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A new proof of Euler's pentagonal number theorem

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Abstract: A new proof of Euler's pentagonal number theorem is obtained.
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1 History and motivation

The classical statement of Euler's pentagonal number theorem is

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{n=-\infty}^{\infty} q^{\frac{n(3n-1)}{2}}, \text{ where } |q| < 1. \tag{1.1}$$

By expanding the left side of the equation (1.1), one can see that

$$\prod_{n=1}^{\infty}(1-q^n) = 1 + \sum_{n=1}^{\infty}(r_e(n) - r_o(n))\, q^n, \tag{1.2}$$

where $r_e(n)$ denotes the number of distinct partitions (partitions with distinct parts) of n with even number of parts, and $r_o(n)$ denotes the number of distinct partitions of n with odd number of parts.

Equations (1.1) and (1.2) together give the following expression:

$$r_e(n) - r_o(n) = \begin{cases} (-1)^k, & \text{if } n = \frac{3k^2 \pm k}{2}; \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

This expression is known as the partition-theoretic interpretation of Euler's pentagonal number theorem. Euler's pentagonal number theorem follows directly from the Jacobi's triple product identity

$$\prod_{m=1}^{\infty}\left(1-q^{2m}\right)\left(1+q^{2m-1}z^{2}\right)\left(1+q^{2m-1}z^{-2}\right)=\sum_{n=-\infty}^{\infty}q^{n^2}z^{2n}$$

for $q=x^{\frac{3}{2}}$ and $z^2=-x^{\frac{1}{2}}$.

Applications of Euler's pentagonal number theorem is manifold. Recently, Chuanan Wei and Dianxuan Gong [10] showed that Euler's pentagonal number theorem implies Jacobi's triple product identity. Applying Jacobi's triple product identity, Ewell [6] obtained Fermat's two squares theorem. Hirschhorn [8] obtained Jacobi's two squares theorem as a consequence of Jacobi's Triple Product Identity.

Euler [5] proved the classical version of his theorem using induction. Many mathematicians obtained proofs for Jacobi's triple product identity (for proof see [1, 2, 3, 9, 11]). Addition to these proofs, Franklin [7] gave a bijective proof for Euler's pentagonal number theorem using Ferrer's diagram of the partition, and F. J. Dyson [4] gave a combinatorial proof involving the idea of the rank of a partition.

In this article, we give a new proof for the partition-theoretic version of Euler's pentagonal number theorem.

Definition 1.1. *Let n be a positive integer. A partition $(a_1, a_2, \ldots, a_k)$ of n is said to be a $distinct$ partition of n if $a_i > a_{i+1}$ for every $i \in \{1, 2, \ldots, k-1\}$.*

2 Proof

Let n be a positive integer. Let Q_n be the set of all distinct partitions of n. Define an operator $\phi : Q_n \to Q_n$ by

$$\phi((a_1, a_2, \ldots, a_k)) = (a_1+1, a_2+1, \ldots, a_{a_k}+1, a_{a_k+1}, \ldots, a_{k-1})$$

when $a_k < k$.

Let $Q_{n,s}$ be the set of all distinct partitions of n with its least part s such that $s <$ number of parts.

Put $A_1 = Q_{n,1}$. Define $\phi : A_1 \to Q_n$. Since every partition in $\phi(A_1)$ has least part greater than 1, we have $\phi(A_1) \cap A_1 = \emptyset$. Since each partition in A_1 has identical least part, ϕ cannot be a many-to-one mapping. Thus, ϕ is an one-to-one mapping. Moreover, we see that image of every partition with even (resp. odd) number of parts in A_1 under ϕ has odd (resp. even) number of parts. Consequently, the number of even partitions (partitions with even number of parts) and odd partitions (partitions with odd number of parts) in $\phi(A_1) \cup A_1$ are same.

Define $A_2 = (Q_n \setminus (A_1 \cup \phi(A_1)) \cap Q_{n,2}$. Consider the mapping $\phi : A_2 \to Q_n$. Following the line of argument in the last paragraph, we again get that $\phi(A_2) \cap A_2 = \emptyset$ and the number of even partitions and odd partitions in $\phi(A_2) \cup A_2$ are same.

For $k \geq 3$, define $A_k = (Q_n \setminus \cup_{i=1}^{k-1}(A_i \cup \phi(A_i))) \cap Q_{n,k}$. We see that there is no possibility for the existence of a distinct partition say π_2 such that $\pi_2 \in A_r$ and $\phi(\pi_2) \in \phi(A_l)$ for some

$l < r$. For otherwise, there will be a distinct partition say π_1 such that $\phi(\pi_1) = \phi(\pi_2)$ with $\pi_1 \neq \pi_2$. This gives $(a_1+1, a_2+1, \ldots, a_l+1, a_{l+1}, \ldots, a_{k-1}) = (b_1+1, b_2+1, \ldots, b_l+1, b_{l+1}+1, \ldots, b_r+1, b_{r+1}, \ldots, b_{k-1})$, where $\pi_1 = (a_1, \ldots, a_k)$ and $\pi_2 = (b_1, \ldots, b_k)$ with $a_k = l$ and $b_k = r$. Consider the partition $\pi^* = (b_1, b_2, \ldots, b_l, b_{l+1}+1, \ldots, b_r+1, b_{r+1}, \ldots, b_{k-1}, b_k, l)$. From the above equality we have $b_{l+1}+1 < b_l$ and since $l < b_k < k$, one can see that π^* is a distinct partition of n with least part l such that l is less than k. Furthermore, $\phi(\pi^*) = \pi_2$. Thus, $\pi_2 \in \phi(A_l)$ which implies $\pi_2 \notin A_r$, which is a contradiction.

Accordingly, we have the following conclusions:

1. $\phi(A_k) \cap A_k = \emptyset$ for every $k \in \{1, 2, \ldots\}$.

2. The number of even and odd partitions in $\cup_{i\geq 1}(A_i \cup \phi(A_i))$ are same.

Let $Q_n^* = \cup_{i\geq 1}(A_i \cup \phi(A_i))$. A closer examination of the set $Q_n \setminus Q_n^*$ completes the proof. Let $\pi = (a_1, a_2, \ldots, a_k) \in Q_n \setminus Q_n^*$. Define $c(\pi)$ to be the largest integer $l \geq 2$ for which $a_1, a_2, \ldots, a_l$ satisfies $a_2 - a_1 = a_3 - a_2 = \cdots = a_l - a_{l-1} = 1$. We claim that $c(\pi) = k$. For if $c(\pi) = s$ for some $s < k$, then it is plain that we can write $\pi = (b, b-1, \ldots, b-(s-1), a_{s+1}, \ldots, a_k)$ with $(b-(s-1)) - a_{s+1} > 1$. Now consider the partition $\pi_1 = (b-1, b-2, \ldots, b-s, a_{s+1}, \ldots, a_k, s)$. From the membership of π, we have $a_k \geq k$. Since $s < k$, we have $a_k - s > 0$. Thus π_1 is a distinct partition of n. Also, we have $\phi(\pi_1) = \pi$. If $\pi_1 \in A_i$ for some i, then we have $\pi \in \phi(A_i)$, which leads to the conclusion that $\pi \in Q_n^*$ which is a contradiction. On the other hand, if $\pi_1 \in \phi(A_j)$ for some j, then there exist a distinct partition say $\pi_2 = (b_1, b_2, \ldots, b_{k+2})$ such that $\phi(\pi_2) = \pi_1$. Note that $1 \leq b_{k+2} < s$. From this it follows that $1 \leq b_{k+2} < k$ and $b_{k+2} < s$. Since $\phi(\pi_2) = \pi_1$, we have the following equalities: $b_1 + 1 = b - 1, b_2 + 1 = b - 2, \ldots, b_{b_{k+2}} + 1 = b - b_{k+2}, b_{b_{k+2}+1} = b - b_{k+2} - 1, \ldots$; which leads to the equality $b_{b_{k+2}} - b_{b_{k+2}+1} = 0$ which is a contradiction. Thus $c(\pi) = k$. Accordingly, π is of the form $\pi = (a_k + k - 1, a_k + k - 2+, \ldots, a_k + 1, a_k)$.

We claim that a_k can assume only two values namely k or $k+1$. From the membership of π it follows that $a_k \geq k$. Suppose that $a_k > k+1$. Then consider the partition $\pi_1 = (a_k + (k-2), \ldots, a_k, a_k - 1, k)$. Clearly, π_1 is a distinct partition of n. We see that $\phi(\pi_1) = \pi$, which implies that, $\pi_1 \notin Q_n \setminus Q_n^*$. This in turn implies that $\pi_1 \in Q_n^*$. If $\pi_1 \in A_i$ for some i, then we would have $\phi(\pi_1) \in \phi(A_i)$, that is, $\pi \in Q_n^*$ which is a contradiction. If $\pi_1 \in \phi(A_j)$ for some j, then there will be a distinct partition of n say $\pi_2 = (b_1, b_2, \ldots, b_{k+2})$ such that $\phi(\pi_2) = \pi_1$. Now we make it a point that $b_{k+2} < k$. Since $\phi(\pi_2) = \pi_1$, we have the equalities $b_1 + 1 = a_k + (k-2), b_2 + 1 = a_k + (k-3), \ldots, b_{b_{k+2}} + 1 = a_k + (k-1) - b_{k+2}, b_{b_{k+2}+1} = a_k + (k-1) - (b_{k+2}+1), \ldots$; this gives $b_{b_{k+2}} = b_{b_{k+2}+1}$, which is absurd. Thus the claim follows.

From these observations, we get that $r_e(n) - r_o(n) = 0$ if n is not of the forms: $k + (k+1) + \cdots + (k + (k-1))$ and $(k+1) + (k+2) + \cdots + (k+k)$, that is, when $n \neq \frac{3k^2 \pm k}{2}$. On the other hand, if $n = \frac{3k^2 \pm k}{2}$ then we have $r_e(n) - r_o(n) = 1$ when k is even, and $r_e(n) - r_o(n) = -1$ when k is odd.

This completes the proof. □

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