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Prime triples p_1, p_2, p_3 in arithmetic progressions such that $p_1 = x^2 + y^2 + 1, p_3 = [n^c]$

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Abstract: In the present paper we prove that there exist infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = 2p_2 - p_1$ such that $p_1 = x^2 + y^2 + 1,\ p_3 = [n^c]$.
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1 Notations

Let X be a sufficiently large positive number. The letter p, with or without subscript, will always denote prime numbers. The letter ε we denote an arbitrary small positive number, not the same in all appearances.

The relation $f(x) \ll g(x)$ means that $f(x) = \mathcal{O}(g(x))$. As usual $[t]$ and $\{t\}$ denote the integer part, respectively, the fractional part of t. Instead of $m \equiv n \pmod k$ we write for simplicity $m \equiv n\,(k)$.

As usual $e(t) = \exp(\pi i t)$. We denote by (d, q), $[d, q]$ the greatest common divisor and the least common multiple of d and q respectively. As usual $\varphi(d)$ is Euler's function; $\mu(d)$ is Möbius' function; $r(d)$ is the number of solutions of the equation $d = m_1^2 + m_2^2$ in integers m_j; $\chi(d)$ is the non-principal character modulo 4 and $L(s, \chi)$ is the corresponding Dirichlet's L-function.

By c_0 we denote some positive number, not necessarily the same in different occurrences. Let c be a real constant such that $1 < c < 73/64$.

Denote

$$\gamma = 1/c\,; \tag{1}$$

$$\psi(t) = \{t\} - 1/2\,; \tag{2}$$

$$\theta_0 = \frac{1}{2} - \frac{1}{4}e\log 2 = 0.0289...\,; \tag{3}$$

$$\sigma_0 = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right)\,; \tag{4}$$

$$\mathfrak{S}_\Gamma = \pi\sigma_0\prod_{p}\left(1 + \frac{\chi(p)}{p(p-1)}\right)\,; \tag{5}$$

$$\Delta(t,h) = \max_{y\le t}\max_{(l,h)=1}\left|\sum_{\substack{p\le y\\ p\equiv l\,(h)}}\log p - \frac{y}{\varphi(h)}\right|\,. \tag{6}$$

2 Introduction and statement of the result

In 1953, Piatetski-Shapiro [9] proved that for any fixed $c \in (1, 12/11)$ the sequence

$$([n^c])_{n\in\mathbb{N}}$$

contains infinitely many prime numbers. Such prime numbers are named in honor of Piatetski-Shapiro. The interval for c was subsequently improved many times and the best result up to now belongs to Rivat and Wu [10] for $c \in (1, 243/205)$.

In 2014, M. Mirek [7] showed that for any fixed $c \in (1, 72/71)$ the set

$$\mathbf{P}_c = \{p \text{ prime} : p = [n^c] \text{ for some } n \in \mathbb{N}\}$$

contains infinitely many non-trivial three-term arithmetic progressions.

On the other hand, in 1960, Linnik [6] showed that there exist infinitely many prime numbers of the form $p = x^2 + y^2 + 1$, where x and y – integers. Recently, the author [1] proved that there exist infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = 2p_2 - p_1$ such that $p_1 = x_1^2 + y_1^2 + 1$, $p_2 = x_2^2 + y_2^2 + 1$. Shortly after that, Joni Teräväinen [12] improved this result by proving that the set

$$\mathcal{P} = \{p \text{ prime} : p = x^2 + y^2 + 1, \ \ (x, y) = 1\}$$

contains infinitely many non-trivial three-term arithmetic progressions.

Motivated by these results, we shall prove that there exist infinitely many arithmetic progressions of three different primes $p_1, p_2, p_3 = 2p_2 - p_1$ such that $p_1 = x^2 + y^2 + 1$, $p_3 = [n^c]$.

Define

$$\Gamma_c(X) = \sum_{\substack{(X/2)^c < p_1, p_2, p_3 \le X^c\\ p_1 + p_3 = 2p_2\\ X/2 < n \le X\\ p_3 = [n^c]}} r(p_1 - 1)p_3^{1-\gamma}\log p_1 \log p_2 \log p_3 \tag{7}$$

We shall prove the following theorem.

Theorem 1. *Assume that* $1 < c < 73/64$*. Then the asymptotic formula*

$$\Gamma_c(X) = \frac{(2^c-1)^2}{c2^{2c+1}}\mathfrak{S}_\Gamma X^{2c} + \mathcal{O}\big(X^{2c}(\log X)^{-\theta_0}(\log\log X)^6\big)\,,$$

holds. Here θ_0 *and* $\mathfrak{S}_\Gamma$ *are defined by* (3) *and* (5)*.*

3 Outline of the proof

Denote

$$D = \frac{X^{c/2}}{(\log X)^A}\,. \tag{8}$$

Using (7) and the well-known identity $r(n) = 4\sum_{d|n}\chi(d)$, we find

$$\Gamma_c(X) = 4\big(\Gamma_c^{(1)}(X) + \Gamma_c^{(2)}(X) + \Gamma_c^{(3)}(X)\big), \tag{9}$$

where

$$\Gamma_c^{(1)}(X) = \sum_{\substack{(X/2)^c<p_1,p_2,p_3\leq X^c\\ p_1+p_3=2p_2\\ X/2<n\leq X\\ p_3=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ d\leq D}}\chi(d)\right) p_3^{1-\gamma}\log p_1\log p_2\log p_3\,, \tag{10}$$

$$\Gamma_c^{(2)}(X) = \sum_{\substack{(X/2)^c<p_1,p_2,p_3\leq X^c\\ p_1+p_3=2p_2\\ X/2<n\leq X\\ p_3=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ D<d<X^c/D}}\chi(d)\right) p_3^{1-\gamma}\log p_1\log p_2\log p_3\,, \tag{11}$$

$$\Gamma_c^{(3)}(X) = \sum_{\substack{(X/2)^c<p_1,p_2,p_3\leq X^c\\ p_1+p_3=2p_2\\ X/2<n\leq X\\ p_3=[n^c]}} \left(\sum_{\substack{d|p_1-1\\ d\geq X^c/D}}\chi(d)\right) p_3^{1-\gamma}\log p_1\log p_2\log p_3\,. \tag{12}$$

In order to estimate $\Gamma_c^{(1)}(X)$ and $\Gamma_c^{(3)}(X)$ we have to consider the sum

$$I_{c,l,d;J}(X) = \sum_{\substack{(X/2)^c<p_2,p_3\leq X^c\\ p_1+p_3=2p_2\\ X/2<n\leq X\\ p_3=[n^c]\\ p_1\equiv l\,(d)\\ p_1\in J}} p_3^{1-\gamma}\log p_1\log p_2\log p_3\,. \tag{13}$$

where d and l are coprime natural numbers, and $J\subset((X/2)^c, X^c]$-interval. If $J = ((X/2)^c, X^c]$ then we write for simplicity $I_{c,l,d}(X)$. We apply the circle method. Clearly,

$$I_{c,l,d;J}(X) = \int\limits_0^1 S_{c,l,d;J}(\alpha)S_c^*(\alpha)S_c(-2\alpha)d\alpha\,, \tag{14}$$

where

$$S_{c,l,d;J}(\alpha)=\sum_{\substack{p\in J\\ p\equiv l\ (d)}} e(\alpha p)\log p\,, \tag{15}$$

$$S_c(\alpha)=S_{c,1,1;((X/2)^c,X^c]}(\alpha)\,, \tag{16}$$

$$S_c^*(\alpha)=\sum_{\substack{(X/2)^c<p\leq X^c\\ X/2<n\leq X\\ p=[n^c]}} p^{1-\gamma}e(\alpha p)\log p\,. \tag{17}$$

We define major and minor arcs by

$$E_1=\bigcup_{q\leq Q}\bigcup_{\substack{a=0\\ (a,q)=1}}^{q-1}\left(\frac{a}{q}-\frac{1}{q\tau},\frac{a}{q}+\frac{1}{q\tau}\right),\quad E_2=\left(-\frac{1}{\tau},1-\frac{1}{\tau}\right)\setminus E_1\,, \tag{18}$$

where

$$Q=(\log X)^B\,,\quad \tau=X^cQ^{-1}\,,\quad A>4B+1\,,\quad B>14\,. \tag{19}$$

Then we have the decomposition

$$I_{c,l,d;J}(X)=I^{(1)}_{c,l,d;J}(X)+I^{(2)}_{c,l,d;J}(X)\,, \tag{20}$$

where

$$I^{(i)}_{c,l,d;J}(X)=\int\limits_{E_i} S_{c,l,d;J}(\alpha)S_c^*(\alpha)S_c(-2\alpha)d\alpha\,,\quad i=1,2. \tag{21}$$

We shall estimate $I^{(1)}_{c,l,d;J}(X)$, $\Gamma^{(3)}_c(X)$, $\Gamma^{(2)}_c(X)$ and $\Gamma^{(1)}_c(X)$, respectively, in Sections 4, 5, 6 and 7. In section 8 we shall complete the proof of the Theorem.

4 Asymptotic formula for $\mathbf{I^{(1)}_{c,l,d;J}(X)}$

We have

$$I^{(1)}_{c,l,d;J}(X)=\sum_{q\leq Q}\sum_{\substack{a=0\\ (a,q)=1}}^{q-1} H(a,q)\,, \tag{22}$$

where

$$H(a,q)=\int\limits_{-1/q\tau}^{1/q\tau} S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)S_c^*\left(\frac{a}{q}+\alpha\right)S_c\left(-2\left(\frac{a}{q}+\alpha\right)\right)d\alpha\,. \tag{23}$$

Arguing as in [8], we find

$$S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)=\frac{c_d(a,q,l)}{\varphi([d,q])}M_J(\alpha)+\mathcal{O}\big(Q\Delta(X^c,[d,q])\big)\,, \tag{24}$$

where

$$c_d(a,q,l)=\sum_{\substack{1\leq m\leq q\\ (m,q)=1\\ m\equiv l\ ((d,q))}} e\left(\frac{am}{q}\right)\,,$$

$$M_J(\alpha)=\sum_{m\in J} e(\alpha m)\,.$$

On the other hand working similar to ([4], Lemma 3, §10) we get

$$S_c\left(-2\left(\frac{a}{q}+\alpha\right)\right)=\frac{c_2(q)}{\varphi(q)}M(-2\alpha)+\mathcal{O}\left(X^c e^{-c_0\sqrt{\log X}}\right)\,, \tag{25}$$

where

$$\begin{aligned} c_2(q)&=\sum_{\substack{a=1\\(a,q)=1}}^{q} e\left(\frac{2a}{q}\right)=\frac{\mu\left(\frac{q}{(2,q)}\right)}{\varphi\left(\frac{q}{(2,q)}\right)}\varphi(q)\,,\\ M(\alpha)&=\sum_{(X/2)^c<m\le X^c} e(\alpha m)\,. \end{aligned} \tag{26}$$

We shall find asymptotic formula for $S_c^*\left(\frac{a}{q}+\alpha\right)$. From (17) we have

$$\begin{aligned} S_c^*(\alpha)&=\sum_{(X/2)^c<p\le X^c} p^{1-\gamma}e(\alpha p)\log p\sum_{\substack{X/2<n\le X\\ p=[n^c]}} 1\\ &=\sum_{(X/2)^c<p\le X^c} p^{1-\gamma}e(\alpha p)\log p\sum_{p^\gamma\le n<(p+1)^\gamma} 1+\mathcal{O}(X^\varepsilon)\\ &=\sum_{(X/2)^c<p\le X^c} p^{1-\gamma}\big([-p^\gamma]-[-(p+1)^\gamma]\big)e(\alpha p)\log p+\mathcal{O}(X^\varepsilon)\\ &=\Omega(\alpha)+\Sigma(\alpha)+\mathcal{O}(X^\varepsilon)\,, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \Omega(\alpha)&=\sum_{(X/2)^c<p\le X^c} p^{1-\gamma}\big((p+1)^\gamma-p^\gamma\big)e(\alpha p)\log p\,,\\ \Sigma(\alpha)&=\sum_{(X/2)^c<p\le X^c} p^{1-\gamma}\big(\psi(-(p+1)^\gamma)-\psi(-p^\gamma)\big)e(\alpha p)\log p\,. \end{aligned}$$

According to Kumchev [5], Theorem 2 we have that

$$\Sigma\left(\frac{a}{q}+\alpha\right)\ll X^{c-\varepsilon}\,. \tag{28}$$

On the other hand,

$$\Omega(\alpha)=\gamma S_c(\alpha)+\mathcal{O}(X^\varepsilon)\,, \tag{29}$$

where $S_c(\alpha)$ is defined by (16).

According to ([4], Lemma 3, §10) we have

$$S_c\left(\frac{a}{q}+\alpha\right)=\frac{\mu(q)}{\varphi(q)}M(\alpha)+\mathcal{O}\left(X^c e^{-c_0\sqrt{\log X}}\right)\,, \tag{30}$$

where $M(\alpha)$ is defined by (26).

Bearing in mind (27)–(30), we obtain

$$S_c^*\left(\frac{a}{q}+\alpha\right)=\gamma\frac{\mu(q)}{\varphi(q)}M(\alpha)+\mathcal{O}\left(X^c e^{-c_0\sqrt{\log X}}\right)\,. \tag{31}$$

Furthermore, we need the trivial estimates

$$\left|S_c^*\left(\frac{a}{q}+\alpha\right)\right| \ll X^c\,, \quad \left|S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)\right| \ll \frac{X^c\log X}{d}\,,$$
$$|M(\alpha)| \ll X^c\,, \quad |c_2(q)| \ll 1\,, \quad |\mu(q)| \ll 1\,. \tag{32}$$

By (24), (25), (31), (32) and the well-known inequality $\varphi(n) \gg n(\log\log n)^{-1}$ we find

$$\begin{aligned}
&S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)S_c^*\left(\frac{a}{q}+\alpha\right)S_c\left(-2\left(\frac{a}{q}+\alpha\right)\right)\\
&=S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)S_c^*\left(\frac{a}{q}+\alpha\right)\frac{c_2(q)}{\varphi(q)}M(-2\alpha)+\mathcal{O}\left(\frac{X^{3c}}{d}e^{-c_0\sqrt{\log X}}\right)\\
&=S_{c,l,d;J}\left(\frac{a}{q}+\alpha\right)\gamma\frac{\mu(q)c_2(q)}{\varphi^2(q)}M(\alpha)M(-2\alpha)+\mathcal{O}\left(\frac{X^{3c}}{d}e^{-c_0\sqrt{\log X}}\right)\\
&=\gamma\frac{c_d(a,q,l)\mu(q)c_2(q)}{\varphi([d,q])\varphi^2(q)}M_J(\alpha)M(\alpha)M(-2\alpha)+\mathcal{O}\left(\frac{X^{3c}}{d}e^{-c_0\sqrt{\log X}}\right)\\
&+\mathcal{O}\left(\frac{X^{2c}Q\log X}{q^2}\Delta(X^c,[d,q])\right)\,.
\end{aligned} \tag{33}$$

Having in mind (19), (23) and (33), we get

$$\begin{aligned}
H(a,q)=\gamma\frac{c_d(a,q,l)\mu(q)c_2(q)}{\varphi([d,q])\varphi^2(q)}\int\limits_{-1/q\tau}^{1/q\tau}M_J(\alpha)M(\alpha)M(-2\alpha)d\alpha\\
+\mathcal{O}\left(\frac{X^{2c}}{qd}e^{-c_0\sqrt{\log X}}\right)+\mathcal{O}\left(\frac{X^cQ^2\log X}{q^3}\Delta(X^c,[d,q])\right)\,.
\end{aligned} \tag{34}$$

Taking into account (22), (23), (34) and following the method in [8] we obtain

$$\begin{aligned}
I^{(1)}_{c,l,d;J}(X)=\gamma\frac{\sigma_0}{\varphi(d)}\sum_{\substack{(X/2)^c<m_1,m_2\le X^c\\ m_1+m_3=2m_2\\ m_3\in J}}1+\mathcal{O}\left(\frac{X^{2c}}{d}\sum_{q>Q}\frac{(d,q)\log^2 q}{q^2}\right)\\
+\mathcal{O}\left(\tau^2(\log X)\sum_{q\le Q}\frac{q}{[d,q]}\right)+\mathcal{O}\left(X^cQ^2(\log X)\sum_{q\le Q}\frac{\Delta(X^c,[d,q])}{q^2}\right)\\
+\mathcal{O}\left(\frac{X^{2c}}{d}e^{-c_0\sqrt{\log X}}\right)\,.
\end{aligned} \tag{35}$$

5 Upper bound for $\Gamma_c^{(3)}(X)$

Consider the sum $\Gamma_c^{(3)}(X)$. Since

$$\sum_{\substack{d|p_1-1\\ d\ge X^c/D}}\chi(d)=\sum_{\substack{m|p_1-1\\ m\le(p_1-1)D/X^c}}\chi\left(\frac{p_1-1}{m}\right)=\sum_{j=\pm1}\chi(j)\sum_{\substack{m|p_1-1\\ m\le(p_1-1)D/X^c\\ \frac{p_1-1}{m}\equiv j\,(4)}}1$$

then from (12) and (13) it follows

$$\Gamma_c^{(3)}(X) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm 1} \chi(j) I_{c,1+jm,4m;J_m}(X)\,,$$

where $J_m = \big(\max\{1 + mX^c/D, (X/2)^c - 1\}, X^c\big]$. Therefore from (20) we get

$$\Gamma_c^{(3)}(X) = \Gamma_c^{(3),(1)}(X) + \Gamma_c^{(3),(2)}(X)\,, \tag{36}$$

where

$$\Gamma_c^{(3),(\nu)}(X) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm 1} \chi(j) I^{(\nu)}_{c,1+jm,4m;J_m}(X)\,, \quad \nu = 1, 2. \tag{37}$$

Let us consider first $\Gamma_c^{(3),(2)}(X)$. Bearing in mind (21) for $i = 2$ and (37) for $\nu = 2$, we have

$$\Gamma_c^{(3),(2)}(X) = \int\limits_{E_2} K(\alpha) S_c^*(\alpha) S_c(-2\alpha) d\alpha\,,$$

where

$$K(\alpha) = \sum_{\substack{m<D \\ 2|m}} \sum_{j=\pm 1} \chi(j) S_{c,1+jm,4m;J_m}(\alpha)\,.$$

Using Cauchy's inequality we obtain

$$\begin{aligned}\Gamma_c^{(3),(2)}(X) &\ll \sup_{\alpha\in E_2} |S_c(-2\alpha)| \int\limits_{E_2} |K(\alpha) S_c^*(\alpha)| d\alpha \\ &\ll \sup_{\alpha\in E_2} |S_c(-2\alpha)| \left(\int\limits_0^1 |K(\alpha)|^2 d\alpha\right)^{1/2} \left(\int\limits_0^1 |S_c^*(\alpha)|^2 d\alpha\right)^{1/2}.\end{aligned} \tag{38}$$

The sum defined by (16) can be estimated over the minor arcs by Vinogradov's method. Using (18) and (19), we can prove in the same way as in ([4], Ch.10, Th.3) that

$$\sup_{\alpha\in E_2} |S_c(-2\alpha)| \ll \frac{X^c}{(\log X)^{B/2-4}}\,. \tag{39}$$

We square out and after straightforward computations find

$$\int\limits_0^1 |S_c^*(\alpha)|^2 d\alpha \ll X^{2c-1} \log X\,. \tag{40}$$

$$\int\limits_0^1 |K(\alpha)|^2 d\alpha \ll X \log^3 X\,. \tag{41}$$

Thus from (38)–(41) it follows

$$\Gamma_c^{(3),(2)}(X) \ll \frac{X^{2c}}{(\log X)^{B/2-6}}\,. \tag{42}$$

Now let us consider $\Gamma_c^{(3),(1)}(X)$. From (35) and (37) for $\nu = 1$ we get

$$\Gamma_c^{(3),(1)}(X) = \Gamma^* + \mathcal{O}\big(X^{2c}\Sigma_1\big) + \mathcal{O}\big(\tau^2(\log X)\Sigma_2\big) + \mathcal{O}\big(X^c Q^2(\log X)\Sigma_3\big) + \mathcal{O}\big(X^{2c}e^{-c_0\sqrt{\log X}}\Sigma_4\big)\,, \tag{43}$$

where

$$\Gamma^* = \gamma\sigma_0 \sum_{\substack{(X/2)^c<m_1,m_2\le X^c\\ m_1+m_3=2m_2\\ m_3\in J_m}} 1 \sum_{\substack{m<D\\ 2|m}} \frac{1}{\varphi(4m)} \sum_{j=\pm1}\chi(j)\,,$$

$$\Sigma_1 = \sum_{m<D}\sum_{q>Q}\frac{(4m,q)\log^2 q}{mq^2}\,,$$

$$\Sigma_2 = \sum_{m<D}\sum_{q\le Q}\frac{q}{[4m,q]}\,,$$

$$\Sigma_3 = \sum_{m<D}\sum_{q\le Q}\frac{\Delta(X^c,[4m,q])}{q^2}\,,$$

$$\Sigma_4 = \sum_{m<D}\frac{1}{m}\,.$$

From the properties of $\chi(k)$ we have that

$$\Gamma^* = 0\,. \tag{44}$$

Arguing as in [8] and using Bombieri–Vinogradov's theorem, we find the following estimates

$$\Sigma_1 \ll \frac{\log^5 X}{Q}\,,\quad \Sigma_2 \ll Q\log^2 X\,, \tag{45}$$

$$\Sigma_3 \ll \frac{X^c}{(\log X)^{A-B-5}}\,,\quad \Sigma_4 \ll \log X\,. \tag{46}$$

Bearing in mind (19), (43)–(46), we obtain

$$\Gamma_c^{(3),(1)}(X) \ll \frac{X^{2c}}{(\log X)^{B-5}}\,. \tag{47}$$

Now from (36), (42) and (47) we find

$$\Gamma_c^{(3)}(X) \ll \frac{X^{2c}}{(\log X)^{B/2-6}}\,. \tag{48}$$

6 Upper bound for $\Gamma_c^{(2)}(X)$

Consider the sum $\Gamma_c^{(2)}(X)$. We have

$$\Gamma_c^{(2)}(X) = \Sigma_1 + \mathcal{O}\big(X^{2c-1+\varepsilon}\big)\,, \tag{49}$$

where

$$\Sigma_1 = \sum_{\substack{(X/2)^c<p_1,p_2,p_3\leq X^c\\ p_1+p_3=2p_2}} \left(\sum_{\substack{d|p_1-1\\ D<d<X^c/D}} \chi(d)\right) p^{1-\gamma}\log p_1 \log p_2 \log p_3 \sum_{p_3^\gamma\leq n<(p_3+1)^\gamma} 1\,.$$

We denote by $\mathcal{F}$ the set of all primes $(X/2)^c < p \leq X^c$ such that $p-1$ has a divisor belongs to the interval $(D, X^c/D)$. By Cauchy's inequality we get

$$\begin{aligned}\Sigma_1^2 &\ll (\log X)^6 \sum_{\substack{(X/2)^c<p_1,\ldots,p_6\leq X^c\\ p_1+p_3=2p_2\\ p_4+p_6=2p_5}} \left|\sum_{\substack{d|p_1-1\\ D<d<X^c/D}} \chi(d)\right|\left|\sum_{\substack{t|p_4-1\\ D<t<X^c/D}} \chi(t)\right|\\ &\ll (\log X)^6 \sum_{\substack{(X/2)^c<p_1,\ldots,p_6\leq X^c\\ p_1+p_3=2p_2\\ p_4+p_6=2p_5\\ p_4\in\mathcal{F}}} \left|\sum_{\substack{d|p_1-1\\ D<d<X^c/D}} \chi(d)\right|^2.\end{aligned}$$

The summands in the last sum for witch $p_1 = p_4$ can be estimated with $\mathcal{O}(X^{3c+\varepsilon})$.

Therefore

$$\Sigma_1^2 \ll (\log X)^6\Sigma_2 + X^{3c+\varepsilon}\,, \tag{50}$$

where

$$\Sigma_2 = \sum_{(X/2)^c<p_1\leq X^c} \left|\sum_{\substack{d|p_1-1\\ D<d<X^c/D}} \chi(d)\right|^2 \sum_{\substack{(X/2)^c<p_4\leq X^c\\ p_4\in\mathcal{F}\\ p_4\neq p_1}} \sum_{\substack{(X/2)^c<p_2,p_3,p_5,p_6\leq X^c\\ p_1+p_3=2p_2\\ p_4+p_6=2p_5}} 1\,.$$

Further we use that if h is an integer such that $1 \leq |h| \leq X^c$, then the number of solutions of the equation $2p_1 - p_2 = h$ in primes $(X/2)^c < p_1, p_2 \leq X^c$ is $\mathcal{O}\big(X^c(\log X)^{-2}\log\log X\big)$. This follows for example from ([2], Ch.2, Th.2.4).

Hence

$$\Sigma_2 \ll \frac{X^{2c}}{\log^4 X}(\log\log X)^2\Sigma_3\Sigma_4\,, \tag{51}$$

where

$$\Sigma_3 = \sum_{(X/2)^c<p\leq X^c} \left|\sum_{\substack{d|p-1\\ D<d<X^c/D}} \chi(d)\right|^2, \quad \Sigma_4 = \sum_{\substack{(X/2)^c<p\leq X^c\\ p\in\mathcal{F}}} 1\,.$$

Arguing as in ([3], Ch.5), we obtain

$$\Sigma_3 \ll \frac{X^c(\log\log X)^7}{\log X}\,, \quad \Sigma_4 \ll \frac{X^c(\log\log X)^3}{(\log X)^{1+2\theta_0}}\,. \tag{52}$$

where θ_0 is denoted by (3).

From (49)–(52) it follows

$$\Gamma_c^{(2)}(X) \ll X^{2c}(\log X)^{-\theta_0}(\log\log X)^6\,. \tag{53}$$

7 Asymptotic formula for $\Gamma_c^{(1)}(X)$

Consider the sum $\Gamma_c^{(1)}(X)$. From (10), (13) and (20) we get

$$\Gamma_c^{(1)}(X) = \Gamma_c^{(1),(1)}(X) + \Gamma_c^{(1),(2)}(X)\,, \tag{54}$$

where

$$\Gamma_c^{(1),(1)}(X) = \sum_{d\le D} \chi(d) I_{c,1,d}^{(1)}(X)\,,$$

$$\Gamma_c^{(1),(2)}(X) = \sum_{d\le D} \chi(d) I_{c,1,d}^{(2)}(X)\,.$$

We estimate the sum $\Gamma_c^{(1),(2)}(X)$ by the same way as the sum $\Gamma_c^{(3),(2)}(X)$ and obtain

$$\Gamma_c^{(1),(2)}(X) \ll \frac{X^{2c}}{(\log X)^{B/2-6}}\,. \tag{55}$$

Now we consider $\Gamma_c^{(1),(1)}(X)$. We use the formula (35) for $J = (X/2, X]$. The error term is estimated by the same way as for $\Gamma_c^{(3),(1)}(X)$. We have

$$\Gamma_c^{(1),(1)}(X) = \frac{(2^c-1)^2}{c2^{2c+1}}\sigma_0 X^{2c} \sum_{d\le D} \frac{\chi(d)}{\varphi(d)} + \mathcal{O}\bigg(\frac{X^{2c}}{(\log X)^{B-5}}\bigg)\,. \tag{56}$$

Denote

$$\Sigma = \sum_{d\le D} f(d)\,, \qquad f(d) = \frac{\chi(d)}{\varphi(d)}\,. \tag{57}$$

We have

$$f(d) \ll d^{-1} \log\log(10d) \tag{58}$$

with absolute constant in the Vinogradov's symbol. Hence the corresponding Dirichlet series

$$F(s) = \sum_{d=1}^{\infty} \frac{f(d)}{d^s}$$

is absolutely convergent in $Re(s) > 0$. On the other hand, $f(d)$ is a multiplicative with respect to d and applying Euler's identity we find

$$F(s) = \prod_p T(p,s)\,, \quad T(p,s) = 1 + \sum_{l=1}^{\infty} f(p^l)p^{-ls}\,. \tag{59}$$

From (57) and (59) we establish that

$$T(p,s) = \bigg(1 - \frac{\chi(p)}{p^{s+1}}\bigg)^{-1} \bigg(1 + \frac{\chi(p)}{p^{s+1}(p-1)}\bigg)\,.$$

Hence we find

$$F(s) = L(s+1,\chi)\mathcal{N}(s)\,, \tag{60}$$

where $L(s+1,\chi)$ is Dirichlet series corresponding to the character χ and

$$\mathcal{N}(s)=\prod_p\left(1+\frac{\chi(p)}{p^{s+1}(p-1)}\right)\,. \tag{61}$$

From the properties of the L-functions it follows that $F(s)$ has an analytic continuation to $Re(s)>-1$. It is well known that

$$L(s+1,\chi)\ll 1+|Im(s)|^{1/6}\quad\text{for}\;\; Re(s)\geq -\frac{1}{2}\,. \tag{62}$$

Moreover

$$\mathcal{N}(s)\ll 1\,. \tag{63}$$

Using (60), (62) and (63), we get

$$F(s)\ll X^{c/6}\quad\text{for}\;\; Re(s)\geq -\frac{1}{2}\,,\quad |Im(s)|\leq X^c\,. \tag{64}$$

We apply Perron's formula given at Tenenbaum ([11], Chapter II.2) and also (58) to obtain

$$\Sigma=\frac{1}{2\pi\imath}\int\limits_{\kappa-\imath X^c}^{\kappa+\imath X^c}F(s)\frac{D^s}{s}ds+\mathcal{O}\left(\sum_{t=1}^{\infty}\frac{D^\kappa\log\log(10t)}{t^{1+\kappa}\left(1+X^c\left|\log\frac{D}{t}\right|\right)}\right)\,, \tag{65}$$

where $\kappa=1/10$. It is easy to see that the error term above is $\mathcal{O}\left(X^{-c/20}\right)$. Applying the residue theorem we see that the main term in (65) is equal to

$$F(0)+\frac{1}{2\pi\imath}\left(\int\limits_{1/10-\imath X^c}^{-1/2-\imath X^c}+\int\limits_{-1/2-\imath X^c}^{-1/2+\imath X^c}+\int\limits_{-1/2+\imath X^c}^{1/10+\imath X^c}\right)F(s)\frac{D^s}{s}ds\,.$$

From (64) it follows that the contribution from the above integrals is $\mathcal{O}\left(X^{-c/20}\right)$.
Hence

$$\Sigma=F(0)+\mathcal{O}\left(X^{-c/20}\right)\,. \tag{66}$$

Using (60) we get

$$F(0)=\frac{\pi}{4}\mathcal{N}(0)\,. \tag{67}$$

Bearing in mind (56), (57), (61), (66) and (67) we find a new expression for $\Gamma_c^{(1),(1)}(X)$

$$\Gamma_c^{(1),(1)}(X)=\frac{(2^c-1)^2}{c2^{2c+3}}\mathfrak{S}_\Gamma X^{2c}+\mathcal{O}\left(\frac{X^{2c}}{(\log X)^{B-5}}\right)\,. \tag{68}$$

where $\mathfrak{S}_\Gamma$ is defined by (5).

From (54), (55) and (68) we obtain

$$\Gamma_c^{(1)}(X)=\frac{(2^c-1)^2}{c2^{2c+3}}\mathfrak{S}_\Gamma X^{2c}+\mathcal{O}\left(\frac{X^{2c}}{(\log X)^{B/2-6}}\right)\,. \tag{69}$$

8 Proof of the Theorem

Therefore using (9), (48), (53) and (69), we find

$$\Gamma_c(X) = \frac{(2^c-1)^2}{c2^{2c+1}}\mathfrak{S}_\Gamma X^{2c} + \mathcal{O}\left(X^{2c}(\log X)^{-\theta_0}(\log\log X)^6\right) .$$

This implies that $\Gamma(X) \to \infty$ as $X \to \infty$.

The theorem is proved. □

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