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Proofs of certain conjectures for means of two arguments

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Abstract: We offer proofs of certain inequalities for means conjectured by N. Elezović [2].
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1 Introduction

Let A, G, H, Q, N denote the classical means of two arguments, where

$$A = A(a,b) = \frac{a+b}{2},\ G = G(a,b) = \sqrt{ab}, H = H(a,b) = \frac{2ab}{a+b},$$

$$Q = Q(a,b) = \sqrt{\frac{a^2+b^2}{2}}, N = N(a,b) = \frac{a^2+b^2}{a+b}.$$

Let L and I denote the famous logarithmic, resp. identric means, defined by

$$L = L(a,b) = \frac{a-b}{\log a - \log b}$$

for a distinct from b, $L(a,a) = a$, and

$$I(a,b) = \frac{1}{e}\left[\frac{b^b}{a^a}\right]^{\frac{1}{b-a}}$$

for a distinct from b, $I(a,a) = a$.

For the identric and logarithmic means there exist in the literature many relations, especially inequalities. We should mention the authors B. C. Carlson, E. B. Leach and M. C. Sholander, A. O. Pittenger, H. Alzer, J. Sándor, E. Neuman, Zs. Páles, T. Trif, S.-L. Qiu, Gh. Toader, I. Rasa, N. Elezović, L. Vuksić, C.-P. Chen, etc.

In 2015, N. Elezovic [2] conjectured certain interesting inequalities of a new type. There are stated three conjectures, each containing a set of (5)–(6) inequalities.

These are the following (where we have preserved all the notations and numbering order of formulas from [2]):

Conjecture 2.3

$$N + 6I < 7A; \tag{2.2}$$

$$Q + 3I < 4A; \tag{2.3}$$

$$A + L < 2I; \tag{2.4}$$

$$2A + G < 7I; \tag{2.5}$$

$$5A + H > 6I. \tag{2.6}$$

Conjecture 4.2

$$eN + (e-2)I > 2(e-1)A; \tag{4.6}$$

$$(e-2)Q + (\sqrt{2}-1)eI > (e\sqrt{2}-2)A; \tag{4.7}$$

$$2A + (e-2)L > eI; \tag{4.8}$$

$$2A + (e-2)G > eI; \tag{4.9}$$

$$2A + (e-2)H < eI. \tag{4.10}$$

Conjecture 4.3

$$eN + (e-1)L > eI; \tag{4.11}$$

$$eN + (e-1)G > eI; \tag{4.12}$$

$$eN + (e-1)H > eI; \tag{4.13}$$

$$\sqrt{2}Q + (e-\sqrt{2})L > eI; \tag{4.14}$$

$$\sqrt{2}Q + (e-\sqrt{2})G > eI; \tag{4.15}$$

$$\sqrt{2}Q + (e-\sqrt{2})H < eI. \tag{4.16}$$

2 Proofs of inequalities

2.1 Inequalities of Conjecture 2.3

We should note that inequality (2.4), written in the form $I > \frac{A+L}{2}$ was proved in 1990 by J. Sándor in [4]; while inequality (2.5), written in the form $I > \frac{2A+G}{3}$ is one of the main results of [5]. It is clear also that (2.5) implies (2.4), see [5].

On the other hand, remark that relation (2.2) may be written in the form $I < \frac{7A-N}{6} = \frac{5A+H}{6}$, which follows by the identity (an easy verification)

$$A = \frac{H+N}{2}. \tag{1}$$

This means that, inequality (2.2) is equivalent with inequality (2.6).

Finally, remark that inequality (2.3) written in the form $I < \frac{4A-Q}{3}$ follows also from inequality (2.6). Indeed, one has to verify that $\frac{5A+H}{6} < \frac{4A-Q}{3}$, or equivalently:

$$H + 2Q < 3A. \tag{2}$$

This is known (see also [2], relation (1.1)). However, for the sake of completeness, we shall give a proof of (2). Letting $z = \frac{a-b}{a+b}$, where $H = H(a,b)$, etc, and $a > b$, we immediately get $\frac{H}{A} = 1 - z^2$, $\frac{Q}{A} = \sqrt{(1+z^2)}$, so inequality (2) becomes $1 - z^2 = 2\sqrt{1+z^2} < 3$, or $2 + t > 2\sqrt{1+t}$, with $t = z^2$. Since t is in $(0,1)$, this is obvious, as $(2+t)^2 > 4(1+t)$ becomes $t(3t-4) <$, which is true.

Now, we shall give a proof of (2.6). We shall use an inequality of Sándor and Trif from [6], namely:

$$I^2 < \frac{2A^2 + G^2}{3}. \tag{3}$$

Since $H = \frac{G^2}{A}$, it is sufficient to prove the following relation:

$$\frac{2A^2+G^2}{3} < \left[\frac{5A^2+G^2}{6A}\right]^2. \tag{4}$$

Put $x = \left(\frac{A}{G}\right)^2$. The (4) becomes $12x(2x+1) < (5x+1)^2$, or equivalently: $24x^2 + 12x < 25x^2 + 10x + 1$, which is $x^2 - 2x + 1 > 0$, or $(x-1)^2 > 0$, true.

Remark 1. *Therefore, the following refinement of* (2.6) *holds true:*

$$I < \sqrt{\frac{2A^2+G^2}{3}} < \frac{5A+H}{6}. \tag{5}$$

We note also that in [6] there are two distinct proofs of (3)*, one of them (by using series expansions) gives in fact a slightly better inequality than* (3)*.*

2.2 Inequalities of Conjecture 4.2

As noted also in [2], relation (4.9) has been proved by H. Alzer and S.-L. Qiu [1].

For the proof of (4.6), remark first that, by identity (1) one has $2A = H + N$, so the inequality may be rewritten as: $eN + (e-2)I > (e-1)N + (e-1)H$, or

$$N + (e-2)I > (e-1)H. \tag{6}$$

Since it is well-known that $I > G$ (see e.g. [4]), (6) follows from $N + (e-2)G > (e-1)H$. This is true, as $N = \frac{Q^2}{A} > H$ (by $Q > A$ and $Q > H$) and $G > H$. The proof is finished.

For the proof of (4.8) remark that the following inequality is well-known: $L > G$ (see e.g. [4]). Therefore, (4.8) is a consequence of (4.9), due to Alzer and Qiu.

In what follows, we shall prove that inequality (4.7) is a consequence of inequality (4.10), and the following auxiliary result:

Lemma 1.

$$Q + \left(\sqrt{2}-1\right) H > \sqrt{2}A. \tag{7}$$

Proof. As in the proof of (2), we have here to prove the inequality

$$\sqrt{1+z^2} + (\sqrt{2}-1)(1-z^2) > \sqrt{2}. \tag{8}$$

Put $1 - z^2 = t$. Then $1 + z^2 = 2 - t$, so (8) becomes $2 - t > \left[\sqrt{2} - (\sqrt{2}-1)t\right]^2$. After elementary computations this becomes $(3-2\sqrt{2})t(t-1) < 0$, which is true, as $3 > 2\sqrt{2}$ and $0 < t < 1$.

Now, by the Lemma at one side, and from (4.10) on the other side we have:

$$(e-2)Q > (e-2)\sqrt{2}A - (e-2)(\sqrt{2}-1)H;$$
$$(\sqrt{2}-1)eI > 2(\sqrt{2}-1)A + (e-2)(\sqrt{2}-1)H.$$

By adding these two inequalities, relation (4.7) follows.

Finally, we prove inequality (4.10).

Since $H = \frac{G^2}{A,}$ we have to prove that

$$I > \frac{2A^2 + (e-2)G^2}{eA}. \tag{9}$$

We will use an inequality of Trif from [7], who proved that for any $p \geq 2$ one has

$$I^p > \left[\left(\frac{2}{e}\right)^p\right] A^p + \left[1 - \left(\frac{2}{e}\right)^p\right] G^p. \tag{10}$$

Let $p = 2$ in (10). Then we get

$$I^2 > \left(\frac{4}{e^2}\right) A^2 + \left[\frac{e^2-4}{e^2}\right] G^2. \tag{11}$$

In order to prove (9), it will be sufficient to show that

$$[4A^2 + (e^2-4)G^2]A^2 > [2A^2 + (e-2)G^2]^2,$$

or after some elementary computations: $A^2G^2(e^2 - 4e + 4) > (e-2)^2G^2$.

As $e^2 - 4e + 4 = (e-2)^2$, the above inequality becomes $A^2 > G^2$, i.e., $A > G$, which is true. $\square$

Remark 2. *Therefore, the following refinement of* (4.10) *holds true:*

$$I > \sqrt{\left(\frac{4}{e^2}\right) A^2 + \left(\frac{e^2-4}{e^2}\right) G^2} > \frac{2A+(e-2)H}{e}. \tag{7'}$$

2.3 Inequalities of Conjecture 4.3

Remark that, by $G < L$, inequality (4.12) implies (4.11).

To prove (4.12), we will use inequality (4.9) of Alzer–Qiu, as well as identity (1). By $eI < 2A+(e-2)G = H+N+(e-2)G < eN+(e-1)G$ iff $H+N < eN+G$, or $H < (e-1)N+G$, which is trivial, as $H < G$, and $(e-1)N > 0$.

As $G < L$, one has similarly that (4.15) imples (4.14).

We will prove (4.15).

By inequality (7) we can state (by $G > H$) that

$$Q + (\sqrt{2}-1)G > \sqrt{2}A. \tag{12}$$

By multiplying with $\sqrt{2}$, one has $\sqrt{2}Q + (2-\sqrt{2})G > 2A$, so we can write by (4.9):

$$\sqrt{2}Q + (e-\sqrt{2})G > 2A + G(e-\sqrt{2}+\sqrt{2}-2) = 2A + G(e-2) > eI,$$

so (4.15) follows.

Related to inequality (4.13), which states that $eN + (e-1)H < eI$, we note that it cannot be true.

Indeed, in $N = N(a,b)$ let $b > a$ and b having values near a (i.e., b tending to a). Then $N(a,b)$ tends to a, $H(a,b)$ to a and $I(a,b)$ again to a.

We get $ea + (e-1)a \le ea$, or $(e-1)a < 0$, which is impossible.

Remark 3. *Inequality* (4.16) *is still open.*

Remark 4. *Meantime, in [3] we learned that, there are certain misprints in inequalities* (4.11)*,* (4.12) *and* (4.13)*, and in fact, the correct versions are the following:*

$$eI < N + (e-1)L; \tag{4.11'}$$

$$eI < N + (e-1)G; \tag{4.12'}$$

$$eI > N + (e-1)H. \tag{4.13'}$$

In what follows, we shall prove also these inequalities.

By relation (4.9) one has $eI < 2A+(e-2)G = 2A+(e-1)G-G < 2A+(e-1)G-H = (2A-H)+(e-1)G = N+(e-1)G$, since $N = 2A-H$ (see relation (1) and $H < G$.) This proves (4.12').

Now, as $N = 2A - H$, it easy to see that, $N + (e-1)H = 2A + (e-2)H$, so inequality (4.13') becomes in fact inequality (4.10), which has been proved above.

Finally, by the known inequality $L > G$, (4.11') is a consequence of (4.12').

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