathrm{n+1}} \right\rangle &= \mathbb{J}_n\mathbb{J}_{n+1} + \mathbb{J}_{n+1}\mathbb{J}_{n+2} + \mathbb{J}_{n+2}\mathbb{J}_{n+3} \\ &= p^2\,[\frac{1}{3}(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}\,J_{n-1}] \\ &\quad + 2\,p\,q\,[\frac{1}{3}(J_{2n+2} + J_{2n+4} + J_{2n+6}) + \,2\,(-1)^{n+1} J_{n-2}] \\ &\quad + q^2\,[\frac{1}{3}(J_{2n+3} + J_{2n+5} + J_{2n+7}) + \,2\,(-1)^{n}\,J_{n}]\end{aligned} \tag{36}$$

and

$$\begin{aligned}\langle \overrightarrow{\mathbb{J}}_{\mathrm{n}}, \overrightarrow{\mathbb{J}}_{\mathrm{n}} \rangle &= \mathbb{J}_n{}^2 + \mathbb{J}_{n+1}{}^2 + \mathbb{J}_{n+2}{}^2 \\ &= p^2[\,J_n^2 + J_{n+1}^2 + J_{n+2}^2\,] \\ &\quad + 2\,p\,q\,[J_nJ_{n+1} + J_{n+1}J_{n+2} + J_{n+}J_{n+3}\,] \\ &\quad + q^2\,[\,J_{n+1}^2 + J_{n+2}^2 + J_{n+3}^2\,] \\ &= p^2[\,\frac{1}{3}(J_{2n} + J_{2n+2} + J_{2n+4}) + 2\,(-1)^{n+1}\,J_n\,] \\ &\quad + 2\,p\,q\,[\frac{1}{3}(J_{2n+1} + J_{2n+3} + J_{2n+5}) + (-1)^{n+1}\,J_{n-1}\,] \\ &\quad + q^2[\,\frac{1}{3}(J_{2n+2} + J_{2n+4} + J_{2n+6}) + 2\,(-1)^{n+2}\,J_{n+1}\,] \\ &\qquad\qquad\qquad or \\ &= p^2[J_{2n+3} - 2^{n+1}J_{n-1}\,] \\ &\quad + \,2\,p\,q\,[\,\frac{1}{3}(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}\,J_{n-1}\,] \\ &\quad + \,q^2[\,J_{2n+5} - 2^{n+2}J_n\,].\end{aligned} \tag{37}$$

□

Theorem 4. Let $\overrightarrow{\mathbb{J}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{J}_{\mathrm{m}}}$ be two generalized Jacobsthal vectors. The cross product of $\overrightarrow{\mathbb{J}_{\mathrm{n}}}$ and $\overrightarrow{\mathbb{J}_{\mathrm{m}}}$ is given by

$$\overrightarrow{\mathbb{J}_{\mathrm{n}}} \times \overrightarrow{\mathbb{J}_{\mathrm{m}}} = (-1)^{n+1}\,2^n\,J_{m-n}\,(p^2 + pq - 2\,q^2)\,(-2\,i - j + k). \tag{38}$$

Proof. The cross product of $\overrightarrow{\mathbb{J}_{\mathrm{n}}} \times \overrightarrow{\mathbb{J}_{\mathrm{m}}}$ is defined by

$$\begin{aligned}\overrightarrow{\mathbb{J}}_{\mathrm{n}} \times \overrightarrow{\mathbb{J}}_{\mathrm{m}} = \begin{vmatrix} i & j & k \\ \mathbb{J}_n & \mathbb{J}_{n+1} & \mathbb{J}_{n+2} \\ \mathbb{J}_m & \mathbb{J}_{m+1} & \mathbb{J}_{m+2}\end{vmatrix} &= i\,(\mathbb{J}_{m+2}\mathbb{J}_{n+1} - \mathbb{J}_{m+1}\mathbb{J}_{n+2}) \\ &\quad -j\,(\mathbb{J}_{m+2}\mathbb{J}_{n} - \mathbb{J}_{m}\mathbb{J}_{n+2}) + k\,(\mathbb{J}_{m+1}\mathbb{J}_{n} - \mathbb{J}_{m}\mathbb{J}_{n+1}).\end{aligned} \tag{39}$$

Now, we calculate the cross products:
Using relations (11) and (13) or the property $J_n^2 - J_{n+1}J_{n-1} = (-1)^{n+1}\,2^{n-1}$, we get

$$\mathbb{J}_{m+2}\mathbb{J}_{n+1} - \mathbb{J}_{m+1}\mathbb{J}_{n+2} = \,(-1)^{n+2}\,2^{n+1}J_{m-n}\,(p^2 + pq - 2\,q^2), \tag{40}$$

$$\mathbb{J}_{m+2}\mathbb{J}_n - \mathbb{J}_m\mathbb{J}_{n+2} = (-1)^{n+1}\, 2^n\, J_{m-n}\, (p^2 + pq - 2\, q^2) \tag{41}$$

and

$$\mathbb{J}_{m+1}\mathbb{J}_n - \mathbb{J}_m\mathbb{J}_{n+1} = (-1)^{n+1}\, 2^n\, J_{m-n}\, (p^2 + pq - 2\, q^2). \tag{42}$$

Then from the equations (40), (41) and (42), we obtain the equation (38). □

Theorem 5. Let $\overrightarrow{\mathbb{J}_{\mathrm{n}}}$, $\overrightarrow{\mathbb{J}_{\mathrm{m}}}$ and $\overrightarrow{\mathbb{J}_l}$ be the generalized Jacobsthal vectors. The mixed product of these vectors is

$$\left\langle \overrightarrow{\mathbb{J}_{\mathrm{n}}} \times \overrightarrow{\mathbb{J}_{\mathrm{m}}}\ ,\ \overrightarrow{\mathbb{J}_l} \right\rangle = 0. \tag{43}$$

Proof. Using $\overrightarrow{\mathbb{J}_l} = (\mathbb{J}_l,\, \mathbb{J}_{l+1},\, \mathbb{J}_{l+2})$, we can write,

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{J}_{\mathrm{n}}} \times \overrightarrow{\mathbb{J}_{\mathrm{m}}}\ ,\ \overrightarrow{\mathbb{J}_l} \right\rangle = \begin{vmatrix} \mathbb{J}_n & \mathbb{J}_{n+1} & \mathbb{J}_{n+2} \\ \mathbb{J}_m & \mathbb{J}_{m+1} & \mathbb{J}_{m+2} \\ \mathbb{J}_l & \mathbb{J}_{l+1} & \mathbb{J}_{l+2} \end{vmatrix} &= \mathbb{J}_n \left(\mathbb{J}_{m+1}\, \mathbb{J}_{l+2} - \mathbb{J}_{m+2}\, \mathbb{J}_{l+1}\right) \\ +\mathbb{J}_{n+1} \left(\mathbb{J}_{m+2}\, \mathbb{J}_l - \mathbb{J}_m\, \mathbb{J}_{l+2}\right) &+ \mathbb{J}_{n+2} \left(\mathbb{J}_m\, \mathbb{J}_{l+1} - \mathbb{J}_{m+1}\, \mathbb{J}_l\right). \end{aligned} \tag{44}$$

Also, using the equations (40), (41) and (42) we obtain

$$\begin{aligned} \mathbb{J}_n \left(\mathbb{J}_{m+1}\, \mathbb{J}_{l+2} - \mathbb{J}_{m+2}\, \mathbb{J}_{l+1}\right) &+ \mathbb{J}_{n+1} \left(\mathbb{J}_{m+2}\, \mathbb{J}_l - \mathbb{J}_m\, \mathbb{J}_{l+2}\right) \\ &+ \mathbb{J}_{n+2} \left(\mathbb{J}_m\, \mathbb{J}_{l+1} - \mathbb{J}_l\, \mathbb{J}_{m+1}\right) \\ &= (-1)^l\, 2^l\, J_{m-l}\, (p^2 + p\, q - 2\, q^2) \\ &\quad (\, -2\, \mathbb{J}_n - \mathbb{J}_{n+1} + \mathbb{J}_{n+2}\,) \\ &= (-1)^l\, 2^l\, J_{m-l}\; e_J\, (\, -\mathbb{J}_{n+2} + \mathbb{J}_{n+2}\,) = 0. \end{aligned} \tag{45}$$

Thus, we have the equation (43). □

3 The generalized complex Jacobsthal sequence

In this section, the generalized complex Jacobsthal sequence, denoted by $\mathbb{C}_n$, will be defined. The generalized complex Jacobsthal sequence is defined by

$$\mathbb{C}_n = \mathbb{J}_n + i\, \mathbb{J}_{n+1}\ , \tag{46}$$

with $\mathbb{C}_0 = q\ + i\, (p + q)\,,\ \mathbb{C}_1 = (p +\ q)\ +\ i\, (p + 3\, q),\ \mathbb{C}_2 = (p + 3\, q)\ +\ i\, (3\, p + 5\ q),$ where p, q are arbitrary integers. That is, the generalized complex Jacobsthal sequence is

$$\begin{gathered} q\ + i\, (p + q)\,,\ (p +\ q)\ +\ i\, (p + 3\, q),\ (p + 3\, q)\ +\ i\, (3\, p + 5\ q), \\ (3\, p + 5\, q)\ +\ i\, (5\, p + 11\, q),\ \ldots\ ,\ (\, p +\ i\, 2\, q\,) J_n +\ (\ q +\ i\, (p + q)\,) J_{n+1}, \ldots \end{gathered} \tag{47}$$

- **Special case 1:** From the generalized complex Jacobsthal sequence $(\mathbb{C}_n)$ for $p = 1,\ q = 0$ in the equation (47), we obtain complex Jacobsthal sequence ($C_n = J_{1\,,\,i}$) as follows:

$$(C_n)\ :\ i\,,\ 1 + i\,,\ 1 + i\, 3\,,\ 3 + i\, 5,\ 5 + i\, 11, \ldots\ , J_n +\ i\, J_{n+1}, \ldots$$

- **Special case 2:** From the generalized complex Jacobsthal sequence ($\mathbb{C}_n$) for $p = -1,\ q = 2$ in the equation (47), we obtain complex Jacobsthal–Lucas sequence ($C_n = j_{-1+4i,2+i}$) as follows:

$$2+\ i\,,\ 1+i\,5\,,\ 5+i\,7\,,\ 7+i\,17,\ 17+i\,31\,,\ldots, j_n+\ i\,j_{n+1},\ldots$$

For the generalized complex Jacobsthal sequence, we have the following properties:

$$\mathbb{C}_n^2\ +\ 2\,\mathbb{C}_{n-1}^2 = [\,(\,2\,p+q\,)+i\,(\,p+5\ q\,)\,]\,\mathbb{C}_{2n-1}\ -(\,3\ +i\,)\ e_J\,J_{2\,n-1} \tag{48}$$

$$\mathbb{C}_{n+1}^2\ +\ 2\,\mathbb{C}_{n}^2 = [\,(\,2\,p+q\,)+i\,(\,p+5\ q\,)\,]\,\mathbb{C}_{2n+1}\ -(\,3\ +i\,)\ e_J\,J_{2\,n+1} \tag{49}$$

$$\mathbb{C}_{n+1}^2\ -\ 4\,\mathbb{C}_{n-1}^2 = [\,(\,2\,p+q\,)+i\,(\,p+5\ q\,)\,]\,\mathbb{C}_{2n}\ -(\,3\ +i\,)\ e_J\,J_{2\,n} \tag{50}$$

$$\mathbb{C}_{n-1}\,\mathbb{C}_{n+1} - \mathbb{C}_n^2 = (-1)^n\,2^{n-1}\,(3+\ i\,)\ e_J, \tag{51}$$

where $e_J = p^2 + p\,q - 2\,q^2$.

4 The generalized dual Jacobsthal sequence

In this section, the generalized dual Jacobsthal sequence, denoted by $\mathbb{D}_n^J$, will be defined. The generalized dual Jacobsthal sequence is defined by

$$\mathbb{D}_n^J = \mathbb{J}_n + \varepsilon\,\mathbb{J}_{n+1}\ , \tag{52}$$

with $\mathbb{D}_0^J = q\ +\varepsilon(p+q)\,,\ \mathbb{D}_1^J = p+\ q\ +\ \varepsilon(p+3\,q),\ \mathbb{D}_2^J = p+3\,q\ +\ \varepsilon(3\,p+5\ q),$ where p, q are arbitrary integers. That is, the generalized dual Jacobsthal sequence is

$$\begin{aligned} &q\ +\varepsilon(p+\ q)\,,\ (\,p+q\,)+\ \varepsilon(\,p+3\,q),\ (\,p+3\,q\,)+\ \varepsilon(3\,p+5\,q\,),\\ &(\,3\,p+5\,q\,)+\varepsilon(5\,p+11\ q),\ (\,5\,p+11\,q\,)+\ \varepsilon(11\,p+21\,q\,),\\ &\ldots,\ [\,(\,p+2\,\varepsilon\,q)J_n+(\,q+\ \varepsilon(p+q)\,)J_{n+1} = \mathbb{J}_n+\varepsilon\,\mathbb{J}_{n+1},\ldots \end{aligned} \tag{53}$$

Using the equations (52) and (53) , we get

$$\begin{aligned} \mathbb{D}_n^J &= (\,p+2\,\varepsilon\,q\,)J_n+\ (\,q+\varepsilon(\,p+\ q\,)\,)\,J_{n+1}\\ \mathbb{D}_{n+1}^J &= 2\,[\,(\,q+\varepsilon(\,p+\ q\,)\,]\,J_n+[\,(p+q)+\varepsilon\,(\,p+3\,q\,)\,]J_{n+1}\\ \mathbb{D}_{n+2}^J &= 2\,[\,(\,p+q\,)+\varepsilon(\,p+3\,q\,)\,]\,J_n+[\,(p+3\,q)+\varepsilon(\,3\,p+5\,q\,)\,]J_{n+1}\\ &\vdots\\ \mathbb{D}_{n+r}^J &= 2\,\mathbb{D}_{n-1}^J\,J_r+\mathbb{D}_n^J\,J_{r+1} \end{aligned} \tag{54}$$

- **Special case 1:** From the generalized dual Jacobsthal sequence ($\mathbb{D}_n^J$) for $p = 1,\ q = 0$ in the equation (53), we obtain dual Jacobsthal sequence (D_n^J) as follows:

$$(D_n^J)\ :\ \varepsilon\,,\,1+\varepsilon\,,\ 1+3\,\varepsilon\,,\ 3+5\,\varepsilon\,,\ 5+11\,\varepsilon,\ldots, J_n+\ \varepsilon\,J_{n+1},\ldots$$

- **Special case 2:** From the generalized dual Jacobsthal sequence ($\mathbb{D}_n^J$) for $p = -1,\ q = 2$ in the equation (53), we obtain dual Jacobsthal–Lucas sequence (D_n^j) as follows:

$$(D_n^j)\ :\ 2+\ \varepsilon\,,\ 1+5\,\varepsilon\,,\ 5+7\,\varepsilon\,,\ 7+17\,\varepsilon,\ 17+31\,\varepsilon\,,\ldots, j_n+\ \varepsilon\,j_{n+1},\ldots$$

For the generalized dual Jacobsthal sequence, we have the following properties:

$$(\mathbb{D}_n^J)^2 + 2\,(\mathbb{D}_n^J)^2 = [\,(2p+q) + \,\varepsilon(p+5\,q)\,]\,\mathbb{D}_{2n-1}^J - (1+\varepsilon)\,e_J\;J_{2n-1}\,, \tag{55}$$

$$(\mathbb{D}_{n+1}^J)^2 + 2\,(\mathbb{D}_n^J)^2 = [\,(2p+q) + \,\varepsilon(p+5\,q)\,]\,\mathbb{D}_{2n+1}^J - (1+\varepsilon)\,e_J\;J_{2n+1}\,, \tag{56}$$

$$(\mathbb{D}_{n+1}^J)^2 - 4\,(\mathbb{D}_{n-1}^J)^2 = [\,(2p+q) + \,\varepsilon(p+5\,q)\,]\,\mathbb{D}_{2n}^J - (1+\varepsilon)\,e_J\;J_{2n}\,, \tag{57}$$

$$\mathbb{D}_{n-1}^J\,\mathbb{D}_{n+1}^J - (\mathbb{D}_n^J)^2 = (-1)^n\,2^{n-1}\,(1+\varepsilon)\,e_J\,, \tag{58}$$

$$4\,(\mathbb{D}_n^J)^2 \;+\; 2\,J_{n+1}^2\,(1+\varepsilon)\,e_J = \;2\,(\,p + 2\,\varepsilon\,q\,)\,\mathbb{D}_{2n+1}^J\,, \tag{59}$$

$$\mathbb{D}_m^J\,\mathbb{D}_{n+1}^J + 2\,\mathbb{D}_{m-1}^J\,\mathbb{D}_n^J = (2p+q) \;+\; \varepsilon(p+5\,q)\mathbb{D}_{m+n}^J - (1+\varepsilon)\,e_J\;J_{m+n}\,, \tag{60}$$

$$\mathbb{D}_m^J\,\mathbb{D}_{n-1}^J - \;\mathbb{D}_{m-1}^J\,\mathbb{D}_n^J = (-1)^n\,2^{n-1}\,(1+\varepsilon)\,e_J\;J_{m-n}\,. \tag{61}$$

where $e_J = (\,p^2 + p\,q\; - 2\,q^2\,)\,.$

- **Special case 3:** From properties of the generalized dual Jacobsthal sequence ($\mathbb{D}_n^J$) for $p\,=\,1,\;\;q\,=\,0$ in the equations (55)–(61), we obtain dual Jacobsthal sequence (D_n^J) as follows:

$$(D_n^J)^2 + 2\,(D_{n-1}^J)^2 = \;(2+\,\varepsilon)\,D_{2n-1}^J - (1+\,\varepsilon\,)\,J_{2n-1}\,, \tag{62}$$

$$(D_{n+1}^J)^2 + 2\,(D_n^J)^2 = \;(2+\,\varepsilon)\,D_{2n+1}^J - (1+\,\varepsilon\,)\,J_{2n+1}\,, \tag{63}$$

$$(D_{n+1}^J)^2 - 4\,(D_{n-1}^J)^2 = (2+\,\varepsilon)\,D_{2n}^J - (1+\,\varepsilon\,)\,J_{2n} \tag{64}$$

$$D_{n-1}^J\,D_{n+1}^J - (D_n^J)^2 = (-1)^n\,2^{n-1}\,(1+\,\varepsilon\,)\,, \tag{65}$$

$$4\,(D_n^J)^2 \;+\; 2\,(1+\varepsilon)\,J_{n+1}^2 \;=\; 2\,D_{2n+1}^J\,, \tag{66}$$

$$D_m^J\,D_{n+1}^J + 2\,D_{m-1}^J\,D_n^J = (2+\,\varepsilon)\,D_{m+n}^J - (1+\,\varepsilon\,)\,J_{m+n}, \tag{67}$$

$$D_m^J\,D_{n-1}^J - \;D_{m-1}^J\,D_n^J = (-1)^n\,2^{n-1}\,(1+\,\varepsilon\,)J_{m-n}\,. \tag{68}$$

Theorem 6. If $\mathbb{D}_n^J$ is the generalized dual Jacobsthal number, then

$$\lim_{n\to\infty}\frac{\mathbb{D}_{n+1}^J}{\mathbb{D}_n^J} = \frac{(\,p\,q+q^2)\,\alpha^2 + (\,p^2 + \,p\,q + \,2\,q^2\,)\alpha + \,2\,p\,q}{q^2\alpha^2 + (\,2\,p\,q\,)\alpha + p^2}$$

where $\alpha = -1$.

Proof. For the generalized dual Jacobsthal number $\mathbb{D}_n^J$, we obtain

$$\begin{aligned}
\lim_{n\to\infty}\frac{\mathbb{D}_{n+1}^J}{\mathbb{D}_n^J} &= \lim_{n\to\infty}\frac{(p+2\,\varepsilon\,q)J_{n+1} + (q+\varepsilon\,(p+q))J_{n+2}}{(p+2\,\varepsilon\,q)J_n + [q+\varepsilon\,(p+q)]J_{n+1}}\\
&= \lim_{n\to\infty}\frac{(p^2+pq+2\,q^2)J_nJ_{n+1} + (p\,q+q^2)(J_{n+1}^2 + 2\,p\,q\,J_n^2)}{p^2J_n^2 + 2\,p\,q\,J_nJ_{n+1} + q^2J_{n+1}^2}\\
&\quad + \lim_{n\to\infty}\varepsilon\,\frac{(-1)^{n+1}\,2^n\,(p^2+p\,q-2\,q^2)}{p^2J_n^2 + 2\,p\,q\,J_nJ_{n+1} + q^2J_{n+1}^2}\\
&= \frac{(p^2+p\,q+2\,q^2)\,\alpha\; + (\,p\,q + \,q^2\,)\alpha^2 + (\,2\,p\,q\,)}{q^2\alpha^2 + (\,2\,p\,q\,)\alpha + p^2}
\end{aligned} \tag{69}$$

where $J_{n+2} = J_{n+1} + 2\,J_n$.

- **Special case 4:** For $p = 1$, $q = 0$ in the equation (69), we obtain

$$\lim_{n\to\infty} \frac{\mathbb{D}_{n+1}^J}{\mathbb{D}_n^J} = \lim_{n\to\infty} \frac{D_{n+1}^J}{D_n^J} = \alpha + 0 = \alpha.$$ □

Theorem 7. The Binet's formula for the generalized dual Jacobsthal sequence is as follows;

$$\mathbb{D}_n^J = \frac{1}{\alpha - \beta} (\bar{\alpha}\, \alpha^n - \bar{\beta}\, \beta^n). \tag{70}$$

Proof. If we use definition of the generalized dual Jacobsthal sequence and substitute first equation in footnote, then we get

$$\begin{aligned} \mathbb{D}_n^J &= (p + 2\varepsilon q) J_n + (q + \varepsilon (p+q)) J_{n+1} \\ &= (p + 2\varepsilon q)(\frac{\alpha^n - \beta^n}{\alpha - \beta}) + (q + \varepsilon (p+q))(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}) \\ &= \frac{\overline{\alpha}\, \alpha^n - \overline{\beta}\, \beta^n}{\alpha - \beta}, \end{aligned} \tag{71}$$

where $\overline{\alpha} = (p + 2\varepsilon q) + \alpha (q + \varepsilon (p+q))$ and $\overline{\beta} = (p + 2\varepsilon q) + \beta (q + \varepsilon (p+q))$. □

5 The generalized dual Jacobsthal vectors

A generalized dual Jacobsthal vector is defined by

$$\overrightarrow{\mathbb{D}^J}_n = (\mathbb{D}_n^J, \mathbb{D}_{n+1}^J, \mathbb{D}_{n+2}^J).$$

From the equations (52), (53) and (54) it can be expressed as

$$\overrightarrow{\mathbb{D}^J}_n = \overrightarrow{\mathbb{J}}_n + \varepsilon\, \overrightarrow{\mathbb{J}}_{n+1} = (p + 2\varepsilon q) \overrightarrow{\mathrm{J}}_n + (q + \varepsilon (p+q)) \overrightarrow{\mathrm{J}}_{n+1} \tag{72}$$

where $\overrightarrow{\mathbb{J}}_n = (\mathbb{J}_n, \mathbb{J}_{n+1}, \mathbb{J}_{n+2})$ and $\overrightarrow{\mathrm{J}}_{\mathrm{n}} = (J_n, J_{n+1}, J_{n+2})$ are the generalized Jacobsthal vector and the Jacobsthal vector, respectively. The product of $\overrightarrow{\mathbb{D}^J}_n$ and $\lambda \in \mathbb{R}$ is given by

$$\lambda \overrightarrow{\mathbb{D}^J}_n = \lambda \overrightarrow{\mathbb{J}}_n + \varepsilon \lambda \overrightarrow{\mathbb{J}}_{n+1}$$

and $\overrightarrow{\mathbb{D}^J}_n$ and $\overrightarrow{\mathbb{D}^J}_m$ are equal if and only if

$$\mathbb{J}_{\mathrm{n}} = \mathbb{J}_{\mathrm{m}},\ \mathbb{J}_{\mathrm{n+1}} = \mathbb{J}_{\mathrm{m+1}},\ \mathbb{J}_{\mathrm{n+2}} = \mathbb{J}_{\mathrm{m+2}}.$$

Some examples of the generalized dual Jacobsthal vectors can be given easily as:

$$\begin{aligned} \overrightarrow{\mathbb{D}^J}_1 &= (\mathbb{D}_1^J, \mathbb{D}_2^J, \mathbb{D}_3^J) = (\mathbb{J}_1, \mathbb{J}_2, \mathbb{J}_3) + \varepsilon(\mathbb{J}_2, \mathbb{J}_3, \mathbb{J}_4) \\ &= [(p+q) + \varepsilon(p+3q),\ (p+3q) + \varepsilon(3p+5q),\ (3p+5q) + \varepsilon(5p+11q)], \\ \overrightarrow{\mathbb{D}^J}_2 &= (\mathbb{D}_2^J, \mathbb{D}_3^J, \mathbb{D}_4^J) + \varepsilon(\mathbb{D}_3^J, \mathbb{D}_4^J, \mathbb{D}_5^J) \\ &= [(p+3q) + \varepsilon(3p+5q),\ (3p+5q) + \varepsilon(5p+11q), (5p+11q) + \varepsilon(11p+21q)]. \end{aligned}$$

Theorem 8. Let $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m$ be two generalized dual Jacobsthal vectors. The dot product of $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m$ is given by

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{\mathbb{J}}}_n, \overrightarrow{\mathbb{D}^{\mathbb{J}}}_m \right\rangle =& p^2[\frac{1}{3}(J_{n+m}+J_{n+m+2}+J_{n+m+4}) \\ &+(-1)^{n+1}J_m+(-1)^{m+1}J_n] \\ &+2\,p\,q\,[\frac{1}{3}(J_{n+m+1}+J_{n+m+3}+J_{n+m+5}) \\ &+(-1)^{n+1}J_{m-1}+(-1)^{m+1}J_{n-1}] \\ &+q^2[\frac{1}{3}(J_{n+m+2}+J_{n+m+4}+J_{n+m+6}) \\ &+(-1)^{n+2}J_{m+1}+(-1)^{m+2}J_{n+1}] \\ &+\ 2\,\varepsilon\,\{\,p^2[\frac{1}{3}(J_{n+m+1}+J_{n+m+3}+J_{n+m+5}) \\ &+(-1)^{n+1}J_{m-1}+(-1)^{m+1}J_{n-1}] \\ &+2\,p\,q\,[\frac{1}{3}(J_{n+m+2}+J_{n+m+4}+J_{n+m+6}) \\ &+(-1)^{n+1}J_{m-2}+(-1)^{m+1}J_{n-2}] \\ &+\ q^2[\frac{1}{3}(J_{n+m+3}+J_{n+m+5}+J_{n+m+7}) \\ &+(-1)^{n+2}J_m+(-1)^{m+2}J_n\,]\}.\end{aligned} \tag{73}$$

Proof. The dot product of $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n=(\mathbb{D}^J_n\,,\,\mathbb{D}^J_{n+1},\,\mathbb{D}^J_{n+2})$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m=(\mathbb{D}^J_m\,,\,\mathbb{D}^J_{m+1},\,\mathbb{D}^J_{m+2})$ is defined by

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{\mathbb{J}}}_n, \overrightarrow{\mathbb{D}^{\mathbb{J}}}_m \right\rangle &= \mathbb{D}^J_n\mathbb{D}^J_m+\mathbb{D}^J_{n+1}\mathbb{D}^J_{m+1}+\mathbb{D}^J_{n+2}\mathbb{D}^J_{m+2} \\ &= \left\langle \overrightarrow{\mathbb{J}}_n, \overrightarrow{\mathbb{J}}_m \right\rangle+\varepsilon\,[\left\langle \overrightarrow{\mathbb{J}}_n\,,\,\overrightarrow{\mathbb{J}}_{m+1} \right\rangle+\left\langle \overrightarrow{\mathbb{J}}_{n+1}\,,\,\overrightarrow{\mathbb{J}}_m \right\rangle],\end{aligned}$$

where $\overrightarrow{\mathbb{J}}_n=(\mathbb{J}_n\,,\,\mathbb{J}_{n+1},\,\mathbb{J}_{n+2})$ is the generalized Jacobsthal vector. Also, the equations (11), (12) and (13), we obtain

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{J}}_n, \overrightarrow{\mathbb{J}}_m \right\rangle =& p^2[\frac{1}{3}(J_{n+m}+J_{n+m+2}+J_{n+m+4}) \\ &+(-1)^{n+1}J_m+(-1)^{m+1}J_n] \\ &+2\,p\,q\,[\frac{1}{3}(J_{n+m+1}+J_{n+m+3}+J_{n+m+5}) \\ &+(-1)^{n+1}J_{m-1}+(-1)^{m+1}J_{n-1}] \\ &+q^2[\frac{1}{3}(J_{n+m+2}+J_{n+m+4}+J_{n+m+6}) \\ &+(-1)^{n+2}J_{m+1}+(-1)^{m+2}J_{n+1}]\,,\end{aligned} \tag{74}$$

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{J}}_n, \overrightarrow{\mathbb{J}}_{m+1}\right\rangle =&p^2[\frac{1}{3}\left(J_{n+m+1}+J_{n+m+3}+J_{n+m+5}\right)\\ &+(-1)^{n+1}J_{m+1}+(-1)^{m+2}J_n]\\ &+2\,p\,q\,[\frac{1}{3}\left(J_{n+m+2}+J_{n+m+4}+J_{n+m+6}\right)\\ &+(-1)^{n+1}J_m+(-1)^{m+2}J_{n-1}]\\ &+q^2[\frac{1}{3}\left(J_{n+m+3}+J_{n+m+5}+J_{n+m+7}\right)\\ &+(-1)^{n+2}J_{m+2}+(-1)^{m+3}J_{n+1}]\,,\end{aligned} \tag{75}$$

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{J}}_{n+1}, \overrightarrow{\mathbb{J}}_{m}\right\rangle =&p^2[\frac{1}{3}\left(J_{n+m+1}+J_{n+m+3}+J_{n+m+5}\right)\\ &+(-1)^{n+2}J_{m}+(-1)^{m+1}J_{n+1}]\\ &+2\,p\,q\,[\frac{1}{3}\left(J_{n+m+2}+J_{n+m+4}+J_{n+m+6}\right)\\ &+(-1)^{n+2}J_{m-1}+(-1)^{m+1}J_{n}]\\ &+q^2[\frac{1}{3}\left(J_{n+m+3}+J_{n+m+5}+J_{n+m+7}\right)\\ &+(-1)^{n+3}J_{m+1}+(-1)^{m+2}J_{n+2}].\end{aligned} \tag{76}$$

Then from equation (74), (75) and (76), we have the equation (73). □

- **Special case 1:** For the dot product of generalized dual Jacobsthal vectors $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_{n+1}$, we get

$$\begin{aligned}\left\langle \overrightarrow{\mathbb{D}^{\mathbb{J}}}_n, \overrightarrow{\mathbb{D}^{\mathbb{J}}}_{n+1}\right\rangle &= \mathbb{D}^J_n\mathbb{D}^J_{n+1}+\mathbb{D}^J_{n+1}\mathbb{D}^J_{n+2}+\mathbb{D}^J_{n+2}\mathbb{D}^J_{n+3}\\ &=\left\langle \overrightarrow{\mathbb{J}}_n, \overrightarrow{\mathbb{J}}_{n+1}\right\rangle+\varepsilon\{\left\langle \overrightarrow{\mathbb{J}}_n, \overrightarrow{\mathbb{J}}_{n+2}\right\rangle+\left\langle \overrightarrow{\mathbb{J}}_{n+1}, \overrightarrow{\mathbb{J}}_{n+1}\right\rangle\}\\ &=\{p^2[\frac{1}{3}\left(J_{2n+1}+J_{2n+3}+J_{2n+5}\right)+2\left(-1\right)^{n+1}J_{n-1}]\\ &\quad+2\,p\,q\,[\frac{2}{3}\left(J_{2n+2}+J_{2n+4}+J_{2n+6}\right)+2\left(-1\right)^{n+1}J_{n-2}]\\ &\quad+q^2\,[\frac{1}{3}\left(J_{2n+3}+J_{2n+5}+J_{2n+7}\right)+\;2\left(-1\right)^{n+2}J_{n}\,]\,\}\\ &+2\varepsilon\,\{\,p^2[\frac{1}{3}\left(J_{2n+2}+J_{2n+4}+J_{2n+6}\right)+2\left(-1\right)^{n+1}\,J_{n-2}\,]\\ &\quad+\;p\,q\,[\frac{2}{3}\left(J_{2n+3}+J_{2n+5}+J_{2n+7}\right)+4\left(-1\right)^{n+1}\,J_{n-3}\,]\\ &\quad+q^2\,[\frac{1}{3}\left(J_{2n+4}+J_{2n+6}+J_{2n+8}\right)+2\left(-1\right)^{n+2}\,J_{n-1}\,]\,\}.\end{aligned} \tag{77}$$

Then for the norm of the generalized dual Jacobsthal vector[3], we have

$$\begin{aligned}
\left\|\overrightarrow{\mathbb{D}^{J}}_{n}\right\| &= \sqrt{\left[\left\langle \overrightarrow{\mathbb{D}^{J}}_{n}, \overrightarrow{\mathbb{D}^{J}}_{n}\right\rangle\right]} = \sqrt{\left[(\mathbb{D}^{J}_{n})^2 + (\mathbb{D}^{J}_{n+1})^2 + (\mathbb{D}^{J}_{n+2})^2\right]} \\
&= \sqrt{[p^2(J_{2n+3} + J_n^2 - J_{n+1}^2)]} \\
&\quad + \sqrt{2\,p\,q[(\frac{1}{3}\,(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}J_{n-1})]} \\
&\quad + \sqrt{q^2[(J_{2n+5} + J_{n+1}^2 - J_{n+2}^2)]} \\
&\quad + \sqrt{\varepsilon\{2p^2(\frac{1}{3}\,(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 4\,(-1)^{n+1}J_{n-1})\}} \\
&\quad + \sqrt{\varepsilon\{4\,p\,q[\frac{1}{3}\,(J_{2n+2} + J_{2n+4} + J_{2n+6}) + 4\,(-1)^{n+1}J_{n-2}]\}} \\
&\quad + \sqrt{\varepsilon\{2q^2(\frac{1}{3}\,(J_{2n+3} + J_{2n+5} + J_{2n+7}) + 4\,(-1)^{n+2}J_{n})]\}}
\end{aligned} \tag{78}$$

where used identity of the Jacobsthal numbers as follows [6]

$$J_n\,J_{n+k} = \frac{1}{3}\,(J_{2n+k} + (-1)^{n+1}J_{n+k} + (-1)^{n+k+1}J_n.$$

$$\begin{aligned}
\left\langle \overrightarrow{\mathbb{D}^{J}}_{n}, \overrightarrow{\mathbb{D}^{J}}_{n}\right\rangle &= (\mathbb{D}^{J}_{n})^2 + (\mathbb{D}^{J}_{n+1})^2 + (\mathbb{D}^{J}_{n+2})^2 \\
&= \left\langle \overrightarrow{\mathbb{J}}_{n}\,,\ \overrightarrow{\mathbb{J}}_{n}\right\rangle + 2\varepsilon\left\langle \overrightarrow{\mathbb{J}}_{n}\,,\ \overrightarrow{\mathbb{J}}_{n+1}\right\rangle \\
&= p^2[J_{2n+3} + J_n^2 - J_{n+1}^2] \\
&\quad + \ 2\,p\,q[\frac{1}{3}\,(J_{2n+1} + J_{2n+3} + J_{2n+5}) \\
&\quad + 2\,(-1)^{n+1}J_{n-1}] + q^2[J_{2n+5} + J_{n+1}^2 - J_{n+2}^2] \\
&\quad + \ \varepsilon\,\{2\,p^2[\frac{1}{3}\,(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}J_{n-1}\,] \\
&\quad + \ 4\,p\,q\,[\,\frac{1}{3}\,(J_{2n+2} + J_{2n+4} + J_{2n+6}) + 2\,(-1)^{n+1}J_{n-2}\,] \\
&\quad + 2\,q^2[\frac{1}{3}\,(J_{2n+3} + J_{2n+5} + J_{2n+7}) + 2\,(-1)^{n+2}J_{n}\,]\}.
\end{aligned} \tag{79}$$

- **Special case 2:** For $p = 1, q = 0$, in the equations (73), (77) and (79), we have

$$\begin{aligned}
\left\langle \overrightarrow{D^{J}}_{n}, \overrightarrow{D^{J}}_{m}\right\rangle &= \left[\frac{1}{3}(J_{n+m} + J_{n+m+2} + J_{n+m+4}) + (-1)^{n+1}J_m + (-1)^{m+1}\,J_n\right] \\
&\quad + 2\,\varepsilon\left[\frac{1}{3}\,(J_{n+m+1} + J_{n+m+3} + J_{n+m+5}) + (-1)^{n+1}J_{m-1} + (-1)^{m+1}\,J_{n-1}\right], \\
\left\langle \overrightarrow{D^{J}}_{n}, \overrightarrow{D^{J}}_{n+1}\right\rangle &= \left[\frac{1}{3}\,(J_{2n+1} + J_{2n+3} + J_{2n+5}) + 2\,(-1)^{n+1}J_{n-1}\right] \\
&\quad + 2\,\varepsilon\left[\frac{1}{3}\,(J_{2n+2} + J_{2n+4} + J_{2n+6}) + 2\,(-1)^{n+1}\,J_{n-2}\right]
\end{aligned}$$

[3] Norm of dual number as follows: $\left\|\overrightarrow{\mathbb{A}}\right\| = \sqrt{a + \varepsilon\,a^*} = \sqrt{a} + \varepsilon\,a^*\,\frac{1}{2\sqrt{a}}, A = a + \varepsilon\,a^*$, [1].

and

$$\begin{aligned}\left\|\overrightarrow{D^J}_n\right\| &= \sqrt{(J_{2n+3}+J_n^2-J_{n+1}^2)}\\ &+ \sqrt{2\,\varepsilon\,[\frac{1}{3}\,(J_{2n+1}+J_{2n+3}+J_{2n+5})+2\,(-1)^{n+1}J_{n-1}]}\\ &= (J_{2n+3}+J_n^2-J_{n+1}^2\,)+\varepsilon\,\frac{\frac{1}{3}\,[\,(J_{2n+1}+J_{2n+3}+J_{2n+5})+2\,(-1)^{n+1}J_{n-1}]}{\sqrt{(J_{2n+3}+J_n^2-J_{n+1}^2\,)}}.\end{aligned}$$

Theorem 9. Let $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m$ be two generalized dual Jacobsthal vectors. The cross product of $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m$ is given by

$$\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n\times\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m=(-1)^{n+1}\,2^n\,J_{m-n}\,(1+\varepsilon)\,(p^2+pq-2\,q^2)(-2\,i-j+k). \tag{80}$$

Proof. The cross product of $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n=\overrightarrow{\mathbb{J}}_n+\varepsilon\overrightarrow{\mathbb{J}}_{n+1}$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m=\overrightarrow{\mathbb{J}}_m+\varepsilon\overrightarrow{\mathbb{J}}_{m+1}$ is defined by

$$\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n\times\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m=(\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m)+\varepsilon\,(\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_{m+1}+\overrightarrow{\mathbb{J}}_{n+1}\times\overrightarrow{\mathbb{J}}_m),$$

where $\overrightarrow{\mathbb{J}}_n$ is the generalized Jacobsthal vector and $\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m$ is the cross product for the generalized Jacobsthal vectors $\overrightarrow{\mathbb{J}}_n$ and $\overrightarrow{\mathbb{J}}_m$.

Now, we calculate the cross products $\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m$, $\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_{m+1}$ and $\overrightarrow{\mathbb{J}}_{n+1}\times\overrightarrow{\mathbb{J}}_m$. Using the property $J_mJ_{n-1}-J_{m-1}J_n=(-1)^n\,2^{n-1}J_{m-n}$, we get

$$\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m=(-1)^{n+1}\,2^nJ_{m-n}(-2\,i-j+k)(p^2+pq-2\,q^2) \tag{81}$$

$$\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_{m+1}=(-1)^{n+1}\,2^n\,J_{m-n+1}(-2\,i-j+k)(p^2+pq-2\,q^2) \tag{82}$$

and

$$\overrightarrow{\mathbb{J}}_{n+1}\times\overrightarrow{\mathbb{J}}_m=(-1)^{n+2}\,2^{n+1}\,J_{m-n-1}(-2\,i-j+k)(p^2+pq-2\,q^2). \tag{83}$$

Then from the equations (81), (82) and (83), we obtain the equation (80).

- **Special case 3:** For $p=1,\ q=0$ in the equations (80), we have

$$\overrightarrow{D^J}_n\times\overrightarrow{D^J}_m=(-1)^{n+1}\,2^n\,J_{m-n}(1+\varepsilon)(-2\,i-j+k).$$

□

Theorem 10. Let $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n$, $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m$ and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_l$ be the generalized dual Jacobsthal vectors. The mixed product of these vectors is

$$\left\langle\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n\times\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m\,,\,\overrightarrow{\mathbb{D}^{\mathbb{J}}}_l\right\rangle=0. \tag{84}$$

Proof. Using the properties

$$\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n\times\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m=(\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m)+\varepsilon\,(\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_{m+1}+\overrightarrow{\mathbb{J}}_{n+1}\times\overrightarrow{\mathbb{J}}_m)$$

and $\overrightarrow{\mathbb{D}^{\mathbb{J}}}_l=\overrightarrow{\mathbb{J}}_l+\varepsilon\overrightarrow{\mathbb{J}}_{l+1}$, we can write,

$$\begin{aligned}\left\langle\overrightarrow{\mathbb{D}^{\mathbb{J}}}_n\times\overrightarrow{\mathbb{D}^{\mathbb{J}}}_m,\ \overrightarrow{\mathbb{D}^{\mathbb{J}}}_l\right\rangle &= \left\langle\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m,\overrightarrow{\mathbb{J}}_l\right\rangle+\varepsilon\,[\left\langle\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_m,\overrightarrow{\mathbb{J}}_{l+1}\right\rangle\\ &+\left\langle\overrightarrow{\mathbb{J}}_n\times\overrightarrow{\mathbb{J}}_{m+1},\overrightarrow{\mathbb{J}}_l\right\rangle+\left\langle\overrightarrow{\mathbb{J}}_{n+1}\times\overrightarrow{\mathbb{J}}_m,\overrightarrow{\mathbb{J}}_{l+1}\right\rangle].\end{aligned}$$

Then from equations (81), (82) and (83), we obtain

$$\left\langle (-2\,i - j + k),\ \overrightarrow{\mathbb{J}}_l \right\rangle = -2\,\mathbb{J}_l - \mathbb{J}_{l+1} + \mathbb{J}_{l+2} = 0,$$
$$\left\langle (-2\,i - j + k),\ \overrightarrow{\mathbb{J}}_{l+1} \right\rangle = -2\,\mathbb{J}_{l+1} - \mathbb{J}_{l+2} + \mathbb{J}_{l+3} = 0.$$

Thus, we have the equation (84). □

6 Conclusions

The generalized Jacobsthal, the generalized complex Jacobsthal and the generalized dual Jacobsthal sequences have been introduced and studied. The use of such special sequences has increased significantly in quantum mechanics, quantum physics, etc.

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