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An indicator characteristic for twin prime formation independent of integer size

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Abstract: The modular ring Z_6 has twin primes located in the same row. This enables the structural mechanisms underlying the formation of twin primes to be summarised by simple equations. The classification system provided by right-end-digits applies equally in all integer domains of any size, and can be used to demonstrate the formation of twin primes in such domains.

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1 Introduction

Twin primes are pairs of primes (p, p+2) and occur up to relatively large values such as the integer $2996863034895 \times 2^{1290000} \pm 1$ which has 388,342 digits. Properties of twin primes may be found in [1, 6]. Marked progress has been made in recent years [2, 7] with modular forms which indicate that the bounded gaps between primes can be quite low. Previously the structure of twin primes had been analysed with the modular ring Z_6 [5]. This paper extends the latter ideas to include integers of any size.

2 Twin primes in the modular ring Z_6

The integer structure analysis utilised the rows of Table 1 in which twin primes can be readily identified. Rows which do not contain twin primes can be easily identified and discarded computationally. The classification system developed in [5] is useful for this.

In considering whether twin primes occur to infinity, it is necessary to show that a gap of 2 between primes is possible anywhere. The value of integer structure analysis in this is that this structure by definition endures throughout the integer system without exception. Understanding the mechanism for the formation and distinction of prime and composite numbers is critical.

An efficient mechanism is provided by analysing twin primes in the modular ring Z_6 [3, 4]. Integers in Z_6 may be represented by $(6r_i + (i - 3))$, in which $i \in \overline{I}_6$, the class, and $r \geq 0$ is the row. Only classes $\overline{2}_6$ and $\overline{4}_6$ contain odd primes. In the odd number class $\overline{6}_6$, all the integers, n, are divisible by 3.

Class	$\overline{1}_6$	$\overline{2}_6$	$\overline{3}_6$	$\overline{4}_6$	$\overline{5}_6$	$\overline{6}_6$	**Comments**
Row, *r*	$6r_1 - 2$	$6r_2 - 1$	$6r_3$	$6r_4 + 1$	$6r_5 + 2$	$6r_6 + 3$	
0				1	2	3	
1	4	5	6	7	8	9	• $6 \mid \overline{3}_6$ and $3 \mid \overline{6}_6$
2	10	11	12	13	14	15	
3	16	17	18	19	20	21	and $\forall a$
4	22	23	24	25	26	27	
5	28	29	30	31	32	33	• $a^2 \notin \{\overline{2}_6, \overline{5}_6\}$
6	34	35	36	37	38	39	• $a^3 \equiv a \pmod 6$
7	40	41	42	43	44	45	
8	46	47	48	49	50	51	• $p > 3$
9	52	53	54	55	56	57	• $p \in \{\overline{2}_6, \overline{4}_6\}$
10	58	59	60	61	62	63	
11	64	65	66	67	68	69	
12	70	71	72	73	74	75	
13	76	77	78	79	80	81	
14	82	83	84	85	86	87	
15	88	89	90	91	92	93	

Table 1. The rows of the modular ring Z_6

The advantage of Z_6 is that twin primes (p_1, p_2) occur in the same row with

$$p_1 = (6r_2 - 1) \text{ and } p_2 = (6r_4 + 1),$$

and when the rows are the same $r_2 = r_4$ $(= R')$, we have

$$(p_1 + p_2) = 6(2R').$$

This shows that the sum of twin primes is an even integer $N \in \bar{3}_6$ in an even row. The integer which separates the twin primes (*tps*) is ½*N*, since (½*N* – 1) + (½*N* + 1) = *N*. The row which contains ½*N* is $R' \in \bar{3}_6$ since ½*N* = 6*R′*.

If either of the integers in a row of $\bar{2}_6$ or $\bar{4}_6$ is composite, then twin primes will not occur. The rows which contain composite integers (designated as *F* or forbidden rows) are easily obtained as follows. A composite odd integer, *M*, is given by [3]

$$M = p^2 + 2p(s-1) \tag{2.1}$$

in which *p* is a prime and *s* = 1,2,3,4,… . Obviously *p*|*M*, so that for composite integers $M_2 \in \bar{2}_6$

$$\begin{aligned} M_2 &= 6r_2 - 1 \\ &= p^2 + 2p(s-1) \end{aligned}$$

so that

$$r_2 = \tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2+1\right) + p(s-1)\right) \tag{2.2}$$

For composite integers $M_4 \in \bar{4}_6$

$$\begin{aligned} M_4 &= 6r_4 + 1 \\ &= p^2 + 2p(s-1) \end{aligned}$$

so that

$$r_4 = \tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2-1\right) + p(s-1)\right) \tag{2.3}$$

and $p \neq 3$ since all *M*: 3|*M* are in $\bar{6}_6$. The *F* rows are therefore represented by

$$R = R_0 + pt,\ t = 0,1,2,3,.... \tag{2.4}$$

R_0 functions are listed in Table 2.

Class of *M*	**Class of *p***	s_0	R_0
$\bar{2}_6$	$\bar{2}_6$	2	$\tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2+1\right)+p\right)$
	$\bar{4}_6$	3	$\tfrac{1}{3}\left(\tfrac{1}{2}\left(p^2+1\right)+2p\right)$
$\bar{4}_6$	$\bar{2}_6 \wedge \bar{4}_6$	1	$\tfrac{1}{6}\left(p^2-1\right)$

Table 2. R_0 functions

Tables of rows which yield or do not yield twin primes may be found in [5]. For example, the first 26 rows with twin primes are

1, 2, 3, 5, 7, 10, 12, 17, 18, 23, 25, 30, 32, 33, 38, 40, 45, 47, 52, 58, 70, 72, 77, 87, 95, 100.

When the right-end-digit, RED, of the row is 1, 4, 6 or 9, one of the integers in class $\bar{2}_6$ or $\bar{4}_6$ will have a RED of 5 so that these integers cannot be primes except for row with RED 1. For twin primes to occur the row must have a RED of 0, 2, 8 or 3, 5, 7. The only RED couples for twin primes are (9, 1), (1, 3) or (7, 9); for instance, {29, 31}, {11, 13} or {17, 19}.

3 RED structure of $(P_1, P_2)^* = (1, 3)$

Integer pairs with REDs of (1, 3) fall into 3 possible class pairs: $\bar{4}_6\bar{6}_6, \bar{6}_6\bar{2}_6$ or $\bar{2}_6\bar{4}_6$ (Table 3). The $\bar{2}_6$, $\bar{4}_6$ class pair is the only one that contains twin primes. Sequences of rows, $\{R_{24}\}$, for this class pair are given by

$$R_{24,k} = 5k+2, \ k = 0, 1, 2, 3, 4, \ldots \tag{3.1}$$

Since a prime in class $\bar{2}_6$ will be found in all integer domains, we can choose such a prime and consider the *possible* character of the following odd integer in class $\bar{4}_6$ after equating equations (3.1) and (2.4), for $R_0 : M \in \bar{4}_6$,

$$30k = p^2 + 6pt - 13 \tag{3.2}$$

or

$$p = -3t + \sqrt{9t^2 + (13 + 30k)} \tag{3.3}$$

Since $(30k)^* = 0$, $(13 + 30k)^* = 3$, we need consider only the values of t^* to find the values of p^* we get from (3.3). When the $\bar{4}_6$ integer is composite, it will satisfy equation (3.3) as in Table 4, but if the $\bar{4}_6$ integer is prime, there will be no integer solution for (3.3) and a twin prime pair will form.

k	N_1, N_2	Z_6	***n***	***k***	N_1, N_2	Z_6	***n***	***k***	N_1, N_2	Z_6	***n***
					261,263	$\bar{6}_6\,\bar{2}_6$	26		531,533	$\bar{6}_6\,\bar{2}_6$	53
0	1,$\underline{3}$	$\bar{4}_6\bar{6}_6$	0	9	$\underline{271}$,273	$\bar{4}_6\,\bar{6}_6$	27	18	541,543	$\bar{4}_6\,\bar{6}_6$	54
	$\underline{11}$,$\underline{13}$	$\bar{2}_6\bar{4}_6$	1		$\underline{281}$,$\underline{283}$	$\bar{2}_6\,\bar{4}_6$	28		551,553	$\bar{2}_6\,\bar{4}_6$	55
	21,$\underline{23}$	$\bar{6}_6\,\bar{2}_6$	2		291,$\underline{293}$	$\bar{6}_6\,\bar{2}_6$	29		561,$\underline{563}$	$\bar{6}_6\,\bar{2}_6$	56
1	$\underline{31}$,33	$\bar{4}_6\,\bar{6}_6$	3	10	301,303	$\bar{4}_6\,\bar{6}_6$	30	19	$\underline{571}$,573	$\bar{4}_6\,\bar{6}_6$	57
	$\underline{41}$,$\underline{43}$	$\bar{2}_6\,\bar{4}_6$	4		$\underline{311}$,$\underline{313}$	$\bar{2}_6\,\bar{4}_6$	31		581,583	$\bar{2}_6\,\bar{4}_6$	58
	51,$\underline{53}$	$\bar{6}_6\,\bar{2}_6$	5		321,323	$\bar{6}_6\,\bar{2}_6$	32		591,$\underline{593}$	$\bar{6}_6\,\bar{2}_6$	59
2	$\underline{61}$,63	$\bar{4}_6\,\bar{6}_6$	6	11	$\underline{331}$,333	$\bar{4}_6\,\bar{6}_6$	33	20	$\underline{601}$,603	$\bar{4}_6\,\bar{6}_6$	60
	$\underline{71}$,$\underline{73}$	$\bar{2}_6\,\bar{4}_6$	7		341,343	$\bar{2}_6\,\bar{4}_6$	34		611,$\underline{613}$	$\bar{2}_6\,\bar{4}_6$	61
	81,$\underline{83}$	$\bar{6}_6\,\bar{2}_6$	8		351,$\underline{353}$	$\bar{6}_6\,\bar{2}_6$	35		621,623	$\bar{6}_6\,\bar{2}_6$	62
3	91,93	$\bar{4}_6\,\bar{6}_6$	9	12	361,363	$\bar{4}_6\,\bar{6}_6$	36	21	$\underline{631}$,633	$\bar{4}_6\,\bar{6}_6$	63
	$\underline{101}$,$\underline{103}$	$\bar{2}_6\,\bar{4}_6$	10		371,$\underline{373}$	$\bar{2}_6\,\bar{4}_6$	37		$\underline{641}$,$\underline{643}$	$\bar{2}_6\,\bar{4}_6$	64
	111,$\underline{113}$	$\bar{6}_6\,\bar{2}_6$	11		381,$\underline{383}$	$\bar{6}_6\,\bar{2}_6$	38		651,$\underline{653}$	$\bar{6}_6\,\bar{2}_6$	65
4	121, 123	$\bar{4}_6\,\bar{6}_6$	12	13	391,393	$\bar{4}_6\,\bar{6}_6$	39	22	$\underline{661}$,663	$\bar{4}_6\,\bar{6}_6$	66
	$\underline{131}$,133	$\bar{2}_6\,\bar{4}_6$	13		$\underline{401}$,403	$\bar{2}_6\,\bar{4}_6$	40		671,$\underline{673}$	$\bar{2}_6\,\bar{4}_6$	67
	141,143	$\bar{6}_6\,\bar{2}_6$	14		411,413	$\bar{6}_6\,\bar{2}_6$	41		681,$\underline{683}$	$\bar{6}_6\,\bar{2}_6$	68
5	$\underline{151}$,153	$\bar{4}_6\,\bar{6}_6$	15	14	$\underline{421}$,423	$\bar{4}_6\,\bar{6}_6$	42	23	$\underline{691}$,693	$\bar{4}_6\,\bar{6}_6$	69
	161,$\underline{163}$	$\bar{2}_6\,\bar{4}_6$	16		$\underline{431}$,$\underline{433}$	$\bar{2}_6\,\bar{4}_6$	43		$\underline{701}$,703	$\bar{2}_6\,\bar{4}_6$	70

(Contd.)

k	N_1, N_2	Z_6	n	k	N_1, N_2	Z_6	n	k	N_1, N_2	Z_6	n
6	171,$\underline{173}$	$\bar{6}_6\ \bar{2}_6$	17	15	441,$\underline{443}$	$\bar{6}_6\ \bar{2}_6$	44	24	711,713	$\bar{6}_6\ \bar{2}_6$	71
	181,183	$\bar{4}_6\ \bar{6}_6$	18		451,453	$\bar{4}_6\ \bar{6}_6$	45		721,723	$\bar{4}_6\ \bar{6}_6$	72
	$\underline{191}$,$\underline{193}$	$\bar{2}_6\ \bar{4}_6$	19		$\underline{461}$,463	$\bar{2}_6\ \bar{4}_6$	46		731,$\underline{733}$	$\bar{2}_6\ \bar{4}_6$	73
7	201,203	$\bar{6}_6\ \bar{2}_6$	20	16	471,473	$\bar{6}_6\ \bar{2}_6$	47	25	741,$\underline{743}$	$\bar{6}_6\ \bar{2}_6$	74
	$\underline{211}$,213	$\bar{4}_6\ \bar{6}_6$	21		481,483	$\bar{4}_6\ \bar{6}_6$	48		$\underline{751}$,753	$\bar{4}_6\ \bar{6}_6$	75
	221,223	$\bar{2}_6\ \bar{4}_6$	22		$\underline{491}$,493	$\bar{2}_6\ \bar{4}_6$	49		$\underline{761}$,763	$\bar{2}_6\ \bar{4}_6$	76
8	231,$\underline{233}$	$\bar{6}_6\ \bar{2}_6$	23	17	501,$\underline{503}$	$\bar{6}_6\ \bar{2}_6$	50	26	771,773	$\bar{6}_6\ \bar{2}_6$	77
	$\underline{241}$,243	$\bar{4}_6\ \bar{6}_6$	24		511,513	$\bar{4}_6\ \bar{6}_6$	51		781,783	$\bar{4}_6\ \bar{6}_6$	78
	$\underline{251}$,253	$\bar{2}_6\ \bar{4}_6$	25		$\underline{521}$,$\underline{523}$	$\bar{2}_6\ \bar{4}_6$	52		791,793	$\bar{2}_6\ \bar{4}_6$	79
								27	801,803	$\bar{6}_6\ \bar{2}_6$	80
									$\underline{811}$,813	$\bar{4}_6\ \bar{6}_6$	81
									821,$\underline{823}$	$\bar{2}_6\ \bar{4}_6$	82

Table 3. Examples of some twin primes; primes underscored

Table 4 shows that $t^* = 8$ is possible. Note $(N^* = 1) \in \bar{2}_6$ is a prime. The t values for the composites with REDs of 3 have $t^* = 2, 3, 7, 8$ (Tables 4, 5).

N	p	t	$k^‡$	N	p	t	$k^‡$	N	p	t	$k^‡$	N	p	t	$k^‡$
133	7	2	4	763	7	17	25	1513	17	12	50	2353	13	28	78
253	11	2	8	913	11	12	30	1573	11	22	52	2413	19	18	80
403	13	3	13	943	23	3	3	1603	7	37	53	2443	7	57	81
493	17	2	16	1183	7	27	39	1813	7	42	60	2533	17	22	84
703	19	3	23	1363	29	3	45	1903	11	27	63	2623	43	3	87

Table 4. Composite $\bar{4}_6$; ‡ from Table 3; $N^* = 3$

t^*	2	3	7	8
$(t^2)^*$	4	9	9	4
$(9t^2)^*$	6	1	1	6
$(9t^2 + 3)^*$	9	4	4	9

Table 5. Structure for composites

These values yield $(9t^2 + 3)^*$ values of 4 and 9, so that the square root function can be a square (Table 5) with the REDs of squares shown in Table 6.

m^*	1	3	5	7	9	0	2	4	6	8
$(m^2)^*$	1	9	5	9	1	0	4	6	6	4

Table 6. REDs of squares

4 Concluding comments

If we consider the remaining t^* values in Table 7, we see that that they all yield REDs of 2, 7 or 8, so that there will be no integer solutions for the square root function of equation (3.3), so that the integer in class $\bar{4}_6$ will be prime, unlike those in Table 4. Hence the formation of twin primes as in Table 3. REDs are independent of the size of an integer: that is, the RED structure is valid *whatever the value of the integers*.

t^*	1	4	5	6	9
$(t^2)^*$	1	6	5	6	1
$(9t^2)^*$	9	4	5	4	9
$(9t^2 + 3)^*$	2	7	8	7	2

Table 7. Continuing structures

Similar analyses can be made with the other RED couples for twin primes (9, 1) or (7, 9) as in Section 2.

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