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# Some properties of the bi-periodic Horadam sequences

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**Abstract:** In this paper, we give some basic properties of the bi-periodic Horadam sequences which generalize the known results for the bi-periodic Fibonacci and Lucas sequences. Also, we obtain some new identities for the bi-periodic Lucas sequences.
**Keywords:** Horadam sequence, Conditional sequence, Bi-periodic Fibonacci sequence.
**AMS Classification:** 11B39, 05A15.

## 1 Introduction

The Horadam sequence $\{W_n\}$ is defined by Horadam [4] as:

$$W_n = pW_{n-1} - qW_{n-2}, \quad n \geq 2 \tag{1.1}$$

with initial conditions $W_0, W_1$ where $W_0, W_1, p, q$ are arbitrary integers. It has considered a generalization of the Fibonacci and Lucas sequences. In particular, if we take $q = -1, W_0 = 0, W_1 = 1$ we obtain the generalized Fibonacci sequence $\{u_n\}$ and if we take $q = -1, W_0 = 2, W_1 = p$ we obtain the generalized Lucas sequence $\{v_n\}$.

Another generalization of the Fibonacci and Lucas sequence, named as the bi-periodic Fibonacci sequence $\{q_n\}$ is defined by

$$q_n = \begin{cases} aq_{n-1} + q_{n-2}, & \text{if } n \text{ is even} \\ bq_{n-1} + q_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1.2}$$

with initial values $q_0 = 0$, $q_1 = 1$ and $a$, $b$ are nonzero numbers (see [3]) and the bi-periodic Lucas sequence $\{p_n\}$ is defined by

$$p_n = \begin{cases} bp_{n-1} + p_{n-2}, & \text{if } n \text{ is even} \\ ap_{n-1} + p_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1.3}$$

with the initial conditions $p_0 = 2$, $p_1 = a$ (see [1]). If we take $a = b = 1$ in $\{q_n\}$, we get the classical Fibonacci sequence and if we take $a = b = 1$ in $\{p_n\}$, we get the classical Lucas sequence. The Binet formulas of the sequences $\{q_n\}$ and $\{p_n\}$ are given by

$$q_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left( \frac{\alpha^n - \beta^n}{\alpha - \beta} \right) \tag{1.4}$$

and

$$p_n = \frac{a^{\zeta(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left( \alpha^n + \beta^n \right), \tag{1.5}$$

respectively, where $\alpha = \frac{ab+\sqrt{a^2b^2+4ab}}{2}$ and $\beta = \frac{ab-\sqrt{a^2b^2+4ab}}{2}$ that is, $\alpha$ and $\beta$ are the roots of the polynomial $x^2 - abx - ab$ and $\zeta(n) = n - 2\left\lfloor \frac{n}{2} \right\rfloor$ is the parity function, i.e., $\zeta(n) = 0$ when $n$ is even and $\zeta(n) = 1$ when $n$ is odd. Let $a^2b^2+4ab \neq 0$. Note that $\alpha+\beta = ab$, $\alpha-\beta = \sqrt{a^2b^2+4ab}$ and $\alpha\beta = -ab$.

A further generalization introduced by Sahin [6] as a Fibonacci conditional sequence $\{f_n\}$:

$$f_n = \begin{cases} af_{n-1} + cf_{n-2}, & \text{if } n \text{ is even} \\ bf_{n-1} + df_{n-2}, & \text{if } n \text{ is odd} \end{cases}, \ n \geq 2 \tag{1.6}$$

with initial conditions $f_0 = 0, f_1 = 1$ where $a, b, c, d$ are nonzero numbers. By taking initial conditions 2 and $b$, authors gave some properties of the Lucas conditional sequence $\{l_n\}$ in [8]. It should be noted that more general case of these sequences can be found in [5] and more results related to these sequences we refer to [1, 2, 5–10].

In this paper, we consider the sequence $\{w_n\}$ which is defined first in [3] as:

$$w_n = \begin{cases} aw_{n-1} + w_{n-2}, & \text{if } n \text{ is even} \\ bw_{n-1} + w_{n-2}, & \text{if } n \text{ is odd} \end{cases}, n \geq 2 \tag{1.7}$$

with arbitrary initial conditions $w_0, w_1$ where $w_0, w_1, a, b$ are nonzero numbers. Here we call the sequence $\{w_n\}$, *the bi-periodic Horadam sequence*. Motivating by Horadam's results in [4], our aim is to obtain some basic properties of the bi-periodic Horadam sequence. Moreover, we give some new identities for the bi-periodic Lucas sequences by using these properties.

Some sequences in the literature can be stated in terms of the sequence $\{w_n\}$ as:

1. If we take $w_0 = 0, w_1 = 1$ in $\{w_n\}$, we get the bi-periodic Fibonacci sequence $\{q_n\}$ in [3].

2. If we take $w_0 = 2, w_1 = b$ in $\{w_n\}$, we get the Lucas conditional sequence $\{l_n\}$ in [8] with the case of $c = d = 1$. If we replace $a$ and $b$ in $\{l_n\}$, we get the bi-periodic Lucas sequence $\{p_n\}$ in [1]. Thus we have the fact

$$l_n = \left( \frac{b}{a} \right)^{\zeta(n)} p_n. \tag{1.8}$$

Note that the above equality gives the relation between the sequences $\{l_n\}$ and $\{p_n\}$. We will use this fact for further results.

3. If we take $a = b = p$ and $w_0 = 0, w_1 = 1$ in $\{w_n\}$, we get the generalized Fibonacci sequence $\{u_n\}$.

4. If we take $a = b = p$ and $w_0 = 2, w_1 = p$ in $\{w_n\}$, we get the generalized Lucas sequence $\{v_n\}$.

## 2 Main results

In this section, we give some basic properties of the bi-periodic Horadam sequences. To obtain these properties, we use the Binet formula of $\{w_n\}$ which can be obtained by using the following result in [3, Theorem 8]. This result gives the relation between the sequences $\{w_n\}$ and $\{q_n\}$.

**Lemma 1.** *[3, Theorem 8] For $n > 0$, we have*

$$w_n = q_n w_1 + \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1} w_0. \tag{2.1}$$

Now, by using Lemma 1 and the Binet formula of $\{q_n\}$ in (1.4), we can easily obtain the Binet formula for the sequence $\{w_n\}$.

**Theorem 1.** *(Binet Formula) For $n > 0$, we have*

$$w_n = \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^{n-1} - B\beta^{n-1}\right), \tag{2.2}$$

*where $A := \left(\frac{\alpha w_1 + b w_0}{\alpha - \beta}\right)$ and $B := \left(\frac{\beta w_1 + b w_0}{\alpha - \beta}\right)$.*

*Proof.* By using Lemma 1 and (1.4), we get

$$\begin{aligned}
w_n &= q_n w_1 + \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1} w_0 \\
&= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right) w_1 + \left(\frac{b}{a}\right)^{\zeta(n)} \frac{a^{\zeta(n)}}{(ab)^{\lfloor \frac{n-1}{2} \rfloor}} \left(\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}\right) w_0 \\
&= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^n - \beta^n}{\alpha - \beta} w_1 + b\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta} w_0\right) \\
&= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(\frac{\alpha^{n-1}(\alpha w_1 + b w_0) - \beta^{n-1}(\beta w_1 + b w_0)}{\alpha - \beta}\right) \\
&= \frac{a^{\zeta(n+1)}}{(ab)^{\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^{n-1} - B\beta^{n-1}\right).
\end{aligned}$$

□

Another relation between the sequences $\{w_n\}$ and $\{q_n\}$ can be given in the following theorem.

**Theorem 2.** *For $n > 0$, we have*

$$w_{n+1}q_n - w_n q_{n+1} = (-1)^{n+1} w_0.$$

*Proof.* By using Lemma 1, we have

$$\begin{aligned}
w_{n+1}q_n - w_n q_{n+1} &= \left(q_{n+1}w_1 + \left(\frac{b}{a}\right)^{\zeta(n+1)} q_n w_0\right) q_n \\
&\quad - \left(q_n w_1 + \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1} w_0\right) q_{n+1} \\
&= \left(q_{n+1}q_n - q_n q_{n+1}\right) w_1 \\
&\quad + \left(\left(\frac{b}{a}\right)^{\zeta(n+1)} q_n^2 - \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1}q_{n+1}\right) w_0 \\
&= \left(\left(\frac{b}{a}\right)^{1-\zeta(n)} q_n^2 - \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1}q_{n+1}\right) w_0 \\
&= \left(\left(\frac{a}{b}\right)^{\zeta(n)} \frac{b}{a} q_n^2 - \left(\frac{b}{a}\right)^{\zeta(n)} q_{n-1}q_{n+1}\right) w_0 \\
&= \left(a^{\zeta(n)} b^{-\zeta(n)} \frac{b}{a} q_n^2 - a^{-\zeta(n)} b^{\zeta(n)} q_{n-1}q_{n+1}\right) w_0.
\end{aligned}$$

By using the Cassini's identity for the sequence $\{q_n\}$

$$a^{\zeta(n)} b^{1-\zeta(n)} q_n^2 - a^{1-\zeta(n)} b^{\zeta(n)} q_{n-1}q_{n+1} = a\,(-1)^{n+1},$$

which is given in [3, Theorem 3], we get the desired result. □

Note that if we take $w_0 = 2$ and $w_1 = b$, then by using the fact (1.8), the above theorem reduces the identity

$$\left(\frac{b}{a}\right)^{\zeta(n+1)} p_{n+1}q_n - \left(\frac{b}{a}\right)^{\zeta(n)} p_n q_{n+1} = 2\,(-1)^{n+1},$$

which can be found in [1, Corallary 3].

Now, we state the Cassini's identity for the bi-periodic Horadam sequences. Since the Cassini's identity is a special case of the Catalan's identity, we only prove the Catalan's identity.

**Theorem 3.** *(Cassini's identity) For any nonnegative integer $n$, we have*

$$a^{1-\zeta(n)} b^{\zeta(n)} w_{n-1}w_{n+1} - a^{\zeta(n)} b^{1-\zeta(n)} w_n^2 = (-1)^n \left[a w_1^2 - (ab)\, w_0 w_1 - b w_0^2\right].$$

**Theorem 4.** *(Catalan's identity) For any nonnegative integer $n$, we have*

$$\begin{aligned}
&a^{\zeta(n-r)} b^{1-\zeta(n-r)} w_{n-r}w_{n+r} - a^{\zeta(n)} b^{1-\zeta(n)} w_n^2 \\
&= (-1)^{n-r+1} \frac{\left(\alpha^r - \beta^r\right)^2}{(ab)^r\,(ab+4)} \left[a w_1^2 - (ab)\, w_0 w_1 - b w_0^2\right].
\end{aligned}$$

*Proof.* By using the Binet formula of $\{w_n\}$, we have

$$
\begin{aligned}
& w_{n-r}w_{n+r} - w_n^2 \\
= & \frac{a^{\zeta(n-r+1)+\zeta(n+r+1)}}{(ab)^{\lfloor \frac{n-r}{2} \rfloor + \lfloor \frac{n+r}{2} \rfloor}} \left(A\alpha^{n-r-1} - B\beta^{n-r-1}\right) A\alpha^{n+r-1} - B\beta \\
& - \frac{a^{2\zeta(n+1)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor}} \left(A\alpha^{n-1} - B\beta^{n-1}\right)^2 \\
= & \frac{a^{2\zeta(n-r-1)}}{(ab)^{\lfloor \frac{n-r}{2} \rfloor + \lfloor \frac{n+r}{2} \rfloor}} \left(A^2\alpha^{2n-2} - AB\left(\alpha^{n-r-1}\beta^{n+r-1} + \beta^{n-r-1}\alpha^{n+r-1}\right) + B^2\beta^{2n-2}\right) \\
& - \frac{a^{2\zeta(n+1)}}{(ab)^{2\lfloor \frac{n}{2} \rfloor}} \left(A^2\alpha^{2n-2} - 2AB\left(\alpha^{n-1}\beta^{n-1}\right) + B^2\beta^{2n-2}\right) \\
= & \frac{a^{2\zeta(n-r-1)}}{(ab)^{n-\zeta(n-r)}} \left(A^2\alpha^{2n-2} - AB\left(\alpha^{n-r-1}\beta^{n+r-1} + \beta^{n-r-1}\alpha^{n+r-1}\right) + B^2\beta^{2n-2}\right) \\
& - \frac{a^{2\zeta(n+1)}}{(ab)^{n-\zeta(n)}} \left(A^2\alpha^{2n-2} - 2AB\left(\alpha^{n-1}\beta^{n-1}\right) + B^2\beta^{2n-2}\right) \\
= & \frac{1}{(ab)^n} \left[a^{1+\zeta(n-r+1)}b^{\zeta(n-r)} \left(A^2\alpha^{2n-2} - AB\left(\alpha^{n-r-1}\beta^{n+r-1} + \beta^{n-r-1}\alpha^{n+r-1}\right) + B^2\beta^{2n-2}\right)\right. \\
& \left. - a^{1+\zeta(n+1)}b^{\zeta(n)} \left(A^2\alpha^{2n-2} - 2AB\left(\alpha^{n-1}\beta^{n-1}\right) + B^2\beta^{2n-2}\right)\right].
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
& a^{1-\zeta(n-r+1)}b^{1-\zeta(n-r)}w_{n-r}w_{n+r} - a^{\zeta(n)}b^{1-\zeta(n)}w_n^2 \\
= & \frac{a^2b}{(ab)^n}\left[-AB\left(\left(\frac{\alpha}{\beta}\right)^{-r}(\alpha\beta)^{n-1} + \left(\frac{\alpha}{\beta}\right)^{r}(\alpha\beta)^{n-1}\right) + 2AB(\alpha\beta)^{n-1}\right] \\
= & \frac{a^2b}{(ab)^n}(\alpha\beta)^{n-1}\left[-AB\left(\left(\frac{\alpha}{\beta}\right)^{-r} + \left(\frac{\alpha}{\beta}\right)^{r} - 2\right)\right] \\
= & (-1)^{n-r}\, a\frac{(\alpha^r - \beta^r)^2}{(ab)^r}AB.
\end{aligned}
$$

By using the definition of $A$ and $B$ in (2.2), we get

$$
\begin{aligned}
(-1)^{n-r}\, a\frac{(\alpha^r - \beta^r)^2}{(ab)^r}AB &= (-1)^{n-r}\, a\frac{(\alpha^r - \beta^r)^2}{(ab)^r}\left(\frac{\alpha w_1 + bw_0}{\alpha - \beta}\right)\left(\frac{\beta w_1 + bw_0}{\alpha - \beta}\right) \\
&= (-1)^{n-r}\, a\frac{(\alpha^r - \beta^r)^2}{(ab)^r}\frac{(-ab)w_1^2 + b(ab)w_0w_1 + b^2w_0^2}{ab(ab+4)} \\
&= (-1)^{n-r+1}\frac{(\alpha^r - \beta^r)^2}{(ab)^r(ab+4)}\left[aw_1^2 - (ab)w_0w_1 - bw_0^2\right].
\end{aligned}
$$

□

Note that if we take $r = 1$ in the above theorem, we obtain the Cassini's identity for the bi-periodic Horadam sequences.

- If we take $w_0 = 0$ and $w_1 = 1$ in the above theorem, we get the Catalan's identity for $\{q_n\}$
$$a^{\zeta(n-r)}b^{1-\zeta(n-r)}q_{n-r}q_{n+r} - a^{\zeta(n)}b^{1-\zeta(n)}q_n^2 = (-1)^{n+1-r}\, a^{\zeta(r)}b^{1-\zeta(r)}q_r^2$$
in [3, Theorem 4].

- If we take $w_0 = 2$ and $w_1 = b$ then by using the fact (1.8), we get the identity
$$\left(\frac{b}{a}\right)^{\zeta(n+r)} p_{n-r}p_{n+r} - \left(\frac{b}{a}\right)^{\zeta(n)} p_n^2 = \frac{(-1)^{n+r}}{(ab)^r}(\alpha^r - \beta^r)^2$$
in [1, Theorem 4].

The following theorem gives the d'Ocagne's identity for the bi-periodic Horadam sequences $\{w_n\}$. It can be proven similarly by using the Binet formula of $\{w_n\}$. Note that if we consider the case $w_0 = 2$ and $w_1 = b$, we get a new identity for the bi-periodic Lucas sequences.

**Theorem 5.** *(d'Ocagne's identity) For any nonnegative integer $n$, we have*
$$\begin{aligned} & a^{\zeta(mn+m)}b^{\zeta(mn+n)}w_m w_{n+1} - a^{\zeta(mn+n)}b^{\zeta(mn+m)}w_{m+1}w_n \\ = \; & (-1)^n\, a^{\zeta(m-n)-1}q_{m-n}\left[aw_1^2 - (ab)\,w_0w_1 - bw_0^2\right]. \end{aligned}$$

- If we take $w_0 = 0$ and $w_1 = 1$ in the above theorem, we get the d'Ocagne's identity for $\{q_n\}$
$$a^{\zeta(mn+m)}b^{\zeta(mn+n)}q_m q_{n+1} - a^{\zeta(mn+n)}b^{\zeta(mn+m)}q_{m+1}q_n = (-1)^n\, a^{\zeta(m-n)}q_{m-n}$$
in [3, Theorem 5].

- If we take $w_0 = 2$ and $w_1 = b$, then by using the fact (1.8), we get
$$\begin{aligned} & a^{\zeta(mn+n)}b^{\zeta(mn+m)}p_m p_{n+1} - a^{\zeta(mn+m)}b^{\zeta(mn+n)}p_{m+1}p_n \\ = \; & (-1)^{n+1}\, a^{\zeta(m-n)}\,(ab+4)\,q_{m-n} \\ = \; & (-1)^{n+1}\, a^{\zeta(m-n)}\,(p_{m-n-1} + p_{m-n+1}) \end{aligned} \tag{2.3}$$
which is a new identity for $\{p_n\}$.

Finally, we state two binomial formula for the bi-periodic Horadam numbers. As a consequence of the following theorem, we can obtain a new identity for the bi-periodic Lucas numbers.

**Theorem 6.** *(Generalized Catalan's identity) For any nonnegative integer $n$, we have*
$$\begin{aligned} w_n = \; & \frac{a^{\zeta(n+1)}}{2^{n-1}(ab)^{\lfloor \frac{n}{2} \rfloor}} \left( (abw_1 + 2bw_0) \sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor} \binom{n-1}{2k+1} (ab)^{n-k-2}(ab+4)^k \right. \\ & \left. + w_1 \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1}{2k} (ab)^{n-k-1}(ab+4)^k \right). \end{aligned}$$

*Proof.* By using the Binet formula of $\{w_n\}$ and the Binomial expansion formula, we have

$$\begin{aligned}
\frac{(ab)^{\lfloor \frac{n}{2} \rfloor}}{a^{\zeta(n+1)}} w_n &= A\alpha^{n-1} - B\beta^{n-1} \\
&= A\left(\frac{ab+\sqrt{\Delta}}{2}\right)^{n-1} - B\left(\frac{ab-\sqrt{\Delta}}{2}\right)^{n-1} \\
&= \frac{1}{2^{n-1}}\left(A\sum_{k=0}^{n-1}\binom{n-1}{k}(ab)^{n-k-1}\left(\sqrt{\Delta}\right)^k \right. \\
&\quad \left. -B\sum_{k=0}^{n-1}\binom{n-1}{k}(ab)^{n-k-1}\left(-\sqrt{\Delta}\right)^k\right) \\
&= \frac{1}{2^{n-1}}\left((A+B)\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor}\binom{n-1}{2k+1}(ab)^{n-2k-2}\Delta^{k+\frac{1}{2}} \right. \\
&\quad \left. +(A-B)\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}(ab)^{n-2k-1}\Delta^{k}\right)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{2^{n-1}}\left(\left(\frac{(\alpha+\beta)w_1+2bw_0}{\alpha-\beta}\right)\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor}\binom{n-1}{2k+1}(ab)^{n-2k-2}\Delta^{k+\frac{1}{2}} \right. \\
&\quad \left. +w_1\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}(ab)^{n-2k-1}\Delta^{k}\right) \\
&= \frac{1}{2^{n-1}}\left((abw_1+2bw_0)\sum_{k=0}^{\lfloor \frac{n-2}{2} \rfloor}\binom{n-1}{2k+1}(ab)^{n-k-2}(ab+4)^{k} \right. \\
&\quad \left. +w_1\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor}\binom{n-1}{2k}(ab)^{n-k-1}(ab+4)^{k}\right).
\end{aligned}$$

Thus, we obtain the desired result. □

- If we take $a = b = 1$, we obtain the result in [4, identity (3.20)] for the classical Horadam sequence.

- If we take $w_0 = 0$ and $w_1 = 1$ we get

$$
\begin{aligned}
2^{n-1}\frac{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}}{a^{\zeta(n+1)}}w_n &= \sum_{k=0}^{\left\lfloor \frac{n-2}{2}\right\rfloor}\binom{n-1}{2k+1}(ab)^{n-k-1}(ab+4)^k \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1}{2k}(ab)^{n-k-1}(ab+4)^k \\
&= \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\left(\binom{n-1}{2k+1}+\binom{n-1}{2k}\right)(ab)^{n-k-1}(ab+4)^k \\
&= \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}(ab)^{n-k-1}(ab+4)^k .
\end{aligned}
$$

Thus, we have

$$
q_n = \frac{a^{\zeta(n+1)}}{2^{n-1}}\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n}{2k+1}(ab)^{\left\lfloor \frac{n-1}{2}\right\rfloor-k}(ab+4)^k ,
$$

which reduces the generalized Catalan's identity in [10, Theorem 5].

- If we take $w_0 = 2$ and $w_1 = b$ we get

$$
\begin{aligned}
2^{n-1}\frac{(ab)^{\left\lfloor \frac{n}{2}\right\rfloor}}{ba^{\zeta(n+1)}}w_n &= \sum_{k=0}^{\left\lfloor \frac{n-2}{2}\right\rfloor}\binom{n-1}{2k+1}(ab)^{n-k-2}(ab+4)^{k+1} \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1}{2k}(ab)^{n-k-1}(ab+4)^k \\
&= \sum_{k=1}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n-1}{2k-1}(ab)^{n-k-1}(ab+4)^k \\
&\quad + \sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor}\binom{n-1}{2k}(ab)^{n-k-1}(ab+4)^k \\
&= \sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\left(\binom{n-1}{2k-1}+\binom{n-1}{2k}\right)(ab)^{n-k-1}(ab+4)^k \\
&= \sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n}{2k}(ab)^{n-k-1}(ab+4)^k .
\end{aligned}
$$

Then by using the fact (1.8), we obtain

$$
p_n = \frac{a^{\zeta(n)}}{2^{n-1}}\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\binom{n}{2k}(ab)^{\left\lfloor \frac{n}{2}\right\rfloor-k}(ab+4)^k , \tag{2.4}
$$

which is a new identity for the bi-periodic Lucas sequences.

**Theorem 7.** *(General Binomial Sum Formula) For any nonnegative integer $n$, we have*

$$\sum_{k=0}^{n}\binom{n}{k}a^{\zeta(k+r)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor+\zeta(k)\zeta(r)}w_{k+r}=a^{\zeta(r)}w_{2n+r}.$$

*Proof.* By using the Binet formula of $\{w_n\}$, we obtain

$$\begin{aligned}
&\sum_{k=0}^{n}\binom{n}{k}a^{\zeta(k+r)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor+\zeta(k)\zeta(r)}w_{k+r}\\
=\;&\sum_{k=0}^{n}\binom{n}{k}a^{\zeta(k+r)+\zeta(k+r+1)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor+\zeta(k)\zeta(r)-\left\lfloor\frac{k+r}{2}\right\rfloor}\left(A\alpha^{k+r-1}-B\beta^{k+r-1}\right)\\
=\;&a\,(ab)^{\frac{\zeta(r)-r}{2}}\left[A\alpha^{r-1}\sum_{k=0}^{n}\binom{n}{k}\alpha^{k}-B\beta^{r-1}\sum_{k=0}^{n}\binom{n}{k}\beta^{k}\right]\\
=\;&a\,(ab)^{\frac{\zeta(r)-r}{2}}\left[A\alpha^{r-1}(1+\alpha)^{n}-B\beta^{r-1}(1+\beta)^{n}\right]\\
=\;&a\,(ab)^{\frac{\zeta(r)-r}{2}}\left[A\alpha^{r-1}\left(\frac{\alpha^{2}}{ab}\right)^{n}-B\beta^{r-1}\left(\frac{\beta^{2}}{ab}\right)^{n}\right]\\
=\;&a\,(ab)^{\frac{\zeta(r)-r-2n}{2}}\left(A\alpha^{2n+r-1}-B\beta^{2n+r-1}\right)\\
=\;&a^{1-\zeta(2n+r+1)}(ab)^{\frac{\zeta(r)-r-2n}{2}+\left\lfloor\frac{2n+r}{2}\right\rfloor}w_{2n+r}\\
=\;&a^{\zeta(r)}w_{2n+r}.
\end{aligned}$$

□

- If we take $w_0=0$ and $w_1=1$, we obtain the identity

$$\sum_{k=0}^{n}\binom{n}{k}a^{\zeta(k+r)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor+\zeta(k)\zeta(r)}q_{k+r}=a^{\zeta(r)}q_{2n+r},$$

  which can be found in [3, Remark 1].

- If we take $w_0=2$ and $w_1=b$ then by using the fact (1.8), we obtain the identity

$$\sum_{k=0}^{n}\binom{n}{k}b^{\zeta(k+r)}(ab)^{\left\lfloor\frac{k}{2}\right\rfloor+\zeta(k)\zeta(r)}p_{k+r}=b^{\zeta(r)}p_{2n+r},$$

  which can be found in [1, Theorem 7].

# 3 Conclusion

In this paper, we considered the bi-periodic Horadam sequences $\{w_n\}$, which is defined by the recurrence $w_n=aw_{n-1}+w_{n-2}$, if $n$ is even, $w_n=bw_{n-1}+w_{n-2}$, if $n$ is odd with arbitrary initial conditions $w_0,w_1$ and nonzero numbers $a,b$. Motivated by Horadam's results in [4], we gave some basic properties of the bi-periodic Horadam sequences which generalize the known results for the bi-periodic Fibonacci and Lucas sequences in [3] and [1]. Moreover, we derived some new identities for the bi-periodic Lucas sequences.

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