Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 24, 2018, No. 1, 53–60
DOI: 10.7546/nntdm.2018.24.1.53-60

New index matrix representations of operations over natural numbers

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Received: 8 July 2017 **Accepted:** 31 January 2018

Abstract: Two new operations over index matrices are introduced. Their possible application in number theory is discussed and illustrated with examples related to the canonical representation of the natural numbers and with two extended Fibonacci sequences.

Keywords: Fibonacci sequence, Index matrix, Natural number.
2010 Mathematics Subject Classification: 11B39.

1 Introduction

The idea of the concept of an Index Matrix (IM) was discussed for the first time in [1] and introduced formally in [4]. There, the first operations over IMs were given. The basic results, related to IMs, were included in [5]. In this book, as examples, IM-representations of some operations in number theory were described.

In the present paper, extensions of some operations discussed in [5] are given and new examples are described.

2 Preliminaries

Following [5], we define the concept of an IM and some operations over them.

Let $\mathcal{I}$ be a fixed set of indices and $\mathcal{R}$ be the set of real numbers. Let operations $\circ, * : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ be fixed. For example, they can be $\circ, * \in \{\times, +, \max, \min\}$, or others.

Let the standard sets K and L satisfy the condition: $K, L \subset \mathcal{I}$. Let over these sets, the standard set-theoretical operations be defined. We call "IM with real number elements" ($\mathcal{R}$-IM) the object:

$$[K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \dots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \dots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \dots & a_{k_m,l_n} \end{array},$$

where

$$K = \{k_1, k_2, \dots, k_m\} \text{ and } L = \{l_1, l_2, \dots, l_n\},$$

and for $1 \le i \le m$, and for $1 \le j \le n : a_{k_i,l_j} \in \mathcal{R}$.

Let the IM A be given and let $k_0 \notin K$ and $l_0 \notin L$ be two indices. Now, following [7] and [5], we introduce the following four aggregation operations over it:

Max-row-aggregation

$$\rho_{max}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \max\limits_{1\le i\le m} a_{k_i,l_1} & \max\limits_{1\le i\le m} a_{k_i,l_2} & \dots & \max\limits_{1\le i\le m} a_{k_i,l_n} \end{array},$$

Min-row-aggregation

$$\rho_{min}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \min\limits_{1\le i\le m} a_{k_i,l_1} & \min\limits_{1\le i\le m} a_{k_i,l_2} & \dots & \min\limits_{1\le i\le m} a_{k_i,l_n} \end{array},$$

Sum-row-aggregation

$$\rho_{sum}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \sum\limits_{i=1}^{m} a_{k_i,l_1} & \sum\limits_{i=1}^{m} a_{k_i,l_2} & \dots & \sum\limits_{i=1}^{m} a_{k_i,l_n} \end{array},$$

Average-row-aggregation

$$\rho_{ave}(A, k_0) = \begin{array}{c|cccc} & l_1 & l_2 & \dots & l_n \\ \hline k_0 & \frac{1}{m}\sum\limits_{i=1}^{m} a_{k_i,l_1} & \frac{1}{m}\sum\limits_{i=1}^{m} a_{k_i,l_2} & \dots & \frac{1}{m}\sum\limits_{i=1}^{m} a_{k_i,l_n} \end{array},$$

3 Main results

As it was mentioned in [5], it is well-known (see, e.g., [9, 10]) that each natural number m has a canonical representation $m = \prod\limits_{i=1}^{k} p_i^{\alpha_i}$, where k, $\alpha_1, \alpha_2, \dots, \alpha_k \ge 1$ are natural numbers and $p_1, p_2, \dots, p_k$ are different prime numbers. Let us always suppose that $p_1 < p_2 < \dots < p_k$. This condition is only for convinience, because there is no specific an order of the rows and columns in an IM, but these are labeled by indices.

Then, as it is shown in [5], the natural number m has the following IM-interpretation:

$$IM(m, a) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline a & \alpha_1 & \alpha_2 & \dots & \alpha_k \end{array},$$

where "a" is an arbitrary symbol, in a particular case – the same "m". In this case, for brevity, we write $IM(m, m) = IM(m)$.

In [2] the function $\underline{set}$ is introduced for the above number m by $\underline{set}(m) = \{p_1, \dots, p_k\}$.

First, we generalize the examples from [5]. Let us have s natural numbers $N_1, N_2, \dots, N_s$ and let

$$\bigcup_{i=1}^{s} \underline{set}(N_i) = \{p_1, \dots, p_k\}.$$

Therefore, for each i $(1 \leq i \leq s) : N_i = \prod_{j=1}^{k} p_j^{\alpha_{i,j}}$, where $\alpha_{i,j} \geq 0$ and $\sum_{j=1}^{k} \alpha_{i,j} \geq 1$. Now, we construct the IM

$$IM(N_1, \dots, N_s) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline N_1 & \alpha_{1,1} & \alpha_{1,2} & \dots & \alpha_{1,k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_s & \alpha_{s,1} & \alpha_{s,2} & \dots & \alpha_{s,k} \end{array}.$$

For example, if $N_1 = 12, N_2 = 27, N_3 = 30, N_4 = 150$, then these numbers have the canonical representation $N_1 = 2^2 \times 3, N_2 = 3^3, N_3 = 2 \times 3 \times 5, N_4 = 2 \times 3 \times 5^2$ and IM-representation

$$IM(N_1, N_2, N_3, N_4) = \begin{array}{c|ccc} & 2 & 3 & 5 \\ \hline N_1 & 2 & 1 & 0 \\ N_2 & 0 & 3 & 0 \\ N_3 & 1 & 1 & 1 \\ N_4 & 1 & 1 & 2 \end{array}.$$

The result of application of the aggregation operations over IM $IM(N_1, \dots, N_s)$ will be, respectively:

$$\rho_{max}(IM(N_1, \dots, N_s), k_0) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline k_0 & \max\limits_{1 \leq i \leq s} \alpha_{i,1} & \max\limits_{1 \leq i \leq s} \alpha_{i,2} & \dots & \max\limits_{1 \leq i \leq m} \alpha_{i,k} \end{array},$$

$$\rho_{min}(IM(N_1, \dots, N_s), k_0) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline k_0 & \min\limits_{1 \leq i \leq s} \alpha_{i,1} & \min\limits_{1 \leq i \leq s} \alpha_{i,2} & \dots & \min\limits_{1 \leq i \leq m} \alpha_{i,k} \end{array},$$

$$\rho_{sum}(IM(N_1, \dots, N_s), k_0) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline k_0 & \sum\limits_{1 \leq i \leq s} \alpha_{i,1} & \sum\limits_{1 \leq i \leq s} \alpha_{i,2} & \dots & \sum\limits_{1 \leq i \leq m} \alpha_{i,k} \end{array},$$

$$\rho_{ave}(IM(N_1, \dots, N_s), k_0) = \begin{array}{c|cccc} & p_1 & p_2 & \dots & p_k \\ \hline k_0 & \frac{1}{s} \sum\limits_{1 \leq i \leq s} \alpha_{i,1} & \frac{1}{s} \sum\limits_{1 \leq i \leq s} \alpha_{i,2} & \dots & \frac{1}{s} \sum\limits_{1 \leq i \leq m} \alpha_{i,k} \end{array}.$$

Now, we see immediately that:

- IM $\rho_{max}(IM(N_1, \ldots, N_s), k_0)$ represents the least common multiple of the numbers $N_1, \ldots, N_s$;
- IM $\rho_{min}(IM(N_1, \ldots, N_s), k_0)$ represents the greatest common divisor of the numbers $N_1, \ldots, N_s$;
- IM $\rho_{sum}(IM(N_1, \ldots, N_s), k_0)$ represents the product of the numbers $N_1, \ldots, N_s$;
- IM $\rho_{ave}(IM(N_1, \ldots, N_s), k_0)$ represents the geometric average of the numbers $N_1, \ldots, N_s$.

It is worth mentioning that the fourth case is not discussed in [5].

For the above example, these formulas obtain the following forms:

$$\rho_{max}(IM(N_1, \ldots, N_4), k_0) = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline k_0 & 2 & 3 & 2 \end{array},$$

$$\rho_{min}(IM(N_1, \ldots, N_4), k_0) = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline k_0 & 0 & 1 & 0 \end{array},$$

$$\rho_{sum}(IM(N_1, \ldots, N_4), k_0) = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline k_0 & 4 & 6 & 3 \end{array},$$

$$\rho_{ave}(IM(N_1, \ldots, N_4), k_0) = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline k_0 & 1 & \frac{3}{2} & \frac{3}{4} \end{array}.$$

The fourth case gives the idea for introducing of the following new aggregation operation:

$$\rho_{geo}(IM(N_1, \ldots, N_s), k_0) = \begin{array}{c|cccc} & p_1 & p_2 & \ldots & p_k \\ \hline k_0 & \sqrt[s]{\prod_{1 \le i \le s} \alpha_{i,1}} & \sqrt[s]{\prod_{1 \le i \le s} \alpha_{i,2}} & \ldots & \sqrt[s]{\prod_{1 \le i \le m} \alpha_{i,k}} \end{array}.$$

For the above example we obtain:

$$\rho_{geo}(IM(N_1, \ldots, N_4), k_0) = \begin{array}{c|ccc} & p_1 & p_2 & p_3 \\ \hline k_0 & 0 & \sqrt[4]{3} & 0 \end{array},$$

but we must mention immediately that the elements of the newly constructed IM do not correspond to geometric average of $N_1, \ldots, N_s$. They do not correspond to any known arithmetic operation.

Second, we introduce two new IM-operations in which the indices, when they are real (natural) numbers, participate with additional role.

Let us have the IM

$$A = [K, L, \{a_{k_i,l_j}\}] \equiv \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} & a_{k_1,l_2} & \ldots & a_{k_1,l_n} \\ k_2 & a_{k_2,l_1} & a_{k_2,l_2} & \ldots & a_{k_2,l_n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} & a_{k_m,l_2} & \ldots & a_{k_m,l_n} \end{array},$$

where $K = \{k_1, k_2, \ldots, k_m\} \subset \mathcal{R}$, and $L = \{l_1, l_2, \ldots, l_n\} \subset \mathcal{R}$, and for $1 \leq i \leq m$, and for $1 \leq j \leq n : a_{k_i,l_j} \in \mathcal{R}$.

Now, we define

$$\downarrow_\circ A = \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} \circ l_1 & a_{k_1,l_2} \circ l_2 & \ldots & a_{k_1,l_n} \circ l_n \\ k_2 & a_{k_2,l_1} \circ l_1 & a_{k_2,l_2} \circ l_2 & \ldots & a_{k_2,l_n} \circ l_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} \circ l_1 & a_{k_m,l_2} \circ l_2 & \ldots & a_{k_m,l_n} \circ l_n \end{array}$$

and

$$\rightarrow_\circ A = \begin{array}{c|cccc} & l_1 & l_2 & \ldots & l_n \\ \hline k_1 & a_{k_1,l_1} \circ k_1 & a_{k_1,l_2} \circ k_1 & \ldots & a_{k_1,l_n} \circ k_1 \\ k_2 & a_{k_2,l_1} \circ k_2 & a_{k_2,l_2} \circ k_2 & \ldots & a_{k_2,l_n} \circ k_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ k_m & a_{k_m,l_1} \circ k_m & a_{k_m,l_2} \circ k_m & \ldots & a_{k_m,l_n} \circ k_m \end{array}.$$

For example, for the IM $IM(N_1, \ldots, N_s)$ we obtain

$$\downarrow_\times IM(N_1, \ldots, N_s) = \begin{array}{c|cccc} & p_1 & p_2 & \ldots & p_k \\ \hline N_1 & \alpha_{1,1}p_1 & \alpha_{1,2}p_2 & \ldots & \alpha_{1,k}p_k \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ N_s & \alpha_{s,1}p_1 & \alpha_{s,2}p_2 & \ldots & \alpha_{s,k}p_k \end{array}.$$

In [5], the following average operation is defined over IM A:

$$\sigma_{sum}(A, l_0) = \begin{array}{c|c} & l_0 \\ \hline k_1 & \sum_{j=1}^{n} a_{k_1,l_j} \\ \vdots & \vdots \\ k_m & \sum_{j=1}^{n} a_{k_m,l_j} \end{array},$$

Now, for our example we obtain

$$\sigma_{sum}(\downarrow_\times IM(N_1, \ldots, N_s), l_0) = \begin{array}{c|c} & l_0 \\ \hline N_1 & \sum_{j=1}^{k} a_{1,j}p_j \\ \vdots & \vdots \\ N_s & \sum_{j=1}^{k} a_{s,j}p_j \end{array}.$$

In [3] function ζ is defined over the natural number m from Section 1 as follows:

$$\zeta(m) = \sum_{i=1}^{k} \alpha_i p_i.$$

Now, for our example we obtain

$$\sigma_{sum}(\downarrow_{\times} IM(N_1, \ldots, N_s), l_0) = \begin{array}{c|c} & l_0 \\ \hline N_1 & \zeta(N_1) \\ \vdots & \vdots \\ N_s & \zeta(N_s) \end{array}.$$

We finish with another example, related to Fibonacci sequence. In [6] the following extension of the Fibonacci sequence, call 2-Fibonacci sequence, was introduced as follows:

$$\alpha_0 = a,\ \beta_0 = b,\ \alpha_1 = c,\ \beta_1 = d$$
$$\alpha_{n+2} = \beta_{n+1} + \beta_n,\ n \geq 0$$
$$\beta_{n+2} = \alpha_{n+1} + \alpha_n,\ n \geq 0$$

The first ten terms of this sequence are:

n	α_n	β_n
0	a	b
1	c	d
2	$b+d$	$a+c$
3	$a+c+d$	$b+c+d$
4	$a+b+2.c+d$	$a+b+c+2.d$
5	$a+2.b+2.c+3.d$	$2.a+b+3.c+2.d$
6	$3.a+2.b+4.c+4.d$	$2.a+3.b+4.c+4.d$
7	$4.a+4.b+7.c+6.d$	$4.a+4.b+6.c+7.d$
8	$6.a+7.b+10.c+11.d$	$7.a+6.b+11.c+10.d$
9	$11.a+10.b+17.c+17.d$	$10.a+11.b+17.c+17.d$

Now, we can construct the following two IM, corresponding, respectively, to the members of sequences $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$, e.g., for $n \leq 9$:

$$IM(\{\alpha_n\}_{0\leq n\leq 9}) = \begin{array}{c|cccc} & a & b & c & d \\ \hline \alpha_0 & 1 & 0 & 0 & 0 \\ \alpha_1 & 0 & 0 & 1 & 0 \\ \alpha_2 & 0 & 1 & 0 & 1 \\ \alpha_3 & 1 & 0 & 1 & 1 \\ \alpha_4 & 1 & 1 & 2 & 1 \\ \alpha_5 & 1 & 2 & 2 & 3 \\ \alpha_6 & 3 & 2 & 4 & 4 \\ \alpha_7 & 4 & 4 & 7 & 6 \\ \alpha_8 & 6 & 7 & 10 & 11 \\ \alpha_9 & 11 & 10 & 17 & 17 \end{array}$$

$$IM(\{\beta_n\}_{0\le n\le 9}) = \begin{array}{c|cccc} & a & b & c & d \\ \hline \beta_0 & 0 & 1 & 0 & 0 \\ \beta_1 & 0 & 0 & 0 & 1 \\ \beta_2 & 1 & 0 & 1 & 0 \\ \beta_3 & 0 & 1 & 1 & 1 \\ \beta_4 & 1 & 1 & 1 & 2 \\ \beta_5 & 2 & 1 & 3 & 2 \\ \beta_6 & 2 & 3 & 4 & 4 \\ \beta_7 & 4 & 4 & 6 & 7 \\ \beta_8 & 7 & 6 & 11 & 10 \\ \beta_9 & 10 & 11 & 17 & 17 \end{array}$$

We see again that

$$\sigma_{sum}(\downarrow_\times IM(\{\alpha_n\}_{0\le n\le 9}, l_0) = \begin{array}{c|c} & l_0 \\ \hline \alpha_0 & a \\ \alpha_1 & c \\ \alpha_2 & b+d \\ \alpha_3 & b+c+d \\ \alpha_4 & a+b+c+2.d \\ \alpha_5 & a+2.b+2.c+3.d \\ \alpha_6 & 3.a+2.b+4.c+4.d \\ \alpha_7 & 4.a+4.b+7.c+6.d \\ \alpha_8 & 6.a+7.b+10.c+11.d \\ \alpha_9 & 11.a+10.b+17.c+17.d \end{array}$$

and

$$\sigma_{sum}(\downarrow_\times IM(\{\beta_n\}_{0\le n\le 9}, l_0) = \begin{array}{c|c} & l_0 \\ \hline \alpha_0 & b \\ \alpha_1 & d \\ \alpha_2 & a+c \\ \alpha_3 & a+c+d \\ \alpha_4 & a+b+2.c+d \\ \alpha_5 & 2.a+b+3.c+2.d \\ \alpha_6 & 2.a+3.b+4.c+4.d \\ \alpha_7 & 4.a+4.b+6.c+7.d \\ \alpha_8 & 7.a+6.b+11.c+10.d \\ \alpha_9 & 10.a+11.b+17.c+17.d \end{array}.$$

4 Conclusion

The apparatus of index matrices has already found some applications in the area of number theory (e.g., in [5, 8] and others), but it is clear that these publications are only the first steps in this direction of research. On one side, the new operations can find applications in a lot of other areas, and on the other side, the above research can be perceived as the first step in applying the new operators to elements of different sequences, which is an object of further research in the future.

References

[1] Atanassov, K. (1984) Conditions in Generalized nets. *Proc. of the XIII Spring Conf. of the Union of Bulg. Math.*, Sunny Beach, April 1984, 219–226.

[2] Atanassov, K. (1984) On one problem of A. Mullin. *Bulletin of Number Theory and Related Topics*, VIII, 3, 1–5.

[3] Atanassov, K. (1987) New integer functions, related to "φ" and "σ" functions. *Bulletin of Number Theory and Related Topics*, XI, 1, 3–26.

[4] Atanassov, K. (1987) Generalized index matrices. *Compt. Rend. de l'Academie Bulgare des Sciences*, 40, 11, 15–18.

[5] Atanassov, K. (2014) *Index Matrices: Towards an Augmented Matrix Calculus*, Springer, Cham.

[6] Atanassov, K., Atanassova, L., & Sasselov, D. (1985) A new perspective to the generalization of the Fibonacci sequence, *Fibonacci Quarterly*, 23, 1, 21–28.

[7] Atanassov, K., Sotirova, E., & Bureva, V. (2013) On index matrices. Part 4: New operations over index matrices. *Advanced Studies in Contemporary Mathematics*, 23, 3, 547–552.

[8] Leyendekkers, J. V., Shannon, A. G., & Rybak, J. (2007) Pattern recognition: Modular Rings & Integer Structure. Raffles KvB Monograph No. 9, North Sydney.

[9] Mitrinović, D. S., Sándor, J., & Crstici, B. (1995) *Handbook of Number Theory*, Kluwer, Dordrecht.

[10] Nagell, T. (1950) *Introduction to Number Theory*, John Wiley & Sons, New York.