Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 4, 85–93

Some variations on Fibonacci matrix graphs

Anthony G. Shannon[1] and Ömür Deveci[2]

[1] Emeritus Professor, University of Technology Sydney, NSW 2007,
Fellow, Warrane College, University of New South Wales, Kensington NSW 2033,
Director, Academic Affairs, Australian Institute of Music, Sydney NSW 2010, Australia
e-mail: tshannon38@gmail.com

[2] Department of Mathematics, Faculty of Science and Letters,
Kafkas University 36100, Turkey
e-mail: odeveci36@hotmail.com

Received: 12 March 2016 **Accepted:** 29 October 2017

Abstract: Matrices are here considered in two ways: arrays containing Fibonacci numbers and their generalizations in the cells, and arrays as graphs where the cells themselves are sub-graphs. Both aspects contain ideas for further development and research.

Keywords: Fibonacci, Pell and Eulerian numbers, Pyramidal numbers, Golden section, Spanning trees, Lattice points.
AMS Classification: 11B39, 05C62.

1 Introduction

With a slight variation of Horadam's classic systematic and simple summary notation [4] we consider aspects of some extensions of some of the well-known sequences in Table 1, though what we elaborate below can be applied to any of these sequences or their generalizations.

Horadam and Mahon studied these together with Chebyshev, Gegenbauer, Humbert and Stirling analogues [5]. The plan of this paper is to consider $\{w_{k,n}(a,b;p,q)\}$ for

- arbitrary k in Section 2;
- fixed $k = 2$ in Section 3;
- variable q in Section 4;
- $k = 2$ & 3 in Section 5.

a	b	p	q	w_n	Sequence
0	1	1	–1	F_n	Fibonacci
2	1	1	–1	L_n	Lucas
0	1	2	–1	P_n	Pell
1	3	2	–1	Q_n	Pell-Lucas
0	1	x+2	+1	$B_n(x)$	Morgan–Voyce Even Fibonacci
1	1	x+2	+1	$b_n(x)$	Morgan–Voyce Odd Fibonacci
2	x+2	x+2	+1	$C_n(x)$	Morgan–Voyce Even Lucas
-1	1	x+2	+1	$c_n(x)$	Morgan–Voyce Odd Lucas
0	1	1	$-x$	$J_n(x)$	Jacobstha–Fibonacci
2	1	1	$-x$	$j_n(x)$	Jacobsthal–Lucas
0	1	x	+1	$V_n(x)$	Vieta–Fibonacci
2	x	x	+1	$v_n(x)$	Vieta–Lucas

Table 1. Integer and polynomial sequences $\{w_{2,n}(a,b;p,q)\}$

2 A Pell variation

Consider the k^{th} order Pell generalization $\{w_{k,n}(1,2,...,k;2,+1)\}$ formed from the recurrence relation

$$u_{k,n} = 2u_{k,n-1} - u_{k,n-k},\ n \geq k, \tag{2.1}$$

with initial values $u_{k,j} = j$, j = 0, 1, 2, 3, ... Some examples are set out in Table 2.

k↓, n→	1	2	3	4	5	6	7	8	9	10	11	12	13
1	1	1	1	1	1	1	1	1	1	1	1	1	1
2	1	2	3	4	5	6	7	8	9	10	11	12	13
3	1	2	3	5	8	13	21	34	55	89	144	233	377
4	1	2	3	4	7	12	21	38	69	126	231	424	779
5	1	2	3	4	5	9	16	29	54	103	201	393	757
6	1	2	3	4	5	6	11	20	37	70	135	264	517
7	1	2	3	4	5	6	7	13	24	45	86	167	328
8	1	2	3	4	5	6	7	8	15	28	53	102	199
9	1	2	3	4	5	6	7	8	9	17	32	61	118
10	1	2	3	4	5	6	7	8	9	10	19	36	69

Table 2. Patterns among $\{w_{k,n}(1,2,...,k;2,1)\}$

If we consider the elements in the backward diagonals in Table 2, then we can establish the sequences in Table 3 with the connections with the Eulerian numbers:

$$E_n = 2^n - n - 1, \ \ n \geq 0. \qquad (2.2)$$

k↓, m→	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**$u_{k,m}$**	**m >**
1	1	1	1	1	1	1	1	1	1	1	$u_{1,m} = 1$	0
2	1	2	3	4	5	6	7	8	9	10	$u_{2,m} = m$	0
3	1	3	5	7	9	11	13	15	17	19	$u_{3,m} = 2m - 1$	1
4	1	4	8	12	16	20	24	28	32	36	$u_{4,m} = 4m - 4$	1
5	1	5	13	21	29	37	45	53	61	69	$u_{5,m} = 8m - 11$	2
6	1	6	21	38	54	70	86	102	118	134	$u_{6,m} = 16m - 26$	3
7	1	7	34	69	103	135	167	199	231	263	$u_{7,m} = 32m - 57$	4
8	1	8	55	126	201	264	328	392	456	520	$u_{8,m} = 64m - 120$	5
9	1	9	89	231	393	517	649	777	905	1033	$u_{9,m} = 128m - 247$	6
10	1	10	144	424	757	1014	1290	1546	1802	2058	$u_{10,m} = 256m - 502$	7

Table 3. Eulerian and generalized Pell numbers

3 Fibonacci matrix variations

The previous tables suggest realigning the columns by dropping the elements in successive columns to produce Fibonacci rectangular and Lucas square 'triangular' matrices as in Tables 4 and 5.

$$F_{10\times 6} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 5 & 2 & 0 & 0 & 0 \\ 1 & 8 & 3 & 1 & 0 & 0 \\ 1 & 13 & 5 & 2 & 0 & 0 \\ 1 & 21 & 8 & 3 & 1 & 0 \\ 1 & 34 & 13 & 5 & 2 & 0 \\ 1 & 55 & 21 & 8 & 3 & 0 \end{bmatrix}$$

Table 4. A Fibonacci triangular matrix

Thus, $F_{5\times 1} = [1, 1, 1, 1, 1]^T$ for instance. The 'left triangle' parts of these matrices lack the symmetry that one finds with Pascal-type triangles of these sorts of numbers [11]. Nevertheless, the Fibonacci triangle set out in Table 4 has several properties similar to these generalizations [9], including the following examples with a variety of row, column and diagonal properties can be discerned.

Sums of cells:

- row sums are Fibonacci numbers;
- partial row sums $\{F_{n+3} - 2\}, j > 1$;
- rising diagonal sums $\{F_{n+1} - \lfloor \frac{1}{2} F_{n+1} \rfloor\}$.

Partial recurrence relations:

- $u_{i,j} = u_{i-1,j} + u_{i-2,j}, \ j > 1$;
- $u_{i,j} = u_{i,j-1} - u_{i-1,j-1}, \ i > 1, j > i - 1$;
- $u_{i,j} = F_{i-2j+4}$.

Analogous variations can also be applied to the other sequences to produce companion matrices [3] and tridiagonal matrices [2]. Instead we now outline a corresponding Lucas illustration.

$$L_{10\times 10} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 3 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 7 & 4 & 3 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 11 & 7 & 4 & 3 & 1 & 2 & 0 & 0 & 0 \\ 1 & 18 & 11 & 7 & 4 & 3 & 1 & 2 & 0 & 0 \\ 1 & 29 & 18 & 11 & 7 & 4 & 3 & 1 & 2 & 0 \\ 1 & 47 & 29 & 18 & 11 & 7 & 4 & 3 & 1 & 1 \end{bmatrix}$$

Table 5. A Lucas triangular matrix

4 Golden ratio variations

Variations of the golden ratio are effectively done by looking at the sequences generated with different values of q. A Fibonacci golden ratio family of sequences is set out in Table 6 [*cf.* 10], and its Lucas counterpart in Table 7 [*cf.* 17].

Thus one can use Sloane's encyclopedia for connections and creations [15]. More deeply one can search for intersections [16] and divisibility properties [7].

$w_n(0,1;1,-q)$	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
$w_n(0,1;1,0)$	1	1	1	1	1	1	1	1	1	1	1	1
$w_n(0,1;1,-1)$	1	1	2	3	5	8	13	21	34	55	89	144
$w_n(0,1;1,-2)$	1	1	3	5	11	21	43	85	171	341	683	1365
$w_n(0,1;1,-3)$	1	1	4	7	19	40	97	217	508	1159	2683	6160
$w_n(0,1;1,-4)$	1	1	5	9	29	65	181	441	1165	2929	7589	19305
$w_n(0,1;1,-5)$	1	1	6	11	41	96	301	781	2286	6191	17621	48576
$w_n(0,1;1,-6)$	1	1	7	13	55	133	463	1261	4039	11605	35839	105469
$w_n(0,1;1,-7)$	1	1	8	15	71	176	673	1905	6616	19951	66263	205920

Table 6. Fibonacci variations

$w_n(2,1;1,-q)$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
$w_n(2,1;1,0)$	2	2	2	2	2	2	2	2	2	2	2	2
$w_n(2,1;1,-1)$	2	1	3	4	7	11	18	29	47	76	123	199
$w_n(2,1;1,-2)$	2	1	5	7	17	31	65	127	257	511	1025	2047
$w_n(2,1;1,-3)$	2	1	7	10	31	61	154	337	799	1810	4207	9637
$w_n(2,1;1,-4)$	2	1	9	13	49	101	297	701	1889	4693	12249	31021
$w_n(2,1;1,-5)$	2	1	11	16	71	151	506	1261	3791	10096	29051	79531
$w_n(2,1;1,-6)$	2	1	13	19	97	211	793	2059	6817	19171	60073	175099
$w_n(2,1;1,-7)$	2	1	15	22	127	281	1170	3137	11327	33286	112575	345577

Table 7. Lucas variations

5 Cells in matrices

We now consider the matrix arrays as graphs in themselves. For simplicity, we start with square matrices which are divided into sub-graphs containing 1, 4, 9, 16, ..., square matrices as illustrated in Figure 1.

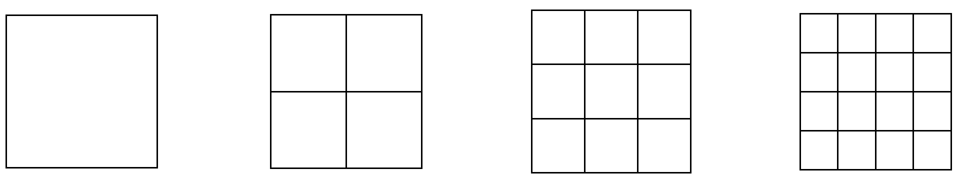

Figure 1. Matrices with 1, 4, 9, 16 cells

Immediately we observe that the number of squares contained in each matrix is 1, 5, 14, 30, 55, ..., the square pyramidal numbers, generated by $n(n + 1)(2n + 1)/6$, where n^2 is the number of cells contained with the whole matrix. There is a wealth of literature on pyramidal numbers [17: M3844] which we do not plan to pursue here. Rather we continue to consider aspects of these subgraphs.

Connections with Fibonacci matrices and graphs occur through spanning trees and the complexity of a graph [6, 12, 14], but many problems remain. For instance, by extending the squares in Figure 1 through their diagonals, we obtain the planar representation of a trellis (or wire mesh) fence consists of sets of 'crosses' or 'squares' as shown in Figure 2.

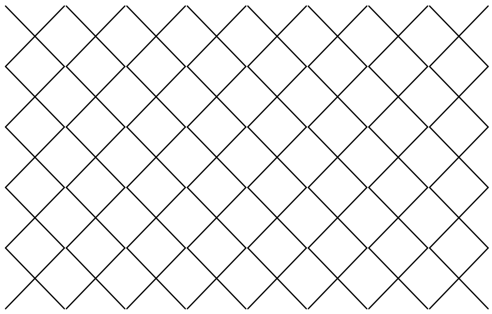

Figure 2. Representation of a section of trellis (wire-mesh)

Immediately a number of non-trivial questions arise, such as how many squares? symmetric crosses? rectangles? lattice points? crosses (symmetric or asymmetric)? spanning trees?

Attempts to solve the problems are probably best illustrated by construction. In general, one would expect the solutions to be functions of the numbers of edges and vertices. We define a trellis of a given size and position in the plane as even or odd:

- an even trellis, $f_{n,m}$, is the set of integer lattice points $\{(x, y) : x + y$ is even, $0 \le x \le 2n,\ 0 \le y \le 2m\}$;
- an odd trellis, $g_{n,m}$, is the set of integer lattice points $\{(x, y) : x + y$ is odd, $0 \le x \le 2n,\ 0 \le y \le 2m\}$.

Figures 3 (a), (b), (c), (d) show the cases for 'fences' $f_{1,m}$, $f_{2,m}$, $f_{3,m}$, m = 1, 2, 3, respectively, where $\{f_{n,m}\}$ represents the set of single-edged symmetric crosses of fences with 'height' n and 'length' m. Thus in Figure 2, $\{f_{2,m}\}$, m = 1, 2, 3, is the set of 3 single-edged symmetric crosses of height 2 such crosses.

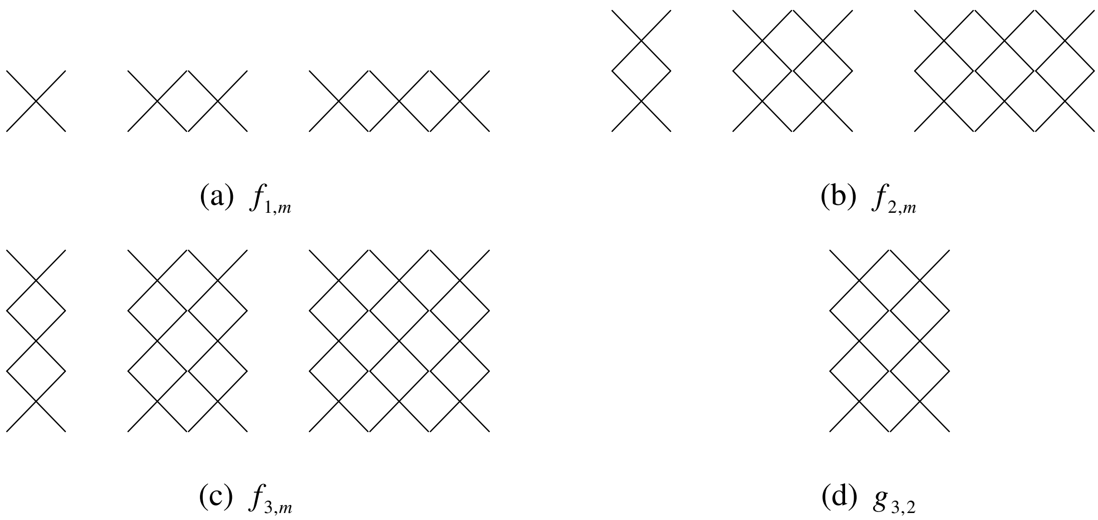

Figure 3. Representation of fences

Let $e_{n,m}$ be the number of edges in $f_{n,m}$. Then, since $f_{n,m}$ is constructed by an $n \times m$ lattice of crosses and each cross contributes four edges, it follows that

$$f_{n,m} = 4nm. \tag{5.1}$$

See the black dots in Figure 4 and the entries in Table 8.

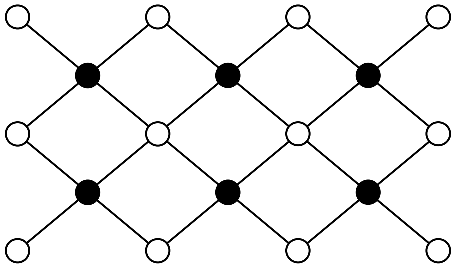

Figure 4. $f_{2,3}$

n↓ m→	**1**	**2**	**3**	**4**	**5**
1	4	8	12	16	20
2	8	16	24	32	40
3	12	24	36	48	60
4	16	32	48	64	80

Table 8. $e_{n,m}$

Similarly let $v_{n,m} \in f_{n,m}$ and $w_{n,m} \in g_{n,m}$ be the corresponding numbers of vertices (Table 9).

n↓ m→	**1**	**2**	**3**	**4**	**5**
1	5	8	11	14	17
2	8	13	18	23	28
3	11	18	25	32	39
4	14	23	32	41	50

Table 9. $v_{n,m}$

For $v_{n,m}$ there are nm black dots and $(n+1)(m+1)$ white dots for a total of

$$v_{n,m} = nm + (n+1)(m+1) = 2nm + n + m + 1. \tag{5.2}$$

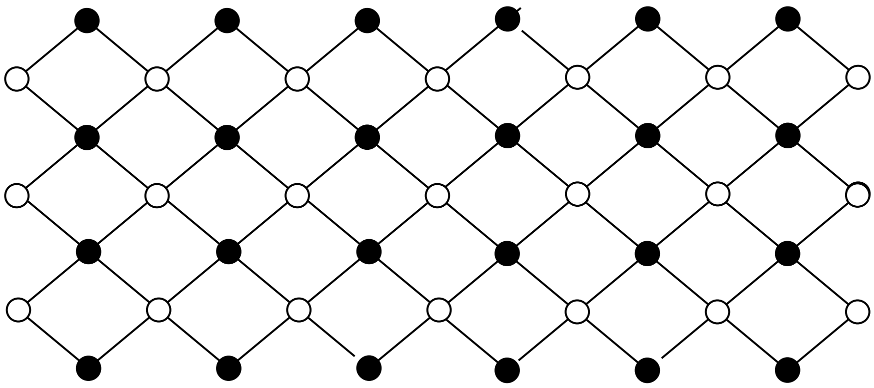

Figure 5. $g_{3,6}$

6 Concluding comments

Various other extensions and generalizations of the sequences in Table 1 can be readily investigated. For example, just as for the second-order Pell sequences

$$\{P_{2,n}\} \equiv \{w_{2,n}(0,1;2,-1)\},\ \{Q_{2,n}\} \equiv \{w_{2,n}(1,3;2,-1)\}$$

there is the connection

$$Q_{2,n} = P_{2,n} + P_{2,n+1} \tag{4.1}$$

so too for the corresponding third-order Pell sequences

$$\{P_{3,n}\} \equiv \{w_{3,n}(0,0,1;2,-1)\},\ \{Q_{3,n}\} \equiv \{w_{3,n}(1,1,3;2,-1)\}$$

there is also the connection

$$Q_{3,n} = P_{3,n-1} + P_{3,n} + P_{3,n+1} \tag{5.2}$$

where the third-order recurrence relation is

$$w_{3,n} = w_{3,n-1} + w_{3,n-2},\ n \geq 2.$$

While at one level almost any desired elegant identity can be obtained by a suitable choice of initial values, the selection can be determined by us with the use of "basic" sequences and corresponding matrices. At order k, there will be k basic fundamental sequences and one primordial sequence, and corresponding matrices [13].

More fundamentally though, these ideas provide a source of multitudes of undergraduate exercises which students generally seem to enjoy both computationally and symbolically, the latter helping to cultivate a feeling for notation as a tool of thought, not only in mathematics but also in music [8], two disciplines which share much in common with the creative processes both in doing and in discovering [1].

References

[1] Benson, D. (2006) *Music: A Mathematical Offering*, Cambridge University Press, Cambridge.

[2] Cahill, N., & Narayan, D. (2004) Fibonacci and Lucas numbers in tridiagonal matrix determinants, *The Fibonacci Quarterly*, 42(3), 216–221.

[3] Deveci, O., & Shannon, A. G. (2017) On the adjacency-type sequences, *International Journal of Advances in Mathematics*, 17(2), 10–24.

[4] Hilton, A. J. W. (1974) Spanning trees and Fibonacci and Lucas numbers, *The Fibonacci Quarterly,* 12(3), 259–262.

[5] Horadam, A. F. (1965) Basic properties of a certain generalized sequence of numbers, *The Fibonacci Quarterly*, 3(3), 161–176.

[6] Horadam, A. F., & Mahon, J. M. (1986) Convolutions for Pell polynomials. In *A.N. Philippou, Gerald E. Bergum and Alwyn F. Horadam (eds). Fibonacci Numbers and Their Applications*, Reidel, Dordecht, 55–80.

[7] Jarden, D. (1958) Recurring Sequences. *Riveon Lematematika, Jerusalem*, 1–4.

[8] Kuijken, B. (2013) *The Notation is Not the Music*, Indiana University Press, Bloomington & Indianapolis, IN.

[9] Ollerton, R. L., & Shannon, A. G. (1998) Some properties of generalized Pascal squares and triangles, *The Fibonacci Quarterly*, 36(2), 98–109.

[10] Pruitt, R. (1967) Fibonacci and Lucas numbers in the sequence of golden numbers, *The Fibonacci Quarterly*, 5(2), 175–178.

[11] Quaintance, J., & Gould, H. W. (2015) *Combinatorial Numbers for Stirling Numbers: The Unpublished Notes of H. W. Gould,* World Scientific, New Jersey, Ch. 10.

[12] Rebman, K. R. (1975) The sequence 1 5 16 45 121 320 ... in Combinatorics, *The Fibonacci Quarterly,* 13(1), 51–55.

[13] Shannon, A. G. (1974) Some properties of a fundamental recursive sequence of arbitrary order, *The Fibonacci Quarterly*, 12(4), 327–335.

[14] Shannon, A. G. (1978), Fibonacci and Lucas Numbers and the Complexity of a Graph, *The Fibonacci Quarterly*, 16(1), 1–4.

[15] Sloane, N. J. A., & Plouffe, S. (1968) *The Encyclopedia of Integer Sequences,* Academic Press, San Diego, CA.

[16] Stein, S. K. (1962) The intersection of Fibonacci sequences, *Michigan Mathematical Journal,* 9(4), 399–402.

[17] Whitford, A. K. (1977) Binet's formula generalized, *The Fibonacci Quarterly*, 15(1), 21, 24, 29.