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Some formulae which match with the prime counting function infinitely often

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Abstract: Recently K. Gaitanas gave a formula which matches with the counting prime function for an infinite set of values of its argument. In this note, we give a construction of an infinite number of such formulae.
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In [1], K. Gaitanas proved that the equality $\pi(n) = \frac{n}{\lfloor \log n - 1/2 \rfloor}$ holds for infinitely many integer values n, where π is the prime counting function. The purpose of this note is to give new examples of equations of this type.

The proof given by Gaitanas uses two theorems. The first was shown by S. W. Golomb [2]:

Theorem 1. *For infinitely many integer values n, $n/\pi(n)$ is an integer.*

The proof is elementary and uses only the facts that $\pi(n) = o(n)$ and $\pi(n+1) - \pi(n) = 0$ or 1. The second theorem is much more technically involved and was proven by J. B. Rosser and L. Schoenfeld [3]:

Theorem 2. *For all $n > 67$,*

$$\frac{n}{\log n - \frac{1}{2}} < \pi(n) < \frac{n}{\log n - \frac{3}{2}}. \tag{1}$$

For our purpose, we will need another theorem of Rosser and Schoenfeld:

Theorem 3. *For all $n > 59$,*

$$|\pi(x) - \mathrm{li}(x)| < 2K\frac{x}{\log^{3/4} x}\exp(-\sqrt{\log x/R}) \tag{2}$$

where $R = 9.645908801$, $K = 0.2197$ and li denotes the log-integral function.

The auxiliary $\mathrm{li}(x)$ function is defined for $x \in \mathbb{R}^{+*} - \{1\}$ by:

$$\mathrm{li}(x) = \begin{cases} \int_0^x \frac{dt}{\log t} & \text{if } 0 < x < 1, \\ \lim\limits_{\varepsilon \to 0}\Big(\int_0^{1-\varepsilon} \frac{dt}{\log t} + \int_{1+\varepsilon}^x \frac{dt}{\log t}\Big) & \text{if } x > 1. \end{cases} \tag{3}$$

The following result will be required.

Theorem 4. *Let x be a real number greater than $e = \exp(1)$ and $k \geq 1$ an integer value. Then if $x > e$, we have:*

$$\mathrm{li}(e) - e\sum_{i=1}^{k}(i-1)! < \mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} \tag{4}$$

Moreover if $x > e^{k+1}$, we have:

$$\mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} - x\frac{k!}{\log^k x} < \mathrm{li}(e) - e\sum_{i=1}^{k+1}(i-1)! \tag{5}$$

Proof. If we define $g_1(x) = \mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x}$ and $g_2(x) = g_1(x) - x\frac{k!}{\log^k x}$, we have $g_1'(x) = \frac{k!}{\log^{k+1} x}$ and $g_2'(x) = \frac{k!}{\log^{k+1} x}(k + 1 - \log x)$. As a consequence, the function $g_1(x)$ increases when $x > 1$ while the function $g_2(x)$ decreases when $x > e^{k+1}$. □

From what precedes, we can prove:

Theorem 5. *If $k \geq 1$, then for x near infinity, we have:*

$$\pi(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} = o\Big(\frac{x}{\log^k x}\Big). \tag{6}$$

Proof. Theorem 3 can be used to prove that, for any integer $k \geq 1$ and x near infinity, we have:

$$\pi(x) - \mathrm{li}(x) = o\Big(\frac{x}{\log^k x}\Big). \tag{7}$$

Then using theorem 4 for $k + 1$, we obtain, for $x > e^{k+2}$:

$$\mathrm{li}(e) - e\sum_{i=1}^{k}(i-1)! + x\frac{(k+1)!}{\log^{k+1} x} < \mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} \tag{8}$$

and

$$\mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} < x\frac{(k+1)! + (k+2)!}{\log^{k+1} x} + \mathrm{li}(e) - e\sum_{i=1}^{k+1}(i-1)! \tag{9}$$

so that:

$$\mathrm{li}(x) - x\sum_{i=1}^{k}\frac{(i-1)!}{\log^i x} = o\Big(\frac{x}{\log^k x}\Big). \tag{10}$$

□

The following theorem will also be needed.

Theorem 6. *Let $k \geq 1$ be an integer and let a_i be real numbers defined by $a_0 = 1$ and $a_i = -\sum_{j=0}^{i-1}(i-j)!\, a_j$ for $i \geq 1$. Then for x near infinity, we have:*

$$\Big(\sum_{i=0}^{k} \frac{i!}{\log^i x}\Big)\Big(\sum_{i=0}^{k} \frac{a_i}{\log^i x}\Big) = 1 + o\Big(\frac{x}{\log^k x}\Big). \tag{11}$$

Proof. We have:

$$\Big(\sum_{i=0}^{k} \frac{i!}{\log^i x}\Big)\Big(\sum_{i=0}^{k} \frac{a_i}{\log^i x}\Big) = \sum_{i=0}^{k}\Big(\sum_{j=0}^{i} i!\, a_{i-j}\Big)\frac{1}{\log^i x} + o\Big(\frac{x}{\log^k x}\Big). \tag{12}$$

By the use of the recurrence equation on the a_i values, the result is obtained. □

At this point, we can prove the main result of this paper:

Theorem 7. *Let $k \geq 2$ be an arbitrary integer. Then the equation:*

$$\pi(n) = \frac{n}{\Big\lfloor \log n\Big(\sum_{i=0}^{k} \frac{a_i}{\log^i n}\Big) + \frac{1}{\log^{k-1} n}\Big\rfloor}. \tag{13}$$

holds for infinitely many integer values n.

Proof. From theorem 5, for x near infinity, we have:

$$\frac{\pi(x)\log x}{x} = \sum_{i=0}^{k-1} \frac{i!}{\log^i x} + o\Big(\frac{1}{\log^{k-1} x}\Big). \tag{14}$$

Multiplying both terms by $\sum_{i=0}^{k-1} \frac{a_i}{\log^i x}$ and using theorem 6, we have then:

$$\frac{\pi(x)\log x}{x}\Big(\sum_{i=0}^{k-1} \frac{a_i}{\log^i x}\Big) = 1 + o\Big(\frac{1}{\log^{k-1} x}\Big). \tag{15}$$

and thus:

$$\frac{x}{\pi(x)\log x} = \Big(\sum_{i=0}^{k-1} \frac{a_i}{\log^i x}\Big) + o\Big(\frac{1}{\log^{k-1} x}\Big). \tag{16}$$

since $\frac{x}{\pi(x)\log x} = O(1)$ by equation 14.

We suppose now that $k \geq 3$. Then for x large enough, we have:

$$\log x\Big(\sum_{i=0}^{k-1} \frac{a_i}{\log^i x}\Big) - \Big(\frac{1}{\log^{k-2} x}\Big) < \frac{x}{\pi(x)} < \log x\Big(\sum_{i=0}^{k-1} \frac{a_i}{\log^i x}\Big) + \Big(\frac{1}{\log^{k-2} x}\Big). \tag{17}$$

The difference between the rightmost and the leftmost terms of equation 17 is equal to $\frac{2}{\log^{k-2} x}$ and is then strictly inferior to 1 when x is large enough.

Invoking Golomb's theorem 1, there are infinitely many integer values n such that $n/\pi(n)$ is an integer and large enough so that all conditions of size in the preceding equations can be met. We have then necessarily:

$$\frac{n}{\pi(n)} = \Big\lfloor \log n\Big(\sum_{i=0}^{k-1} \frac{a_i}{\log^i n}\Big) + \Big(\frac{1}{\log^{k-2} n}\Big)\Big\rfloor. \tag{18}$$

Finally, replacing $k-1$ by k in the preceding equation proves the theorem. □

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