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Generalized dual Pell quaternions

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Abstract: In this paper, we defined the generalized dual Pell quaternions. Also, we investigated the relations between the generalized dual Pell quaternions. Furthermore, we gave the Binet's formulas and Cassini-like identities for these quaternions.
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1 Introduction

The real quaternions are a number system which extends to the complex numbers. They are first described by Irish mathematician William Rowan Hamilton in 1843.

Hamilton [1] introduced the set of real quaternions which can be represented as

$$H = \{\, q = q_0 + i\,q_1 + j\,q_2 + k\,q_3 \;\mid\; q_0,\, q_1,\, q_2,\, q_3 \in \mathbb{R}\,\} \tag{1.1}$$

where

$$i^2 = j^2 = k^2 = -1\,, \;\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j\,.$$

Several authors worked on different quaternions and their generalizations. ([2–22, 24–26]). In 2013, Akyiğit et al. [17] defined split Fibonacci and split Lucas quaternions and obtained some identities for them. Complex split quaternions were defined by Kula and Yayli [13] in 2007.

In 1961, Horadam [3] firstly introduced the generalized Fibonacci sequence (H_n) and used this sequence in 1963, Horadam [4] defined the n-th Fibonacci quaternion which can be represented as

$$Q_F = \{Q_n = F_n + i\,F_{n+1} + j\;F_{n+2} + k\,F_{n+3} \mid F_n,\, n-th\text{Fibonacci number}\,\} \tag{1.2}$$

where

$$\begin{aligned} i^2 =& j^2 = k^2 = i\,j\,k = -1\,,\;\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \\ & k\,i = -i\,k = j \end{aligned}$$

and $n \geq 1$.

In 1969, Iyer [5, 6] derived many relations for the Fibonacci quaternions.

In 1973, Swamy [8] considered generalized Fibonacci quaternions as a new quaternion as follows:

$$P_n = H_n + i\,H_{n+1} + j\;H_{n+2} + k\,H_{n+3} \tag{1.3}$$

where

$$\begin{cases} H_n = H_{n-1} + \;H_{n-2}, \\ H_1 = p, \\ H_2 = p + q, \\ H_n = (p-q)F_n + qF_{n+1},\;\; n \geq 1 \end{cases}$$

where H_n is the $n-th$ generalized Fibonacci number that is defined in [4].

(See [8] for generalized Fibonacci quaternions).

In 1977, Iakin [9, 10] introduced higher order quaternions and gave some identities for these quaternions.

In 1993, Horadam [12] extend to quaternions to the complex Fibonacci numbers defined by Harman [11].

In 2006, Majernik [18] defined dual quaternions as follows:

$$H_{\mathbb{D}} = \left\{ \begin{array}{c} Q = a + b\,i\, + c\,j\, + d\,k \mid a,\, b,\, c,\, d \in \mathbb{R},\; i^2 = j^2 = k^2 = i\,j\,k = 0, \\ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0 \end{array} \right\}. \tag{1.4}$$

In 2009, Ata and Yaylı [14] defined dual quaternions with dual numbers coefficient $(\,a + \varepsilon\,b\,,\; a, b \in \mathbb{R}\,,\; \varepsilon^2 = 0\,,\; \varepsilon \neq 0\,)$ as follows:

$$H(\mathbb{D}) = \left\{ Q = A + Bi\, + Cj\, + Dk \mid A,\, B,\, C,\, D \in \mathbb{D}\,, i^2 = j^2 = k^2 = -1 = ijk \right\} \tag{1.5}$$

In 2014, Nurkan and Güven [20] defined dual Fibonacci quaternions as follows:

$$H(\mathbb{D}) = \{\tilde{Q}_n = \tilde{F}_n + i\tilde{F}_{n+1} + j\tilde{F}_{n+2} + k\tilde{F}_{n+3} \mid \tilde{F}_n = F_n + \epsilon F_{n+1},\, \epsilon^2 = 0, \epsilon \neq 0\}, \tag{1.6}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = -1\,,\;\; i\,j = -j\,i = k\,, \quad j\,k = -k\,j = i\,, \quad k\,i = -i\,k = j$$

$n \geq 1$ and $\tilde{Q}_n = Q_n + \varepsilon\, Q_{n+1}$. Essentially, these quaternions in equations (1.5) and (1.6) must be called dual coefficient quaternion and dual coefficient Fibonacci quaternions, respectively. For more details on dual quaternions, see [19]. It is clear that $H(\mathbb{D})$ and $H_{\mathbb{D}}$ are different sets.

In 2016, Yüce and Torunbalcı Aydın [21] defined dual Fibonacci quaternions as follows:

$$H_{\mathbb{D}} = \{\, Q_n = F_n + i\, F_{n+1} + j\;\, F_{n+2} + k\, F_{n+3} \mid F_n,\, n\text{-}th \text{ Fibonacci number}\}, \tag{1.7}$$

where

$$i^2 = j^2 = k^2 = i\, j\, k = 0\,,\;\; i\, j = -j\, i = j\, k = -k\, j = k\, i = -i\, k = 0.$$

In 2016, Yüce and Torunbalcı Aydın [22] defined generalized dual Fibonacci quaternions as follows:

$$\begin{aligned} Q_{\mathbb{D}} = \{\mathbb{D}_{\mathbf{n}} = & H_n + i\, H_{n+1} + j\;\, H_{n+2} + k\, H_{n+3} \mid H_n,\, n\text{-}th \\ & \text{Generalized Fibonacci number}\} \end{aligned} \tag{1.8}$$

where

$$i^2 = j^2 = k^2 = i\, j\, k = 0\,,\;\; i\, j = -j\, i = j\, k = -k\, j = k\, i = -i\, k = 0.$$

In 1971, Horadam studied on the Pell and Pell–Lucas sequences and he gave Cassini-like formula as follows [27]:

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n, \tag{1.9}$$

and Pell identities

$$\begin{cases} P_r P_{n+1} + P_{r-1} P_n = P_{n+r}, \\ P_n(P_{n+1} + P_{n-1}) = P_{2n}, \\ P_{2n+1} + P_{2n} = 2P_{n+1}^2 - 2P_n^2 - (-1)^n, \\ P_n^2 + P_{n+1}^2 = P_{2n+1}, \\ P_n^2 + P_{n+3}^2 = 5(P_{n+1}^2 + P_{n+2}^2), \\ P_{n+a}P_{n+b} - P_n P_{n+a+b} = (-1)^n P_n P_{n+a+b}, \\ P_{-n} = (-1)^{n+1} P_n. \end{cases} \tag{1.10}$$

In 1985, Horadam and Mohan [28] obtained Cassini-like formula as follows:

$$q_{n+1}\, q_{n-1} - q_n^2 = 8\,(-1)^{n+1}. \tag{1.11}$$

First the idea to consider Pell quaternions it was suggested by Horadam in paper [12].

In 2017 (arXiv), Torunbalcı Aydın and Köklü [23] defined generalized Pell sequence as follows:

$$\begin{cases} \mathbb{P}_0 = & q,\, \mathbb{P}_1 = p,\, \mathbb{P}_2 = 2p + q,\;\, p\, q \in \mathbb{Z} \\ \mathbb{P}_n = & 2P_{n-1} + \mathbb{P}_{n-2},\;\, n \geq 2 \\ or & \\ \mathbb{P}_n = & (p - 2q)P_n + q\, P_{n+1} = p\, P_n + q\, P_{n-1} \end{cases} \tag{1.12}$$

where $\mathbb{P}_n$ is the n-th generalized Pell number that defined in [23] as follows:

$$(\mathbb{P}_n) :\; q\,,\;\, p\,,\;\, 2\,p + q\,,\;\, 5\,p + 2\,q\,,\;\, 12\,p + 5\,q\,,\;\, 29\,p + 12\,q\,,\; \ldots\;, p\,P_n + qP_{n-1}, \ldots \tag{1.13}$$

In 2016, Torunbalcı Aydın and Yüce [24] defined dual Pell quaternions and dual Pell–Lucas quaternions as follows respectively:

$$P_D = \{\, D_n^P = P_n + i\, P_{n+1} + j\, P_{n+2} + k\, P_{n+3} \mid \, P_n \; n\text{-}th \,\text{Pell number}\}, \tag{1.14}$$

where

$$i^2 = j^2 = k^2 = i\, j\, k = 0\,, \;\; i\, j = \, -j\, i = \, j\, k = \, -k\, j = \, k\, i = \, -i\, k = 0$$

and

$$p_D = \{\, D_n^p = q_n + i\, q_{n+1} + j\, q_{n+2} + k\, q_{n+3} \mid \, q_n \; n\text{-}th \,\text{Pell–Lucas number}\}, \tag{1.15}$$

$$i^2 = j^2 = k^2 = i\, j\, k = 0\,, \;\; i\, j = \, -j\, i = \, j\, k = \, -k\, j = \, k\, i = \, -i\, k = 0\,.$$

Here, the Pell–Lucas sequence (q_n) and q_n which is the n-th term of the dual Pell–Lucas quaternion sequence (D_n^q) are defined by the following recurrence relations:

$$(q_n): \;\; 2\,, \; 2\,, \; 6\,, \; 14, \; 34, \; 82, \; 198, \; 478, \; 1154, \; 2786, \ldots, q_n, \ldots$$

$$\begin{cases} q_n = & 2q_{n-1} + q_{n-2}, \;\; n \geq 3, \\ q_0 = & 2, \;\; q_1 = 2, \;\; q_2 = 6. \end{cases} \tag{1.16}$$

In 2016, Çimen and İpek [25] worked on Pell quaternions and Pell–Lucas quaternions and defined as follows respectively:

$$QP_n = \{QP_n = P_n\, e_0 + P_{n+1}\, e_1 + P_{n+2}\, e_2 + P_{n+3}\, e_3 \mid P_n, \; n\text{-}th \,\text{Pell number}\} \tag{1.17}$$

and

$$QPL_n = \{QPL_n = q_n\, e_0 + q_{n+1}\, e_1 + q_{n+2}\, e_2 + q_{n+3}\, e_3 \mid \, q_n, \; n\text{-}th \;\text{Pell–Lucas number}\} \tag{1.18}$$

where

$$\begin{cases} e_0^2 = 1, \;\; e_1^2 = e_2^2 = e_3^2 = -1, \\ e_0\, e_1 = e_1\, e_0 = e_1, \;\; e_0\, e_2 = e_2\, e_0 = e_2, \;\; e_0\, e_3 = e_3\, e_0 = e_3, \\ e_1\, e_2 = -e_2\, e_1 = e_3, \;\; e_2\, e_3 = -e_3\, e_2 = e_1, \;\; e_3\, e_1 = -e_1\, e_3 = e_2. \end{cases}$$

In 2016, Anetta and Iwona [26] worked on the Pell quaternions and the Pell octanions.

In this paper, we define the generalized dual Pell quaternions as follows:

$$P_{\mathbb{D}} = \{\mathbb{D}_{\mathbf{n}}^{\mathbf{P}} = \mathbb{P}_n + i\, \mathbb{P}_{n+1} + j\, \mathbb{P}_{n+2} + k\, \mathbb{P}_{n+3} \mid \mathbb{P}_n, \; n\text{-}th \,\text{Gen.Pell number}\} \tag{1.19}$$

where

$$i^2 = j^2 = k^2 = i\; j\; k = 0\,, \;\; i\; j = -j\; i = j\; k = -k\; j = k\; i = -i\; k = 0.$$

Furthermore, we give Binet's Formula and Cassini-like identities for the generalized dual Pell quaternions.

2 Generalized dual Pell quaternions

The generalized Pell sequence $\mathbb{P}_n$ is defined as

$$\begin{cases} \mathbb{P}_0 = & q,\, \mathbb{P}_1 = p,\, \mathbb{P}_2 = 2p+q,\ \ p, q \in \mathbb{Z} \\ \mathbb{P}_n = & 2\,\mathbb{P}_{n-1} + \mathbb{P}_{n-2},\ \ n \geq 2 \\ \quad or & \\ \mathbb{P}_n = & (p-2q)P_n + q\,P_{n+1} = p\,P_n + q\,P_{n-1}. \end{cases} \tag{2.1}$$

Here, P_n is the n-th Pell number and $\mathbb{P}_n$ is the n-th generalized Pell number that defined in [23] as follows:

$$(\mathbb{P}_n) :\ q\,,\ p\,,\ 2\,p+q\,,\ 5\,p+2\,q\,,\ 12\,p+5\,q\,,\ 29\,p+12\,q\,,\ \ldots\ , p\,P_n + qP_{n-1}, \ldots$$

We can define the generalized dual Pell quaternions by using generalized Pell numbers as follows

$$Q_{\mathbb{D}} = \{\mathbb{D}^{\mathbf{P}}{}_n = \mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\ \mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}\ \mid \mathbb{P}_n,\, n\text{-}th \text{ Gen. Pell number}\}, \tag{2.2}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

The scaler and the vector part of $\mathbb{D}^{\mathbf{P}}{}_n$ which is the n-th term of the generalized dual Pell quaternion $(\mathbb{D}^{\mathbf{P}}{}_n)$ are denoted by

$$S_{\mathbb{D}^{\mathbf{P}}{}_n} = \mathbb{P}_n \text{ and } \ V_{\mathbb{D}^{\mathbf{P}}{}_n} = i\mathbb{P}_{n+1} + j\mathbb{P}_{n+2} + k\mathbb{P}_{n+3}. \tag{2.3}$$

Thus, the generalized dual Pell quaternion $\mathbb{D}^{\mathbf{P}}_{\mathbf{n}}$ is given by $\mathbb{D}^{\mathbf{P}}{}_n = S_{\mathbb{D}^{\mathbf{P}}{}_n} + V_{\mathbb{D}^{\mathbf{P}}{}_n}$. Let $\mathbb{D}^{\mathbf{P_1}}{}_n$ and $\mathbb{D}^{\mathbf{P_2}}{}_n$ be n-th terms of the generalized dual Pell quaternion sequences $(\mathbb{D}^{\mathbf{P_1}}{}_n)$ and $(\mathbb{D}^{\mathbf{P_2}}{}_n)$ such that

$$\mathbb{D}^{\mathbf{P_1}}{}_n = \mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\ \mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3} \tag{2.4}$$

and

$$\mathbb{D}^{\mathbf{P_2}}{}_n = \mathbb{K}_n + i\,\mathbb{K}_{n+1} + j\ \mathbb{K}_{n+2} + k\,\mathbb{K}_{n+3}. \tag{2.5}$$

Then, the addition and subtraction of the generalized dual Pell quaternions is defined by

$$\begin{aligned} \mathbb{D}^{\mathbf{P_1}}{}_n \pm \mathbb{D}^{\mathbf{P_2}}{}_n = & \ (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & \pm(\mathbb{K}_n + i\,\mathbb{K}_{n+1} + j\,\mathbb{K}_{n+2} + k\,\mathbb{K}_{n+3}) \\ = & \ (\mathbb{P}_n \pm \mathbb{K}_n) + i\,(\mathbb{P}_{n+1} \pm \mathbb{K}_{n+1}) + j\,(\mathbb{P}_{n+2} \pm \mathbb{K}_{n+2}) \\ & +k\,(\mathbb{P}_{n+3} \pm \mathbb{K}_{n+3})\,. \end{aligned} \tag{2.6}$$

Multiplication of the generalized dual Pell quaternions is defined by

$$\begin{aligned} \mathbb{D}^{\mathbf{P_1}}{}_n \,.\, \mathbb{D}^{\mathbf{P_2}}{}_n = & \ (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & \ (\mathbb{K}_n + i\,\mathbb{K}_{n+1} + j\,\mathbb{K}_{n+2} + k\,\mathbb{K}_{n+3}) \\ = & \ (\mathbb{P}_n\,\mathbb{K}_n) + \mathbb{P}_n(i\,\mathbb{K}_{n+1} + j\,\mathbb{K}_{n+2} + k\,\mathbb{K}_{n+3}) \\ & +(i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3})\mathbb{K}_n. \end{aligned} \tag{2.7}$$

or

$$\mathbb{D}^{\mathbf{P_1}}{}_n \, . \, \mathbb{D}^{\mathbf{P_2}}{}_n = S_{\mathbb{D}^{\mathbf{P_1}}{}_n} \; S_{\mathbb{D}^{\mathbf{P_2}}{}_n} + S_{\mathbb{D}^{\mathbf{P_1}}{}_n} \, V_{\mathbb{D}^{\mathbf{P_2}}{}_n} + S_{\mathbb{D}^{\mathbf{P_2}}{}_n} \, V_{\mathbb{D}^{\mathbf{P_1}}{}_n}. \tag{2.8}$$

The conjugate of generalized dual Pell quaternion $\mathbb{D}^{\mathbf{P}}_{\mathbf{n}}$ is denoted by $\overline{\mathbb{D}^{\mathbf{P}}_{\mathbf{n}}}$ and it is

$$\overline{\mathbb{D}^{\mathbf{P}}{}_n} = \mathbb{P}_n - i\,\mathbb{P}_{n+1} - j\,\mathbb{P}_{n+2} - k\,\mathbb{P}_{n+3}. \tag{2.9}$$

The norm of $\mathbb{D}^{\mathbf{P}}{}_n$ is defined as

$$\|\mathbb{D}^{\mathbf{P}}{}_n\|^2 = \mathbb{D}^{\mathbf{P}}{}_n \; \overline{\mathbb{D}^{\mathbf{P}}{}_n} = (\mathbb{P}_n)^2. \tag{2.10}$$

Then, we give the following theorem using statements (2.1), (2.2) and the generalized Pell number in [23] as follows

$$\mathbb{P}_m\,\mathbb{P}_{n+1} + \mathbb{P}_{m-1}\,\mathbb{P}_n = (2p-2q)\mathbb{P}_{m+n} - e_P\,P_{m+n} \tag{2.11}$$

where

$$e_P = p^2 - 2p\,q - q^2.$$

Theorem 2.1. *Let $\mathbb{P}_n$ and $\mathbb{D}^{\mathbf{P}}{}_n$ be the $n-th$ terms of generalized Pell sequence $(\mathbb{P}_n)$ and the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$, respectively. In this case, for $n \geq 1$ we can give the following relations:*

$$\mathbb{D}^{\mathbf{P}}{}_n + 2\,\mathbb{D}^{\mathbf{P}}{}_{n+1} = \mathbb{D}^{\mathbf{P}}{}_{n+2} \tag{2.12}$$

$$(\mathbb{D}^{\mathbf{P}}{}_n)^2 = 2\,\mathbb{P}_n\,\mathbb{D}^{\mathbf{P}}{}_n - (\mathbb{P}_n)^2 \tag{2.13}$$

$$\mathbb{D}^{\mathbf{P}}{}_n - i\,\mathbb{D}^{\mathbf{P}}{}_{n+1} - j\,\mathbb{D}^{\mathbf{P}}{}_{n+2} - k\,\mathbb{D}^{\mathbf{P}}{}_{n+3} = \mathbb{P}_n \tag{2.14}$$

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_n\,\mathbb{D}^{\mathbf{P}}{}_m + \mathbb{D}^{\mathbf{P}}{}_{n+1}\,\mathbb{D}^{\mathbf{P}}{}_{m+1} =& (2p-2q)\,[2\,\mathbb{D}^{\mathbf{P}}{}_{n+m+1} - \mathbb{P}_{n+m+1}] \\ & - e_P\,[2\,D^P_{n+m+1} - P_{n+m+1}]. \end{aligned} \tag{2.15}$$

where D^P_{n+m+1} is the dual Pell quaternion [24].

Proof. (2.12): By the equations

$$\mathbb{D}^{\mathbf{P}}{}_n = \mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\;\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3} \tag{2.16}$$

and

$$\mathbb{D}^{\mathbf{P}}{}_{n+1} = \mathbb{P}_{n+1} + i\,\mathbb{P}_{n+2} + j\;\mathbb{P}_{n+3} + k\,\mathbb{P}_{n+4} \tag{2.17}$$

we get,

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_n + 2\,\mathbb{D}^{\mathbf{P}}{}_{n+1} = \;& (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & +2(\mathbb{P}_{n+1} + i\,\mathbb{P}_{n+2} + j\,\mathbb{P}_{n+3} + k\,\mathbb{P}_{n+4}\,) \\ = \;& (\mathbb{P}_n + 2\mathbb{P}_{n+1}) + i\,(\mathbb{P}_{n+1} + 2\mathbb{P}_{n+2}) + j\,(\mathbb{P}_{n+2} + 2\mathbb{P}_{n+3}) \\ & +k\,(\mathbb{P}_{n+3} + 2\mathbb{P}_{n+4}) \\ = \;& \mathbb{P}_{n+2} + i\,\mathbb{P}_{n+3} + j\,\mathbb{P}_{n+4} + k\,\mathbb{P}_{n+5} \\ = \;& \mathbb{D}^{\mathbf{P}}_{\mathbf{n+2}}\,. \end{aligned} \tag{2.18}$$

(2.13):

$$\begin{aligned}(\mathbb{D}^{\mathbf{P}}{}_n)^2 = \;& (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ = \;& (\mathbb{P}_n)^2 + 2i\,(\mathbb{P}_n\,\mathbb{P}_{n+1}) + 2j\,(\mathbb{P}_n\,\mathbb{P}_{n+2}) + 2k\,(\mathbb{P}_n\,\mathbb{P}_{n+3}) \\ = \;& 2\,\mathbb{P}_n\,\mathbb{D}^{\mathbf{P}}{}_n - (\mathbb{P}_n)^2.\end{aligned} \tag{2.19}$$

(2.14): By using (2.3) and conditions in the equation (2.2), we get

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n - i\,\mathbb{D}^{\mathbf{P}}{}_{n+1} - j\,\mathbb{D}^{\mathbf{P}}{}_{n+2} - k\,\mathbb{D}^{\mathbf{P}}{}_{n+3} = \;& (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & -i\,(\mathbb{P}_{n+1} + i\,\mathbb{P}_{n+2} + j\,\mathbb{P}_{n+3} + k\,\mathbb{P}_{n+4}) \\ & -j\,(\mathbb{P}_{n+2} + i\,\mathbb{P}_{n+3} + j\,\mathbb{P}_{n+4} + k\,\mathbb{P}_{n+5}) \\ & -k(\mathbb{P}_{n+3} + i\,\mathbb{P}_{n+4} + j\,\mathbb{P}_{n+5} + k\,\mathbb{P}_{n+6}) \\ = \;& \mathbb{P}_n\,.\end{aligned} \tag{2.20}$$

(2.15): By using (2.6) and (2.11)

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n\,\mathbb{D}^{\mathbf{P}}{}_m = \;& \mathbb{P}_n\,\mathbb{P}_m + i\,(\mathbb{P}_n\,\mathbb{P}_{m+1} + \mathbb{P}_{n+1}\,\mathbb{P}_m) \\ & +j\,(\mathbb{P}_n\,\mathbb{P}_{m+2} + \mathbb{P}_{n+2}\,\mathbb{P}_m) + k\,(\mathbb{P}_n\,\mathbb{P}_{m+3} + \mathbb{P}_{n+3}\,\mathbb{P}_m).\end{aligned} \tag{2.21}$$

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_{n+1}\,\mathbb{D}^{\mathbf{P}}{}_{m+1} = \;& \mathbb{P}_{n+1}\,\mathbb{P}_{m+1} + i\,(\mathbb{P}_{n+1}\,\mathbb{P}_{m+2} + \mathbb{P}_{n+2}\,\mathbb{P}_{m+1}) \\ & +j\,(\mathbb{P}_{n+1}\,\mathbb{P}_{m+3} + \mathbb{P}_{n+3}\,\mathbb{P}_{m+1}) \\ & +k\,(\mathbb{P}_{n+1}\,\mathbb{P}_{m+4} + \mathbb{P}_{n+4}\,\mathbb{P}_{m+1}).\end{aligned} \tag{2.22}$$

Finally, adding equations (2.21) and (2.22) side by side, we obtain

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n\,\mathbb{D}^{\mathbf{P}}{}_m + \mathbb{D}^{\mathbf{P}}{}_{n+1}\,\mathbb{D}^{\mathbf{P}}{}_{m+1} = \;& (\mathbb{P}_n\,\mathbb{P}_m + \mathbb{P}_{n+1}\,\mathbb{P}_{m+1}) \\ & +i\,[\,\mathbb{P}_n\mathbb{P}_{m+1} + \mathbb{P}_{n+1}\mathbb{P}_m + \mathbb{P}_{n+1}\mathbb{P}_{m+2} + \mathbb{P}_{n+2}\mathbb{P}_{m+1}] \\ & +j\,[\,\mathbb{P}_n\mathbb{P}_{m+2} + \mathbb{P}_{n+2}\mathbb{P}_m + \mathbb{P}_{n+1}\mathbb{P}_{m+3} + \mathbb{P}_{n+3}\mathbb{P}_{m+1}] \\ & +k\,[\,\mathbb{P}_n\mathbb{P}_{m+3} + \mathbb{P}_{n+3}\mathbb{P}_m + \mathbb{P}_{n+1}\mathbb{P}_{m+4} + \mathbb{P}_{n+4}\mathbb{P}_{m+1}] \\ \\ = \;& (2p-2q)[\,\mathbb{P}_{n+m+1} + 2i\,\mathbb{P}_{n+m+2} + 2j\,\mathbb{P}_{n+m+3} \\ & +2k\,\mathbb{P}_{n+m+4}] \\ & -e\,[\,P_{n+m+1} + 2iP_{n+m+2} + 2jP_{n+m+3} + 2kP_{n+m+4}] \\ \\ = \;& (2p-2q)[\,2\,\mathbb{D}^{\mathbf{P}}{}_{n+m+1} - \mathbb{P}_{n+m+1}] \\ & -e_P[\,2\,D^P_{n+m+1} - P_{n+m+1}]\end{aligned} \tag{2.23}$$

where D^P_{n+m+1} is the dual Pell quaternion [24]. □

Theorem 2.2. *Let $\mathbb{D}^{\mathbf{P}}{}_n$, D^P_n and D^q_n be n-th terms of the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$, the dual Pell quaternion sequence (D^P_n) and the dual Pell–Lucas quaternion sequence (D^q_n), respectively. The following relations are satisfied*

$$\begin{aligned}&\mathbb{D}^{\mathbf{P}}{}_{n+1} + \mathbb{D}^{\mathbf{P}}{}_{n-1} = p\,D^q_n + q\,D^q_{n-1}\,, \\ &\mathbb{D}^{\mathbf{P}}{}_n + \mathbb{D}^{\mathbf{P}}{}_{n+1} = \frac{p}{2}\,D^q_{n+1} + \frac{q}{2}\,D^q_n\,, \\ &\mathbb{D}^{\mathbf{P}}{}_{n+1} - \mathbb{D}^{\mathbf{P}}{}_n = \frac{p}{2}\,D^q_n + \frac{q}{2}\,D^q_{n-1}\,, \\ &\mathbb{D}^{\mathbf{P}}{}_{n+1} - \mathbb{D}^{\mathbf{P}}{}_{n-1} = 2\,[\,p\,D^P_n + q\,D^P_{n-1}]\,, \\ &\mathbb{D}^{\mathbf{P}}{}_{n+2} - \mathbb{D}^{\mathbf{P}}{}_{n-2} = 2\,[\,p\,D^q_n + q\,D^q_{n-1}]\,.\end{aligned} \tag{2.24}$$

Proof. From equations (2.16), (2.17) and identities between the generalized Pell number $\mathbb{P}_n$ [23],

$$\begin{cases} \mathbb{P}_n = (p-2q)P_n + q\,P_{n+1} = p\,P_n + q\,P_{n-1}, \\ \mathbb{P}_n + \mathbb{P}_{n+1} = \frac{p}{2}\,q_{n+1} + \frac{q}{2}\,q_n, \\ \mathbb{P}_{n+1} - \mathbb{P}_n = \frac{p}{2}\,q_n + \frac{q}{2}\,q_{n-1}, \\ \mathbb{P}_{n+1} + \mathbb{P}_{n-1} = p\,q_n + q\,q_{n-1}, \\ \mathbb{P}_{n+1} - \mathbb{P}_{n-1} = 2\,(p\,P_n + q\,P_{n-1}), \\ \mathbb{P}_{n+2} - \mathbb{P}_{n-2} = 2\,(p\,q_n + q\,q_{n-1}). \end{cases} \tag{2.25}$$

also, from the relations of between Pell and Pell–Lucas numbers as follows:

$$\begin{cases} P_{n+1} + P_{n-1} = q_n, \\ P_{n+1} - P_{n-1} = 2\,P_n, \\ P_n + P_{n+1} = \frac{1}{2}\,q_{n+1}, \\ P_{n+2} + P_{n-2} = 6\,P_n, \\ P_{n+2} - P_{n-2} = 2\,q_n. \end{cases}$$

it follows that

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_{n+1} + \mathbb{D}^{\mathbf{P}}{}_{n-1} = &\ (\mathbb{P}_{n+1} + \mathbb{P}_{n-1}) + i\,(\mathbb{P}_{n+2} + \mathbb{P}_n) + j\,(\mathbb{P}_{n+3} + \mathbb{P}_{n+1}) \\ &+k\,(\mathbb{P}_{n+4} + \mathbb{P}_{n+2}) \\ = &\ [p\,(P_{n+1} + P_{n-1}) + q\,(P_n + P_{n-2})] \\ &+i\,[p\,(P_{n+2} + P_n) + q\,(P_{n+1} + P_{n-1})] \\ &+j\,[p\,(P_{n+3} + P_{n+1}) + q\,(P_{n+2} + P_n)] \\ &+k\,[p\,(P_{n+4} + P_{n+2}) + q\,(P_{n+3} + P_{n+1})] \\ = &\ p\,(q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\ &+q\,(q_{n-1} + i\,q_n + j\,q_{n+1} + k\,q_{n+2}) \\ = &\ p\,D^q_n + q\,D^q_{n-1}\,, \end{aligned} \tag{2.26}$$

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_{n} + \mathbb{D}^{\mathbf{P}}{}_{n+1} = &\ (\mathbb{P}_n + \mathbb{P}_{n+1}) + i\,(\mathbb{P}_{n+1} + \mathbb{P}_{n+2}) + j\,(\mathbb{P}_{n+2} + \mathbb{P}_{n+3}) \\ &+k\,(\mathbb{P}_{n+3} + \mathbb{P}_{n+4}) \\ = &\ [p\,(P_n + P_{n+1}) + q\,(P_{n-1} + P_n)] \\ &+i\,[p\,(P_{n+1} + P_{n+2}) + q\,(P_n + P_{n+1})] \\ &+j\,[p\,(P_{n+2} + P_{n+3}) + q\,(P_{n+1} + P_{n+2})] \\ &+k\,[p\,(P_{n+3} + P_{n+4}) + q\,(P_{n+2} + P_{n+3})] \\ = &\ \tfrac{p}{2}\,(q_{n+1} + i\,q_{n+2} + j\,q_{n+3} + k\,q_{n+4}) \\ &+\tfrac{q}{2}\,(q_n + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\ = &\ \tfrac{p}{2}\,D^q_{n+1} + \tfrac{q}{2}\,D^q_n\,, \end{aligned} \tag{2.27}$$

$$
\begin{aligned}
\mathbb{D}^{\mathbf{P}}{}_{n+1} - \mathbb{D}^{\mathbf{P}}{}_{n} = \;& (\mathbb{P}_{n+1} - \mathbb{P}_{n}) + i\,(\mathbb{P}_{n+2} - \mathbb{P}_{n+1}) + j\,(\mathbb{P}_{n+3} - \mathbb{P}_{n+2}) \\
& + k\,(\mathbb{P}_{n+4} - \mathbb{P}_{n+3}) \\
= \;& [p\,(P_{n+1} - P_{n}) + q\,(P_{n} - P_{n-1})] \\
& + i\,[p\,(P_{n+2} - P_{n+1}) + q\,(P_{n+1} - P_{n})] \\
& + j\,[p\,(P_{n+3} - P_{n+2}) + q\,(P_{n+2} - P_{n+1})] \\
& + k\,[p\,(P_{n+4} - P_{n+3}) + q\,(P_{n+3} - P_{n+2})] \\
= \;& \tfrac{p}{2}\,(q_{n} + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\
& + \tfrac{q}{2}\,(q_{n-1} + i\,q_{n} + j\,q_{n+1} + k\,q_{n+2}) \\
= \;& \tfrac{p}{2}\,D_{n}^{q} + \tfrac{q}{2}\,D_{n-1}^{q}\,,
\end{aligned} \tag{2.28}
$$

$$
\begin{aligned}
\mathbb{D}^{\mathbf{P}}{}_{n+1} - \mathbb{D}^{\mathbf{P}}{}_{n-1} = \;& (\mathbb{P}_{n+1} - \mathbb{P}_{n-1}) + i\,(\mathbb{P}_{n+2} - \mathbb{P}_{n}) + j\,(\mathbb{P}_{n+3} - \mathbb{P}_{n+1}) \\
& + k\,(\mathbb{P}_{n+4} - \mathbb{P}_{n+2}) \\
= \;& [p\,(P_{n+1} - P_{n-1}) + q\,(P_{n} - P_{n-2})] \\
& + i\,[p\,(P_{n+2} - P_{n}) + q\,(P_{n+1} - P_{n-1})] \\
& + j\,[p\,(P_{n+3} - P_{n+1}) + q\,(P_{n+2} - P_{n})] \\
& + k\,[p\,(P_{n+4} - P_{n+2}) + q\,(P_{n+3} - P_{n+1})] \\
= \;& 2\,p\,(P_{n} + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\
& + 2\,q\,(P_{n-1} + i\,P_{n} + j\,P_{n+1} + k\,P_{n+2}) \\
= \;& 2\,[p\,D_{n}^{P} + q\,D_{n-1}^{P}]\,,
\end{aligned} \tag{2.29}
$$

and

$$
\begin{aligned}
\mathbb{D}^{\mathbf{P}}{}_{n+2} - \mathbb{D}^{\mathbf{P}}{}_{n-2} = \;& (\mathbb{P}_{n+2} - \mathbb{P}_{n-2}) + i\,(\mathbb{P}_{n+3} - \mathbb{P}_{n-1}) + j\,(\mathbb{P}_{n+4} - \mathbb{P}_{n}) \\
& + k\,(\mathbb{P}_{n+5} - \mathbb{P}_{n+1}) \\
= \;& [p\,(P_{n+2} - P_{n-2}) + q\,(P_{n+1} - P_{n-3})] \\
& + i\,[p\,(P_{n+3} - P_{n-1}) + q\,(P_{n+2} - P_{n-2})] \\
& + j\,[p\,(P_{n+4} - P_{n}) + q\,(P_{n+3} - P_{n-1})] \\
& + k\,[p\,(P_{n+5} - P_{n+1}) + q\,(P_{n+4} - P_{n})] \\
= \;& 2\,p\,(q_{n} + i\,q_{n+1} + j\,q_{n+2} + k\,q_{n+3}) \\
& + 2\,q\,(q_{n-1} + i\,q_{n} + j\,q_{n+1} + k\,q_{n+2}) \\
= \;& 2\,[p\,D_{n}^{q} + q\,D_{n-1}^{q}]\,.
\end{aligned} \tag{2.30}
$$

□

Theorem 2.3. *Let $\mathbb{D}^{\mathbf{P}}{}_{n}$ be the $n-th$ term of the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_{n})$. Then, we have the following relations between these quaternions:*

$$\mathbb{D}^{\mathbf{P}}{}_{n} + \overline{\mathbb{D}^{\mathbf{P}}{}_{n}} = 2\,\mathbb{P}_{n} \tag{2.31}$$

$$\mathbb{D}^{\mathbf{P}}{}_{n}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n}} + \mathbb{D}^{\mathbf{P}}{}_{n-1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n-1}} = (\mathbb{P}_{n})^{2} + (\mathbb{P}_{n-1})^{2} = (2p-2q)\mathbb{P}_{2n-1} - e_{P}\,P_{2n-1} \tag{2.32}$$

$$\mathbb{D}^{\mathbf{P}}{}_{n}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n}} + \mathbb{D}^{\mathbf{P}}{}_{n+1}\overline{\mathbb{D}^{\mathbf{P}}{}_{n+1}} = (\mathbb{P}_{n})^{2} + (\mathbb{P}_{n+1})^{2} = (2p-2q)\mathbb{P}_{2n+1} - e_{P}\,P_{2n+1} \tag{2.33}$$

$$\mathbb{D}^{\mathbf{P}}{}_{n+1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n+1}} - \mathbb{D}^{\mathbf{P}}{}_{n-1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n-1}} = (\mathbb{P}_{n+1})^{2} - (\mathbb{P}_{n-1})^{2} = 2\,[(2p-2q)\mathbb{P}_{2n} - e_{P}\,P_{2n}\,] \tag{2.34}$$

$$
\begin{aligned}
(\mathbb{D}^{\mathbf{P}}{}_{n})^{2} + (\mathbb{D}^{\mathbf{P}}{}_{n-1})^{2} &= 2\,\mathbb{D}^{\mathbf{P}}{}_{n}\,\mathbb{P}_{n} - (\mathbb{P}_{n})^{2} + 2\,\mathbb{D}^{\mathbf{P}}{}_{n-1}\,\mathbb{P}_{n-1} - (\mathbb{P}_{n-1})^{2} \\
&= (2p-2q)\,[\,2\,\mathbb{D}^{\mathbf{P}}{}_{2n-1} - \mathbb{P}_{2n-1}] - e_{P}\,[\,2\,D_{2n-1}^{P} - P_{2n-1}\,]
\end{aligned} \tag{2.35}
$$

where D_{2n-1}^{P} is the dual Pell quaternion [24].

Proof. (2.31): By using (2.9), we get

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n + \overline{\mathbb{D}^{\mathbf{P}}{}_n} = & \; (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ & +(\mathbb{P}_n - i\,\mathbb{P}_{n+1} - j\,\mathbb{P}_{n+2} - k\,\mathbb{P}_{n+3}\,) \\ = & \; 2\,\mathbb{P}_n + i\,(\mathbb{P}_{n+1} - \mathbb{P}_{n+1}) + j\,(\mathbb{P}_{n+2} - \mathbb{P}_{n+2}) \\ & +k\,(\mathbb{P}_{n+3} - \mathbb{P}_{n+3}) \\ = & \; 2\,\mathbb{P}_n\,. \end{aligned}$$

(2.32): By using (2.9) and (2.10), we get

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} + \mathbb{D}^{\mathbf{P}}{}_{n-1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n-1}} = & \; (\mathbb{P}_n)^2 + (\mathbb{P}_{n-1})^2 \\ = & \; (2p-2q)\mathbb{P}_{2n-1} - \,e_P\,P_{2n-1}\end{aligned}$$

(2.33): By using (2.9) and (2.10) and [23], we get

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_n\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} + \mathbb{D}^{\mathbf{P}}{}_{n+1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n+1}} = & \; (\mathbb{P}_n)^2 + (\mathbb{P}_{n+1})^2 \\ = & \; (2p-2q)\mathbb{P}_{2n+1} - \,e_P\,P_{2n+1}\end{aligned}$$

(2.34): By using (2.9) and (2.10) and [23], we get

$$\begin{aligned}\mathbb{D}^{\mathbf{P}}{}_{n+1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n+1}} - \mathbb{D}^{\mathbf{P}}{}_{n-1}\,\overline{\mathbb{D}^{\mathbf{P}}{}_{n-1}} = & \; (\mathbb{P}_{n+1})^2 - (\mathbb{P}_{n-1})^2 \\ = & \; (4p-4q)\mathbb{P}_{2n} - \,2\,e_P\,P_{2n}\end{aligned}$$

(2.35): By using (2.10) and [23], we get

$$\begin{aligned}(\mathbb{D}^{\mathbf{P}}{}_n)^2 + (\mathbb{D}^{\mathbf{P}}{}_{n-1})^2 & = [2\,\mathbb{D}^{\mathbf{P}}{}_n\,\mathbb{P}_n - (\mathbb{P}_n)^2] + [2\,\mathbb{D}^{\mathbf{P}}{}_{n-1}\,\mathbb{P}_{n-1} - (\mathbb{P}_{n-1})^2] \\ & = 2\,\mathbb{D}^{\mathbf{P}}{}_n\,\mathbb{P}_n + 2\,\mathbb{D}^{\mathbf{P}}{}_{n-1}\,\mathbb{P}_{n-1} - (\mathbb{P}_n)^2 + (\mathbb{P}_{n-1})^2 \\ & = (2p-2q)\,[\,2\,\mathbb{D}^{\mathbf{P}}{}_{2n-1} - \mathbb{P}_{2n-1}] - e_P\,[2\,D^P_{2n-1} - P_{2n-1}].\end{aligned}$$

where D^P_{2n-1} is the dual Pell quaternion [24]. □

Theorem 2.4. *Let $\mathbb{D}^{\mathbf{P}}{}_n$ be the $n-th$ term of the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$. Then, we have the following identities*

$$\sum_{s=1}^{n} \mathbb{D}^{\mathbf{P}}{}_s = \frac{1}{4}\,[p\,D^q_{n+1} + q\,D^q_n] - \frac{p}{4}\,D^q_1 - \frac{q}{4}\,D^q_0, \tag{2.36}$$

$$\sum_{s=0}^{p} \mathbb{D}^{\mathbf{P}}{}_{n+s} = \frac{p}{4}\,[D^q_{n+p+1} - D^q_n] + \frac{q}{4}\,[D^q_{n+p} - D^q_{n-1}], \tag{2.37}$$

$$\sum_{s=1}^{n} \mathbb{D}^{\mathbf{P}}{}_{2s-1} = \frac{1}{2}\,[\mathbb{D}^{\mathbf{P}}{}_{2n} - p\,D^P_0 - q\,D^P_{-1}].,\tag{2.38}$$

$$\sum_{s=1}^{n} \mathbb{D}^{\mathbf{P}}{}_{2s} = \frac{1}{2}\,[\mathbb{D}^{\mathbf{P}}{}_{2n+1} - p\,D^P_1 - q\,D^P_0]. \tag{2.39}$$

where D^P_n and D^q_n are the dual Pell quaternion and the dual Pell–Lucas quaternion respectively [24].

Proof. (2.36): Using $\sum\limits_{t=1}^{n}\mathbb{P}_t=\frac{1}{2}(\mathbb{P}_n+\mathbb{P}_{n+1}-\mathbb{P}_0-\mathbb{P}_1)$ [23], we get

$$\begin{aligned}\sum_{s=1}^{n}\mathbb{D}^{\mathbf{P}}{}_s &= \sum_{s=1}^{n}\mathbb{P}_s+i\sum_{s=1}^{n}\mathbb{P}_{s+1}+j\sum_{s=1}^{n}\mathbb{P}_{s+2}+k\sum_{s=1}^{n}\mathbb{P}_{s+3}\\
&=\frac{1}{2}[(\mathbb{P}_n+\mathbb{P}_{n+1}-p-q)+i\,(\mathbb{P}_{n+1}+\mathbb{P}_{n+2}-3\,p-q)\\
&+j\,(\mathbb{P}_{n+2}+\mathbb{P}_{n+3}-7p-3q)+k\,(\mathbb{P}_{n+3}+\mathbb{P}_{n+4}-17p-7q)]\\
&=\frac{1}{2}(\mathbb{P}_n+i\,\mathbb{P}_{n+1}+j\,\mathbb{P}_{n+2}+k\,\mathbb{P}_{n+3})\\
&+\frac{1}{2}(\mathbb{P}_{n+1}+i\,\mathbb{P}_{n+2}+j\,\mathbb{P}_{n+3}+k\,\mathbb{P}_{n+4})\\
&-\frac{p}{2}(1+3i+7j+17k)-\frac{q}{2}(1+i+3j+7k)\\
&=\frac{1}{2}\,[\mathbb{D}^{\mathbf{P}}{}_n+\mathbb{D}^{\mathbf{P}}{}_{n+1}]-\frac{p}{4}\,D_1^q-\frac{q}{4}\,D_0^q\\
&=\frac{1}{4}\,[p\,D_{n+1}^q+q\,D_n^q]-\frac{p}{4}\,D_1^q-\frac{q}{4}\,D_0^q\,.\end{aligned}$$

(2.37): Hence, we can write

$$\begin{aligned}\sum_{s=0}^{p}\mathbb{D}^{\mathbf{P}}{}_{n+s} &= \sum_{s=0}^{p}\mathbb{P}_{n+s}+i\sum_{s=0}^{p}\mathbb{P}_{n+s+1}+j\sum_{s=0}^{p}\mathbb{P}_{n+s+2}+k\sum_{s=0}^{p}\mathbb{P}_{n+s+3}\\
&=\tfrac{1}{2}[(\mathbb{P}_{n+p}+\mathbb{P}_{n+p+1}-\mathbb{P}_1-\mathbb{P}_0)+i\,(\mathbb{P}_{n+p+1}-\mathbb{P}_{n+p+2}-\mathbb{P}_2-\mathbb{P}_1)\\
&+j\,(\mathbb{P}_{n+p+2}+\mathbb{P}_{n+p+3}-\mathbb{P}_3-\mathbb{P}_2)+k\,(\mathbb{P}_{n+p+3}+\mathbb{P}_{n+p+4}-\mathbb{P}_4-\mathbb{P}_3)]\\
&=\tfrac{1}{2}(\mathbb{P}_{n+p}+i\,\mathbb{P}_{n+p+1}+j\,\mathbb{P}_{n+p+2}+k\,\mathbb{P}_{n+p+3})\\
&+\tfrac{1}{2}(\mathbb{P}_{n+p+1}+i\,\mathbb{P}_{n+p+2}+j\,\mathbb{P}_{n+p+3}+k\,\mathbb{P}_{n+p+4})\\
&-\tfrac{p}{2}(1+3i+7j+17k)-\tfrac{q}{2}(1+i+3j+7k)\\
&=\tfrac{1}{2}\,[\mathbb{D}^{\mathbf{P}}{}_{n+p}+\mathbb{D}^{\mathbf{P}}{}_{n+p+1}]-\tfrac{p}{4}\,D_n^q-\tfrac{q}{4}\,D_{n-1}^q\\
&=\tfrac{p}{4}\,[D_{n+p+1}^q-D_n^q]+\tfrac{q}{4}\,[D_{n+p}^q-\,D_{n-1}^q]\,.\end{aligned}$$

(2.38): Using $\sum\limits_{i=1}^{n}\mathbb{P}_{2i-1}=\frac{1}{2}(\mathbb{P}_{2n}-q)$ and $\sum\limits_{i=1}^{n}\mathbb{P}_{2i}=\frac{1}{2}(\mathbb{P}_{2n+1}-p)$ [23], we get

$$\begin{aligned}\sum_{s=1}^{n}\mathbb{D}^{\mathbf{P}}{}_{2s-1} &= \tfrac{1}{2}\,[(\mathbb{P}_{2n}-q)+i\,(\mathbb{P}_{2n+1}-p)+j\,(\mathbb{P}_{2n+2}-q-2p)\\
&\quad+k\,(\mathbb{P}_{2n+3}-2q-5p)]\\
&= \tfrac{1}{2}\,[\mathbb{P}_{2n}+i\,\mathbb{P}_{2n+1}+j\,\mathbb{P}_{2n+2}+k\,\mathbb{P}_{2n+3}]\\
&\quad-\tfrac{1}{2}\,[q+ip+j(2p+q)+k(5p+2q)]\\
&= \tfrac{1}{2}\,\mathbb{D}^{\mathbf{P}}{}_{2n}-p\,(0+i+2j+5k)-q\,(1+0i+j+2k)]\\
&= \tfrac{1}{2}\,[\mathbb{D}^{\mathbf{P}}{}_{2n}-p\,D_0^P-q\,D_{-1}^P]\,.\end{aligned}$$

(2.39): Using $\sum\limits_{i=1}^{n} \mathbb{P}_{2i} = \frac{1}{2}(\mathbb{P}_{2n+1} - p)$ [23], we obtain

$$\begin{aligned}\sum_{s=1}^{n} \mathbb{D}^{\mathbf{P}}{}_{2s} = & \tfrac{1}{2}\left[(\mathbb{P}_{2n+1} - p) + i\,(\mathbb{P}_{2n+2} - 2p - q)\right. \\ & \left.+j\,(\mathbb{P}_{2n+3} - 5p - 2q) + k\,(\mathbb{P}_{2n+4} - 12p - 5q)\right] \\ = & \tfrac{1}{2}\left[\mathbb{P}_{2n+1} + i\,\mathbb{P}_{2n+2} + j\,\mathbb{P}_{2n+3} + k\,\mathbb{P}_{2n+4}\right] \\ & -\tfrac{p}{2}\left[1 + 2i + 5j + 12k\right] - \tfrac{q}{2}\left[0 + i + 2j + 5k\right] \\ = & \tfrac{1}{2}\left[\mathbb{D}^{\mathbf{P}}{}_{2n+1} - p\,D_1^P - q\,D_0^P\right].\end{aligned}$$

□

Theorem 2.5. *Let $\mathbb{D}^{\mathbf{P}}{}_n$ and D_n^P be the $n-th$ terms of the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$ and the dual Pell quaternion sequence (D_n^P), respectively. Then, we have*

$$D_n^P\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} - \overline{D_n^P}\,\mathbb{D}^{\mathbf{P}}{}_n = 2\left[\mathbb{P}_n D_n^P - P_n \mathbb{D}^{\mathbf{P}}_{\mathbf{n}}\right] \tag{2.40}$$

$$D_n^P\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} + \overline{D_n^P}\,\mathbb{D}^{\mathbf{P}}{}_n = 2\,P_n\,\mathbb{P}_n \tag{2.41}$$

$$D_n^P\,\mathbb{D}^{\mathbf{P}}{}_n - \overline{D_n^P}\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} = 2\left[P_n\,\mathbb{D}^{\mathbf{P}}{}_n + \mathbb{P}_n D_n^P - 2\,P_n\,\mathbb{P}_n\right] \tag{2.42}$$

Proof. (2.40): By using (2.3) and (2.9), we get

$$\begin{aligned}D_n^P\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} - \overline{D_n^P}\,\mathbb{D}^{\mathbf{P}}{}_n = & (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ & (\mathbb{P}_n - i\,\mathbb{P}_{n+1} - j\,\mathbb{P}_{n+2} - k\,\mathbb{P}_{n+3}) \\ & -(P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}) \\ & (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ = & (P_n\mathbb{P}_n - P_n\mathbb{P}_n) + 2i\,(-P_n\mathbb{P}_{n+1} + P_{n+1}\mathbb{P}_n) \\ & +2j\,(-P_n\mathbb{P}_{n+2} + P_{n+2}\mathbb{P}_n) \\ & +2k\,(-P_n\mathbb{P}_{n+3} + P_{n+3}\mathbb{P}_n) \\ = & -2\,P_n[\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}] \\ & +2\,\mathbb{P}_n[P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}] \\ = & 2\left[\mathbb{P}_n D_n^P - P_n\mathbb{D}^{\mathbf{P}}{}_n\right].\end{aligned}$$

(2.41): By using (2.3) and (2.9), we get

$$\begin{aligned}D_n^P\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} + \overline{D_n^P}\,\mathbb{D}^{\mathbf{P}}{}_n = & (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ & (\mathbb{P}_n - i\,\mathbb{P}_{n+1} - j\,\mathbb{P}_{n+2} - k\,\mathbb{P}_{n+3}) \\ & +(P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}) \\ & (\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ = & (P_n\mathbb{P}_n + P_n\mathbb{P}_n) \\ & +i\,(-P_n\mathbb{P}_{n+1} + P_{n+1}\mathbb{P}_n + P_n\mathbb{P}_{n+1} - P_{n+1}\mathbb{P}_n) \\ & +j\,(-P_n\mathbb{P}_{n+2} + P_{n+2}\mathbb{P}_n + P_n\mathbb{P}_{n+2} - P_{n+2}\mathbb{P}_n) \\ & +k\,(-P_n\mathbb{P}_{n+3} + P_{n+3}\mathbb{P}_n + P_n\mathbb{P}_{n+3} - P_{n+3}\mathbb{P}_n) \\ = & 2\,P_n\mathbb{P}_n.\end{aligned}$$

(2.42): By using (2.3) and (2.9), we get

$$\begin{aligned} D_n^P\,\mathbb{D}^{\mathbf{P}}{}_n - \overline{D_n^P}\,\overline{\mathbb{D}^{\mathbf{P}}{}_n} = &\ (P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &(\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ &-(P_n - i\,P_{n+1} - j\,P_{n+2} - k\,P_{n+3}) \\ &(\mathbb{P}_n - i\,\mathbb{P}_{n+1} - j\,\mathbb{P}_{n+2} - k\,\mathbb{P}_{n+3}) \\ = &\ (P_n\mathbb{P}_n - P_n\mathbb{P}_n) + i\,(2\,P_n\mathbb{P}_{n+1} + 2\,P_{n+1}\mathbb{P}_n) \\ &+j\,(2\,P_n\mathbb{P}_{n+2} + 2\,P_{n+2}\mathbb{P}_n) \\ &+k\,(2\,P_n\mathbb{P}_{n+3} + 2\,P_{n+3}\mathbb{P}_n) \\ = &\ 2P_n(\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}) \\ &+2\mathbb{P}_n(P_n + i\,P_{n+1} + j\,P_{n+2} + k\,P_{n+3}) \\ &-4\mathbb{P}_n\,P_n \\ = &\ 2\,[P_n\mathbb{D}^{\mathbf{P}}_{\mathbf{n}} + \mathbb{P}_n D_n^P - 2\,P_n\mathbb{P}_n]. \end{aligned}$$

□

Theorem 2.6 (Binet's Formulas). *Let $\mathbb{D}^{\mathbf{P}}{}_n$ and $\mathbb{D}^{\mathbf{q}}{}_n$ be $n-th$ terms of the generalized dual Pell quaternion sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$ and the generalized dual Pell–Lucas quaternion sequence $(\mathbb{D}^{\mathbf{q}}{}_n)$ respectively. For $n \geq 1$, the Binet's formulas for these quaternions are as follows:*

$$\mathbb{D}^{\mathbf{P}}{}_n = \frac{1}{\alpha - \beta}\left(\hat{\alpha}\ \alpha^n - \hat{\beta}\ \beta^n\right) \tag{2.43}$$

and

$$\mathbb{D}^{\mathbf{q}}{}_n = (\overline{\alpha}\ \alpha^n + \overline{\beta}\,\beta^n) \tag{2.44}$$

respectively, where

$$\begin{aligned} \hat{\alpha} = &\ (p - q\beta) + i\,[p\,(2-\beta) + q] + j\,[p\,(5-2\beta) + q(2-\beta)] \\ &+k\,[p\,(12-5\beta) + q\,(5-2\beta)], \quad \alpha = 1+\sqrt{2}, \end{aligned}$$

$$\begin{aligned} \hat{\beta} = &\ (q\alpha - p) + i\,[p\,(\alpha-2) - q] + j\,[p\,(2\alpha-5) + q(\alpha-2)] \\ &+k\,[(p\,(5\alpha-12) + q\,(2\alpha-5)], \quad \beta = 1-\sqrt{2}. \end{aligned}$$

and

$$\begin{aligned} \overline{\alpha} = &\ [p\,(2-2\beta) + q(2+2\beta)] + i\,[p\,(6-2\beta) + q\,(2-2\beta)] \\ &+j\,[p\,(14-6\beta) + q(6-2\beta)] + k\,[p\,(34-14\beta) + q\,(14-6\beta)], \quad \alpha = 1+\sqrt{2}, \end{aligned}$$

$$\begin{aligned} \overline{\beta} = &\ [p\,(2\alpha-2) - q\,(2\alpha+2)] + i\,[p\,(2\alpha-6) + q\,(2\alpha-2)] \\ &+j\,[p\,(6\alpha-14) + q(2\alpha-6)] + k\,[(p\,(14\alpha-34) + q\,(6\alpha-14)], \quad \beta = 1-\sqrt{2}. \end{aligned}$$

respectively.

Proof. The Binet's formulas for Pell sequence, generalized Pell sequence and dual Pell quaternion sequence respectively, are as follows

$P_n = \frac{1}{2\sqrt{2}}\left(\alpha^n - \beta^n\right), \mathbb{P}_n = \frac{1}{2\sqrt{2}}\left(l\,\alpha^n - m\,\beta^n\right)$ and $D_n^P = \frac{1}{2\sqrt{2}}\left(\underline{\alpha}\alpha^n - \underline{\beta}\beta^n\right)$ [3],[23],[24].

Using the recurrence relations for generalized dual Pell number and generalized dual Pell quaternion $\mathbb{D}^{\mathbf{P}}{}_n$ respectively, $\mathbb{P}_{n+2} = 2\mathbb{P}_{n+1} + \mathbb{P}_n$, $\mathbb{D}^{\mathbf{P}}{}_{n+2} = 2\mathbb{D}^{\mathbf{P}}{}_{n+1} + \mathbb{D}^{\mathbf{P}}{}_n$, we can write the characteristic equation as follows:

$$t^2 - 2t - 1 = 0.$$

The roots of this equation are

$$\alpha = 1 + \sqrt{2} \quad \text{and} \quad \beta = 1 - \sqrt{2},$$

where $\alpha + \beta = 2$, $\alpha - \beta = 2\sqrt{2}$, $\alpha\beta = -1$.

Using recurrence relation and initial values $\mathbb{D}^{\mathbf{p}}{}_0 = (q,\, p,\, 2p+q,\, 5p+2q)$, $\mathbb{D}^{\mathbf{p}}{}_1 = (p,\, 2p+q,\, 5p+2q,\, 12p+5q)$, the Binet's formula for $\mathbb{D}^{\mathbf{p}}_{\mathbf{n}}$ is

$$\mathbb{D}^{\mathbf{p}}{}_n = A\,\alpha^n + B\,\beta^n = \frac{1}{2\sqrt{2}}\left[\,\hat{\alpha}\ \alpha^n - \hat{\beta}\ \beta^n\right],$$

where $A = \frac{\mathbb{D}^{\mathbf{p}}_{\mathbf{1}} - \mathbb{D}^{\mathbf{p}}_{\mathbf{0}}\,\beta}{\alpha-\beta}$, $B = \frac{\alpha\,\mathbb{D}^{\mathbf{p}}_{\mathbf{0}} - \mathbb{D}^{\mathbf{p}}_{\mathbf{1}}}{\alpha-\beta}$ and

$$\hat{\alpha} = \ (p - q\beta) + i\,[p\,(2-\beta) + q] + j\,[p\,(5-2\beta) + q(2-\beta)] + k\,[(12-5\beta) + q\,(5-2\beta)],$$

$$\hat{\beta} = \ (q\alpha - p) + i\,[p\,(\alpha-2) - q] + j\,[p\,(2\alpha-5) + q\,(\alpha-2)] + k\,[p\,(5\alpha-12) + q\,(2\alpha-5)].$$

Similarly, using recurrence relation $\mathbb{D}^{\mathbf{q}}{}_{n+2} = 2\mathbb{D}^{\mathbf{q}}{}_{n+1} + \mathbb{D}^{\mathbf{q}}{}_n$, the Binet's formula for generalized Pell–Lucas quaternion $\mathbb{D}^{\mathbf{q}}{}_n$ is obtained as follows:

$$\mathbb{D}^{\mathbf{q}}{}_n = (\overline{\alpha}\ \alpha^n + \overline{\beta}\ \beta^n) \tag{2.45}$$

where initial values

$$\mathbb{D}^{\mathbf{q}}{}_0 = (2p-2q,\, 2p+2q,\, 6p+2q,\, 14p+6q),$$
$$\mathbb{D}^{\mathbf{q}}{}_1 = (2p+2q,\, 6p+2q,\, 14p+6q,\, 34p+14q).$$ □

Theorem 2.7 (Cassini-like Identity). *Let $\mathbb{D}^{\mathbf{P}}{}_n$ and $\mathbb{D}^{\mathbf{q}}{}_n$ be $n-th$ terms of the generalized dual Pell sequence $(\mathbb{D}^{\mathbf{P}}{}_n)$ and the generalized dual Pell–Lucas sequence $(\mathbb{D}^{\mathbf{p}}{}_n)$ respectively. For $n \geq 1$, the Cassini-like identity for $\mathbb{D}^{\mathbf{P}}{}_n$ and $\mathbb{D}^{\mathbf{p}}{}_n$ are as follows:*

$$\mathbb{D}^{\mathbf{P}}{}_{n-1}\,\mathbb{D}^{\mathbf{P}}{}_{n+1} - (\mathbb{D}^{\mathbf{P}}{}_n)^2 = (-1)^n\,e_P\,(1 + 2i + 6j + 14k) \tag{2.46}$$

and

$$\mathbb{D}^{\mathbf{q}}{}_{n-1}\,\mathbb{D}^{\mathbf{q}}{}_{n+1} - (\mathbb{D}^{\mathbf{q}}{}_n)^2 = 8(-1)^{n+1}\,e_q\,(1 + 2i + 6j + 14k) \tag{2.47}$$

where

$$e_P = e_q = p^2 - 2\,p\,q - q^2.$$

Proof. (2.46): By using (2.16) and (2.17) we get

$$\begin{aligned}
\mathbb{D}^{\mathbf{P}}{}_{n-1}\,\mathbb{D}^{\mathbf{P}}{}_{n+1} - (\mathbb{D}^{\mathbf{P}}{}_n)^2 = \ & (\mathbb{P}_{n-1} + i\,\mathbb{P}_n + j\,\mathbb{P}_{n+1} + k\,\mathbb{P}_{n+2}) \\
& (\mathbb{P}_{n+1} + i\,\mathbb{P}_{n+2} + j\,\mathbb{P}_{n+3} + k\,\mathbb{P}_{n+4}) \\
& -(\,\mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}\,)^2 \\
= \ & [\mathbb{P}_{n-1}\mathbb{P}_{n+1} - (\mathbb{P}_n)^2] \\
& +i\,[\mathbb{P}_{n-1}\mathbb{P}_{n+2} + \mathbb{P}_n\mathbb{P}_{n+1} - 2\,\mathbb{P}_n\mathbb{P}_{n+1}] \\
& +j\,[\mathbb{P}_{n-1}\mathbb{P}_{n+3} - 2\,\mathbb{P}_n\mathbb{P}_{n+2} + (\mathbb{P}_{n+1})^2] \\
& +k\,[\mathbb{P}_{n-1}\mathbb{P}_{n+4} + \mathbb{P}_{n+1}\mathbb{P}_{n+2} - 2\,\mathbb{P}_n\mathbb{P}_{n+3}] \\
= \ & (-1)^n\,e_P\,(1 + 2i + 6j + 14k).
\end{aligned}$$

where we use identity of the Pell number $P_m P_{n+1} - P_{m+1} P_n = (-1)^n P_{m-n}$ and identities of the generalized Pell numbers as follows:

$$\mathbb{P}_{n+1}\mathbb{P}_{n-1} - (\mathbb{P}_n)^2 = (-1)^n\, e_P, \tag{2.48}$$

$$\mathbb{P}_{n+2}\mathbb{P}_{n-1} - \mathbb{P}_n\mathbb{P}_{n+1} \quad = 2(-1)^n\, e_P, \tag{2.49}$$

$$\mathbb{P}_{n+3}\mathbb{P}_{n-1} + \mathbb{P}_{n+1}\mathbb{P}_{n+1} - 2\,\mathbb{P}_n\mathbb{P}_{n+2} \quad = 6\,(-1)^n\, e_P, \tag{2.50}$$

$$\mathbb{P}_{n+4}\mathbb{P}_{n-1} + \mathbb{P}_{n+2}\mathbb{P}_{n+1} - 2\,\mathbb{P}_n\mathbb{P}_{n+3} \quad = 14\,(-1)^n\, e_P, \tag{2.51}$$

$$e_P = p^2 - 2p\,q - q^2.$$

Let the generalized Pell–Lucas sequence $(\mathfrak{q}_n)$ be defined as follows:

$$\begin{cases} \mathfrak{q}_0 = & 2p - 2q,\ \mathfrak{q}_1 = 2p + 2q,\ \mathfrak{q}_2 = 6p + 2q,\ \ p\,q \in \mathbb{Z} \\ \mathfrak{q}_n = & 2\mathfrak{q}_{n-1} + \ \mathfrak{q}_{n-2},\ \ n \geq 2 \\ \quad or & \\ \mathfrak{q}_n = & (p - 2q)\, q_n + q\, q_{n+1} = p\, q_n + q\, q_{n-1}. \end{cases} \tag{2.52}$$

Here, $\mathfrak{q}_n$ is the n-th generalized Pell–Lucas number that defined as follows:

$$(\mathfrak{q}_n):\ 2\,p - 2\,q\,,\ 2\,p + 2\,q\,,\ 6\,p + 2\,q\,,\ 14\,p + 6\,q\,,\ 34\,p + 14\,q\,,\ ,\ \ldots\,, p\,q_n + q\,q_{n-1}, \ldots \tag{2.53}$$

and let the generalized dual Pell–Lucas quaternion be defined as follows:

$$\{\, \mathbb{D}^{\mathfrak{q}}{}_n = \mathfrak{q}_n + i\,\mathfrak{q}_{n+1} + j\ \mathfrak{q}_{n+2} + k\,\mathfrak{q}_{n+3}\ \mid \mathfrak{q}_n,\ n\text{-th gen. Pell–Lucas number}\,\} \tag{2.54}$$

where

$$i^2 = j^2 = k^2 = i\,j\,k = 0\,,\ \ i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0.$$

(2.47): By using (2.53) and (2.54) we get

$$\begin{aligned} \mathbb{D}^{\mathfrak{q}}{}_{n-1}\,\mathbb{D}^{\mathfrak{q}}{}_{n+1} \ - (\mathbb{D}^{\mathfrak{q}}{}_n)^2 = &\ \ (\mathfrak{q}_{n-1} + i\,\mathfrak{q}_n + j\,\mathfrak{q}_{n+1} + k\,\mathfrak{q}_{n+2}) \\ &\ \ (\mathfrak{q}_{n+1} + i\,\mathfrak{q}_{n+2} + j\,\mathfrak{q}_{n+3} + k\,\mathfrak{q}_{n+4}) \\ &\ \ -(\,\mathfrak{q}_n + i\,\mathfrak{q}_{n+1} + j\,\mathfrak{q}_{n+2} + k\,\mathfrak{q}_{n+3}\,)^2 \\ = &\ \ [\,\mathfrak{q}_{n-1}\,\mathfrak{q}_{n+1} - (\mathfrak{q}_n)^2] \\ &\ \ +i\,[\,\mathfrak{q}_{n-1}\,\mathfrak{q}_{n+2} + \mathfrak{q}_n\,\mathfrak{q}_{n+1} - 2\,\mathfrak{q}_n\,\mathfrak{q}_{n+1}] \\ &\ \ +j\,[\,\mathfrak{q}_{n-1}\,\mathfrak{q}_{n+3} - 2\,\mathfrak{q}_n\,\mathfrak{q}_{n+2} + (\mathfrak{q}_{n+1})^2\,] \\ &\ \ +k\,[\,\mathfrak{q}_{n-1}\,\mathfrak{q}_{n+4} + \mathfrak{q}_{n+1}\,\mathfrak{q}_{n+2} - 2\,\mathfrak{q}_n\,\mathfrak{q}_{n+3}] \\ = &\ \ 8\,(-1)^{n+1}\,e_q\,(1 + 2i + 6j + 14k). \end{aligned}$$

where we use identity of the Pell–Lucas number $q_{n-1}\,q_{n+1} - q_n\,q_n = 8\,(-1)^{n+1}$ and identities of the generalized Pell–Lucas numbers as follows:

$$\mathfrak{q}_{n+1}\,\mathfrak{q}_{n-1} - (\mathfrak{q}_n)^2 = 8\,(-1)^{n+1}\,e_q, \tag{2.55}$$

$$\mathfrak{q}_{n+2}\,\mathfrak{q}_{n-1} - \mathfrak{q}_n\,\mathfrak{q}_{n+1} \quad = 16(-1)^{n+1}\,e_q, \tag{2.56}$$

$$\mathfrak{q}_{n+3}\,\mathfrak{q}_{n-1}+\mathfrak{q}_{n+1}\,\mathfrak{q}_{n+1}-2\,\mathfrak{q}_n\,\mathfrak{q}_{n+2} \;=48\,(-1)^{n+1}\,e_q, \tag{2.57}$$

$$\mathfrak{q}_{n+4}\,\mathfrak{q}_{n-1}+\mathfrak{q}_{n+2}\,\mathfrak{q}_{n+1}-2\,\mathfrak{q}_n\,\mathfrak{q}_{n+3} \;=112\,(-1)^{n+1}\,e_q, \tag{2.58}$$

$$e_q=p^2-2p\,q-q^2.$$

Special Case: From the equations (2.46) and (2.47) for $p=1,\ q=0$ and $e_P=e_q=1$, we obtain all results in [24] as a special case as follows:

$$D^P_{n-1}\,D^P_{n+1}-(D^P_n)^2=(-1)^n\,(1+2i+6j+14k) \tag{2.59}$$

and

$$D^q_{n-1}\,D^q_{n+1}-(D^q_n)^2=8(-1)^{n+1}\,(1+2i+6j+14k). \tag{2.60}$$

We will give an example in which we check in a particular case the Cassini-like identity for the generalized dual Pell quaternions. □

Example 1. Let $\mathbb{D}^{\mathbf{P}}{}_1$, $\mathbb{D}^{\mathbf{P}}{}_2$, $\mathbb{D}^{\mathbf{P}}{}_3$ and $\mathbb{D}^{\mathbf{P}}{}_4$ be the generalized dual Pell quaternions such that

$$\begin{cases} \mathbb{D}^{\mathbf{P}}{}_1=p+i(2p+q)+j(5p+2q)+k(12p+5q) \\ \mathbb{D}^{\mathbf{P}}{}_2=(2p+q)+i(5p+2q)+j(12p+5q)+k(29p+12q) \\ \mathbb{D}^{\mathbf{P}}{}_3=(5p+2q)+i(12p+5q)+j(29p+12q)+k(70p+29q) \\ \mathbb{D}^{\mathbf{P}}{}_4=(12p+5q)+i(29p+12q)+j(70p+29q)+k(169p+70q). \end{cases}$$

In this case,

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_1\,\mathbb{D}^{\mathbf{P}}{}_3\,-(\mathbb{D}^{\mathbf{P}}{}_2)^2 =\; & [p+i\,(2p+q)+j\,(5p+2q)+k\,(12p+5q)] \\ & [(5p+2q)+i\,(12p+5q)+j\,(29p+12q)+k\,(70p+29q)] \\ & -[(2p+q)+i\,(5p+2q)+j\,(12p+5q)+k\,(29p+12q)]^2 \\ =\; & (p^2-2p\,q-q^2)+i\,(2p^2-4p\,q-2q^2) \\ & +j\,(6p^2-12\,p\,q-6\,q^2)+k\,(14p^2-28\,p\,q-14\,q^2) \\ =\; & (p^2-2p\,q-q^2)(1+2i+6j+14k) \\ =\; & (-1)^2\,e_P\,(1+2i+6j+14k) \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}^{\mathbf{P}}{}_2\,\mathbb{D}^{\mathbf{P}}{}_4\,-(\mathbb{D}^{\mathbf{P}}{}_3)^2 =\; & [(2p+q)+i\,(5p+2q)\,+j\,(12p+5q)+k\,(29p+12q)] \\ & [(12p+5q)+i\,(29p+12q)+j\,(70p+29q)+k\,(169p+70q)] \\ & -[(5p+2q)+i\,(12p+5q)+j\,(29p+12q)+k\,(70p+29q)]^2 \\ =\; & (-p^2+2p\,q+q^2)+i\,(-2p^2+4p\,q+2q^2) \\ & +j\,(-6p^2+12\,p\,q+6\,q^2)+k\,(-14p^2+28\,p\,q+14\,q^2) \\ =\; & -(p^2-2p\,q-q^2)(1+2i+6j+14k) \\ =\; & (-1)^3\,e_P\,(1+2i+6j+14k). \end{aligned}$$

Example 2. Let $\mathbb{D}^{\mathbf{q}}{}_1$, $\mathbb{D}^{\mathbf{q}}{}_2$, $\mathbb{D}^{\mathbf{q}}{}_3$ and $\mathbb{D}^{\mathbf{q}}{}_4$ be the generalized dual Pell–Lucas quaternions such that

$$\begin{cases} \mathbb{D}^{\mathbf{q}}{}_1 = (2p+2q) + i(6p+2q) + j(14p+6q) + k(34p+14q) \\ \mathbb{D}^{\mathbf{q}}{}_2 = (6p+2q) + i(14p+6q) + j(34p+14q) + k(82p+34q) \\ \mathbb{D}^{\mathbf{q}}{}_3 = (14p+6q) + i(34p+14q) + j(82p+34q) + k(198p+82q) \\ \mathbb{D}^{\mathbf{q}}{}_4 = (34p+14q) + i(82p+34q) + j(198p+82q) + k(478p+198q). \end{cases}$$

In this case,

$$\begin{aligned} \mathbb{D}^{\mathbf{q}}{}_1\,\mathbb{D}^{\mathbf{q}}{}_3 - (\mathbb{D}^{\mathbf{q}}{}_2)^2 = & \; [(2p+2q) + i\,(6p+2q) + j\,(14p+6q) + k\,(34p+14q)] \\ & [(14p+6q) + i\,(34p+14q) + j\,(82p+34q) \\ & +k\,(198p+82q)] \\ & -[(6p+2q) + i\,(14p+6q) + j\,(34p+14q) \\ & +k\,(82p+34q)]^2 \\ = & \; -(8\,p^2 - 16\,p\,q - 8\,q^2) - i\,(16p^2 - 32p\,q - 16q^2) \\ & -j\,(48p^2 - 160\,p\,q - 48\,q^2) - k\,(112p^2 - 224\,p\,q - 112\,q^2) \\ = & \; -8(p^2 - 2p\,q - q^2)(1 + 2i + 6j + 14k) \\ = & \; 8\,(-1)^3\,e_q\,(1 + 2i + 6j + 14k) \end{aligned}$$

and

$$\begin{aligned} \mathbb{D}^{\mathbf{q}}{}_2\,\mathbb{D}^{\mathbf{q}}{}_4 - (\mathbb{D}^{\mathbf{q}}{}_3)^2 = & \; [(6p+2q) + i\,(14p+6q) + j\,(34p+14q) + k\,(82p+34q)] \\ & [(34p+14q) + i\,(82p+34q) + j\,(198p+82q) \\ & +k\,(478p+198q)] \\ & -[(14p+6q) + i\,(34p+14q) + j\,(82p+34q) \\ & +k\,(198p+82q)]^2 \\ = & \; 8\,(p^2 - 2p\,q - q^2) + 16\,i\,(p^2 - 2p\,q - q^2) \\ & +48\,j\,(p^2 - 2\,p\,q - q^2) + 112\,k\,(p^2 - 2\,p\,q - q^2) \\ = & \; 8\,(p^2 - 2p\,q - q^2)(1 + 2i + 6j + 14k) \\ = & \; 8\,(-1)^4\,e_q\,(1 + 2i + 6j + 14k). \end{aligned}$$

3 Conclusion

The generalized dual Pell quaternions is given by

$$\mathbb{D}^{\mathbf{P}}{}_n = \mathbb{P}_n + i\,\mathbb{P}_{n+1} + j\,\mathbb{P}_{n+2} + k\,\mathbb{P}_{n+3}, \tag{3.1}$$

where $\mathbb{P}_n$ is the n-th generalized Pell number and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = 0\,, \quad i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\,.$$

The generalized dual Pell–Lucas quaternions is given by

$$\mathbb{D}^{\mathfrak{q}}{}_n = \mathfrak{q}_n + i\,\mathfrak{q}_{n+1} + j\,\mathfrak{q}_{n+2} + k\,\mathfrak{q}_{n+3}, \tag{3.2}$$

where $\mathfrak{q}_n$ is the n-th generalized Pell–Lucas number and i, j, k are quaternionic units which satisfy the equalities

$$i^2 = j^2 = k^2 = 0\,, \;\; i\,j = -j\,i = j\,k = -k\,j = k\,i = -i\,k = 0\,.$$

Also, from the generalized dual Pell quaternions and the generalized dual Pell–Lucas quaternions for $p = 1,\ q = 0$, we obtain results of the dual Pell quaternions and the dual Pell–Lucas quaternions given by Torunbalcı Aydın and Yüce [24] as a special case.

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