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On a curious biconditional involving the divisors of odd perfect numbers

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Abstract: We investigate the implications of a curious biconditional involving the divisors of odd perfect numbers, if Dris conjecture that $q^k < n$ holds, where $q^k n^2$ is an odd perfect number with Euler prime q. We then show that this biconditional holds unconditionally. Lastly, we prove that the inequality $q < n$ holds unconditionally.
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1 Introduction

If N is a positive integer, then we write $\sigma(N)$ for the sum of the divisors of N. A number N is *perfect* if $\sigma(N) = 2N$. We denote the abundancy index I of the positive integer w as $I(w) = \sigma(w)/w$. We also denote the deficiency D of the positive integer x as $D(x) = 2x - \sigma(x)$ [12].

Euclid and Euler showed that an even perfect number E must have the form

$$E = (2^p - 1)\, 2^{p-1},$$

where $2^p - 1$ is a *Mersenne prime*. On the other hand, Euler showed that an odd perfect number O must have the form

$$O = q^k n^2,$$

where q is an *Euler prime* (i.e., $q \equiv k \equiv 1 \pmod 4$ and $\gcd(q, n) = 1$).

It is currently unknown whether there are any odd perfect numbers. On the other hand, only 49 even perfect numbers have been found, a couple of which were discovered by the Great Internet Mersenne Prime Search [9]. It is conjectured that there are infinitely many even perfect numbers, and that there are no odd perfect numbers.

Descartes, Frenicle and subsequently Sorli conjectured that $k = 1$ [2]. Sorli conjectured $k = 1$ after testing large numbers with eight distinct prime factors for perfection [14].

Dris conjectured in [5] and [6] that the divisors q^k and n are related by the inequality $q^k < n$. Brown [3] and Starni [13] have recently uploaded preprints claiming a proof for the weaker inequality $q < n$.

Holdener presented some conditions equivalent to the existence of odd perfect numbers in [10]. In this paper, we prove the following results:

Lemma 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then the sum*

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above if and only if the sum

$$\frac{q^k}{n} + \frac{n}{q^k}$$

is bounded from above.

The following lemma is proved in the preprint [4]. (We will not need to use this result in the present paper. Hence, we will not be proving this lemma here.)

Lemma 1.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q and $3 \nmid N$, then $\sigma(q) \neq \sigma(n)$.*

Using Lemma 1.1, we are able to prove the following unconditional result.

Theorem 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $\sigma(q^k) \neq \sigma(n)$.*

Lemma 1.3. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then the inequality*

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

holds if and only if the biconditional

$$q^k < n \iff \sigma(n) < \sigma(q^k)$$

holds.

The following result is trivial. The proof is easy, and is left for the interested reader.

Lemma 1.4. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then either $q^k < n$, $\sigma(q^k) < n$ or $\sigma(n) < q^k$ imply that the biconditional*

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

holds.

The following corollary follows easily from Theorem 1.1 and Lemma 1.3.

Corollary 1.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then the biconditional*

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}$$

holds.

All of the proofs given in this note are elementary.

2 Preliminaries

Let $N = q^k n^2$ be an odd perfect number with Euler prime q.

First, we show that the following equations hold. (The proof is taken from the paper [7].) This will serve as motivation for trying to prove the inequality $q^k < n$ or the stronger inequality $\sigma(q^k) < n$.

Lemma 2.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then*

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$

Proof. Since $N = q^k n^2$ is an odd perfect number, we have

$$\sigma(q^k)\sigma(n^2) = \sigma(N) = 2N = 2q^k n^2,$$

from which it follows that $q^k \mid \sigma(n^2)$ (because $\gcd\left(q^k, \sigma(q^k)\right) = 1$). Hence,

$$\frac{\sigma(n^2)}{q^k} = \frac{\sigma(N/q^k)}{q^k}$$

is an integer.

First, we prove that

$$\frac{D(n^2)}{\sigma(q^{k-1})} = \frac{\sigma(N/q^k)}{q^k}.$$

We rewrite the equation

$$\sigma(q^k)\sigma(n^2) = 2q^k n^2$$

as

$$\left(q^k + \sigma(q^{k-1})\right)\sigma(n^2) = 2q^k n^2$$
$$\sigma(q^{k-1})\sigma(n^2) = q^k\left(2n^2 - \sigma(n^2)\right) = q^k \cdot D(n^2)$$
$$\frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})},$$

and we are done.

Next, we show that

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})}.$$

We already know that

$$\sigma(n^2) = q^k \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right).$$

Since $\sigma(q^k)\sigma(n^2) = 2q^k n^2$, we also obtain

$$\frac{2n^2}{\sigma(q^k)} = \frac{\sigma(n^2)}{q^k} = \frac{D(n^2)}{\sigma(q^{k-1})}.$$

This implies that

$$n^2 = \frac{\sigma(q^k)}{2} \cdot \left(\frac{D(n^2)}{\sigma(q^{k-1})}\right).$$

It follows that

$$\gcd\left(n^2, \sigma(n^2)\right) = \frac{D(n^2)}{\sigma(q^{k-1})}$$

since

$$\gcd\left(q^k, \frac{\sigma(q^k)}{2}\right) = \gcd(q^k, \sigma(q^k)) = 1.$$

This concludes the proof. □

Remark 2.1. *Dris obtained the lower bound 3 for $\sigma(N/q^k)/q^k$ in [5] and [6].*

Remark 2.2. *Notice that*

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} > \frac{8}{5} \cdot \left(\frac{n^2}{q^k}\right)$$

since $I(q^k) < 5/4$ holds unconditionally (i.e., for $k \geq 1$). Additionally, note that

$$\frac{8}{5} \cdot \left(\frac{n^2}{q^k}\right) > \frac{8n}{5}$$

is true if $q^k < n$. Furthermore, note that we then have the estimate $n > \sqrt[3]{N}$.

Lastly, note that we have

$$\frac{\sigma(n^2)}{q^k} = \frac{2n^2}{\sigma(q^k)} > 2n > \sigma(n)$$

if the stronger inequality $\sigma(q^k) < n$ holds.

3 The proof of Lemma 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q. We want to show that the sum

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above if and only if the sum

$$\frac{q^k}{n} + \frac{n}{q^k}$$

is bounded from above.

To this end, note that we have the trivial inequalities

$$q^k < \sigma(q^k) < 2q^k$$

and

$$n < \sigma(n) < 2n$$

since both q^k and n are greater than one, and because q^k and n are deficient (being proper divisors of the perfect number $N = q^k n^2$. These two sets of inequalities imply that

$$\frac{q^k}{n} < \frac{\sigma(q^k)}{n} < 2 \cdot \frac{q^k}{n}$$

and

$$\frac{n}{q^k} < \frac{\sigma(n)}{q^k} < 2 \cdot \frac{n}{q^k}$$

so that we obtain

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 \cdot \left(\frac{q^k}{n} + \frac{n}{q^k}\right).$$

First, we show that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \text{ is bounded from above } \implies \frac{q^k}{n} + \frac{n}{q^k} \text{ is bounded from above.}$$

Suppose that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above. This implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \leq C_1$$

for some absolute constant C_1. But since

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

this implies that

$$\frac{q^k}{n} + \frac{n}{q^k} < \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \leq C_1$$

which means that

$$\frac{q^k}{n} + \frac{n}{q^k} < C_1.$$

We conclude that

$$\frac{q^k}{n} + \frac{n}{q^k}$$

is bounded from above.

Next, we prove that

$$\frac{q^k}{n} + \frac{n}{q^k} \text{ is bounded from above} \implies \frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} \text{ is bounded from above.}$$

Suppose that

$$\frac{q^k}{n} + \frac{n}{q^k}$$

is bounded from above. This implies that

$$\frac{q^k}{n} + \frac{n}{q^k} \leq C_2$$

for some absolute constant C_2. But since

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 \cdot \left(\frac{q^k}{n} + \frac{n}{q^k}\right)$$

this implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 \cdot \left(\frac{q^k}{n} + \frac{n}{q^k}\right) \leq 2C_2$$

which means that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2C_2.$$

We conclude that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above.

This finishes the proof of Lemma 1.1.

Remark 3.1. *In general, the function $f(z) = z + (1/z)$ is not bounded from above. (To see why, it suffices to consider the cases $z \to 0^+$ and $z \to \infty$.)*

This means that we do not expect the sum

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

to be bounded from above.

4 The proof of Theorem 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q. We want to show that $\sigma(q^k) \neq \sigma(n)$.

Suppose to the contrary that $\sigma(q^k) = \sigma(n)$. Then we obtain

$$\frac{\sigma(q^k)}{q^k} = \frac{\sigma(n)}{q^k}$$

and

$$\frac{\sigma(n)}{n} = \frac{\sigma(q^k)}{n}$$

from which it follows that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} = \frac{\sigma(q^k)}{q^k} + \frac{\sigma(n)}{n} = I(q^k) + I(n) < I(q^k) + I(n^2).$$

But Dris proved in [5] and [6] that

$$I(q^k) + I(n^2) < 3$$

so that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 3.$$

This means that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above. This contradicts Lemma 1.1 (see Remark 3.1).

This finishes the proof of Theorem 1.1. □

Remark 4.1. *Similarly, we can show that $\sigma(n) \neq q^k$. For suppose to the contrary that $\sigma(n) = q^k$.*

Then we have

$$2 > \frac{\sigma(q^k)}{n} \cdot \frac{\sigma(n)}{q^k} = \frac{\sigma(q^k)}{n}$$

since $q^k n$ is deficient (being a proper divisor of the perfect number $N = q^k n^2$). But this implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} = \frac{\sigma(q^k)}{n} + 1 < 3$$

from which it follows that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above. This contradicts Lemma 1.1 (see Remark 3.1).

5 The proof of Lemma 1.3

Let $N = q^k n^2$ be an odd perfect number with Euler prime q. We want to show that the inequality

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

holds if and only if the biconditional

$$q^k < n \iff \sigma(n) < \sigma(q^k)$$

holds.

To this end, observe that we have the series of biconditionals

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n) \iff q^k \sigma(q^k) + n\sigma(n) < n\sigma(q^k) + q^k \sigma(n)$$

$$\iff \Big(q^k - n\Big)\sigma(q^k) + \Big(n - q^k\Big)\sigma(n) < 0 \iff \Big(q^k - n\Big) \cdot \Big(\sigma(q^k) - \sigma(n)\Big) < 0$$

$$\iff \Big(q^k < n \implies \sigma(n) < \sigma(q^k)\Big) \land \Big(n < q^k \implies \sigma(q^k) < \sigma(n)\Big)$$

$$\iff \Big(q^k < n \iff \sigma(n) < \sigma(q^k)\Big).$$

Notice that we have used the facts that $q^k \neq n$ (since $\gcd(q, n) = 1$) and $\sigma(q^k) \neq \sigma(n)$ (from Theorem 1.1) as underlying assumptions throughout.

This finishes the proof of Lemma 1.3. □

6 The proof of Corollary 1.1

Let $N = q^k n^2$ be an odd perfect number with Euler prime q. We want to give an unconditional proof for the truth of the biconditional

$$q^k < n \iff \sigma(q^k) < \sigma(n) \iff \frac{\sigma(q^k)}{n} < \frac{\sigma(n)}{q^k}.$$

It suffices to show only the first biconditional

$$q^k < n \iff \sigma(q^k) < \sigma(n).$$

We consider three cases:

Case 1

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < I(q^k) + I(n)$$

We know (from [5] and [6]) that $I(q^k) + I(n) < I(q^k) + I(n^2) < 3$. This implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$

is bounded from above, which contradicts Lemma 1.1.

Case 2

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} = I(q^k) + I(n)$$

This equation is equivalent to

$$\Big(q^k - n\Big) \cdot \Big(\sigma(q^k) - \sigma(n)\Big) = 0.$$

Since $q^k \neq n$, we must have $\sigma(q^k) = \sigma(n)$, contradicting Theorem 1.1.

Case 3

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} > I(q^k) + I(n)$$

This is equivalent to the inequality

$$\Big(q^k - n\Big) \cdot \Big(\sigma(q^k) - \sigma(n)\Big) > 0,$$

which in turn is equivalent to the truth of the biconditional

$$q^k < n \iff \sigma(q^k) < \sigma(n).$$

This finishes the proof of Corollary 1.1. $\square$

7 Concluding remarks

Since $q^k n$ is deficient if $N = q^k n^2$ is an odd perfect number, then $I(q^k n) < 2$. This implies that

$$\frac{1}{2} \cdot \frac{\sigma(q^k)}{n} < \frac{q^k}{\sigma(n)}$$

and

$$\frac{1}{2} \cdot \frac{\sigma(n)}{q^k} < \frac{n}{\sigma(q^k)},$$

from which it follows that

$$\frac{1}{2} \cdot \left(\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}\right) < \frac{q^k}{\sigma(n)} + \frac{n}{\sigma(q^k)}.$$

Since the arithmetic mean is never less than the harmonic mean, and since

$$\frac{\sigma(q^k)}{n} \neq \frac{\sigma(n)}{q^k}$$

(see [8] for a proof of this inequation and some related considerations), then we have

$$\frac{2}{\frac{n}{\sigma(q^k)} + \frac{q^k}{\sigma(n)}} = \frac{2}{\frac{1}{\sigma(q^k)/n} + \frac{1}{\sigma(n)/q^k}} < \frac{1}{2} \cdot \left(\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}\right),$$

from which we obtain

$$\frac{2}{\frac{n}{\sigma(q^k)} + \frac{q^k}{\sigma(n)}} < \frac{q^k}{\sigma(n)} + \frac{n}{\sigma(q^k)}.$$

We conclude that

$$\sqrt{2} < \frac{q^k}{\sigma(n)} + \frac{n}{\sigma(q^k)}.$$

We now claim that either

$$\frac{\sigma(q^k)}{n} < \sqrt{2} < \frac{\sigma(n)}{q^k}$$

or

$$\frac{\sigma(n)}{q^k} < \sqrt{2} < \frac{\sigma(q^k)}{n}$$

holds. (It suffices to prove one inequality, as the proof for the other one is very similar.)

To this end, assume that

$$\sqrt{2} < \frac{\sigma(n)}{q^k}.$$

This implies that
$$\sqrt{2} \cdot \frac{\sigma(q^k)}{n} < \frac{\sigma(q^k)}{n} \cdot \frac{\sigma(n)}{q^k} = I(q^k n) < 2,$$
which finally gives
$$\frac{\sigma(q^k)}{n} < \frac{2}{\sqrt{2}} = \sqrt{2} < \frac{\sigma(n)}{q^k}.$$

This proves our claim.

We now consider whether the following further refinements are possible:

Case A
$$1 < \frac{\sigma(q^k)}{n} < \sqrt{2} < \frac{\sigma(n)}{q^k} < 2$$
In this case,
$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 + \sqrt{2}$$
so that
$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$
is bounded from above. This contradicts Lemma 1.1.

Case B
$$1 < \frac{\sigma(n)}{q^k} < \sqrt{2} < \frac{\sigma(q^k)}{n} < 2$$
Similarly, in this case,
$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} < 2 + \sqrt{2}$$
so that
$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k}$$
is bounded from above. This contradicts Lemma 1.1.

Consequently, since $\sigma(q^k) \neq n$ (because $\sigma(q^k) \equiv k + 1 \equiv 2 \pmod 4$) and $\sigma(n) \neq q^k$, then we either have
$$\sigma(q^k) < n$$
or
$$\sigma(n) < q^k.$$

Remark 7.1. *The result in Corollary 1.1 together with the main findings in the preprint [4] shows that*
$$3 \nmid q^k n^2 \implies q < n.$$
This conclusion is derived independently of Brown's and Starni's methods.

Remark 7.2. *By Corollary 1.1, if $N = q^k n^2$ is an odd perfect number with Euler prime q, then there are a total of four cases to consider:*

Case $\alpha : q^k < \sigma(q^k) < n < \sigma(n)$

Case $\beta : q^k < n < \sigma(q^k) < \sigma(n)$

$$\textit{Case } \gamma : n < q^k < \sigma(n) < \sigma(q^k)$$

$$\textit{Case } \delta : n < \sigma(n) < q^k < \sigma(q^k)$$

Note that Cases β and γ imply that $k \neq 1$. Also, from previous considerations, we know that $n < \sigma(q^k)$ and $q^k < \sigma(n)$ cannot both be true. Consequently, Cases β and γ do not hold.

We are left with the scenarios:

$$\textit{Case } \alpha : q^k < \sigma(q^k) < n < \sigma(n)$$

$$\textit{Case } \delta : n < \sigma(n) < q^k < \sigma(q^k)$$

It turns out we can dispose of Case δ when $k = 1$. We obtain

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} = \frac{\sigma(q)}{n} + \frac{\sigma(n)}{q} < \left(\sqrt{3} + (\sqrt[6]{3} \cdot 10^{-500})\right) + 1,$$

where the estimate

$$\frac{\sigma(q)}{n} < \sqrt{3} + (\sqrt[6]{3} \cdot 10^{-500})$$

uses Acquaah and Konyagin's estimate $q < n\sqrt{3}$ [1] and Ochem and Rao's lower bound $N > 10^{1500}$ for the magnitude of an odd perfect number [11]. This implies that

$$\frac{\sigma(q^k)}{n} + \frac{\sigma(n)}{q^k} = \frac{\sigma(q)}{n} + \frac{\sigma(n)}{q}$$

is bounded from above, which contradicts Lemma 1.1.

Consequently, $k \neq 1$ must hold in Case δ. From the papers [6] and [8], this implies that $q < n$.

Since $\sigma(q^k) < n$ holds in Case α, and since $q \leq q^k < \sigma(q^k)$, we also have $q < n$ under Case α.

We summarize the results we proved in Remark 7.2 in the following theorems.

Theorem 7.1. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $q < n$ holds unconditionally.*

Theorem 7.2. *If $N = q^k n^2$ is an odd perfect number with Euler prime q, then $k = 1$ implies $\sigma(q^k) < n$.*

8 Further research

Let $N = q^k n^2$ be an odd perfect number with Euler prime q. Suppose that the Descartes–Frenicle–Sorli conjecture that $k = 1$ is true.

By Theorem 7.2 and Lemma 1.1, $q + 1 = \sigma(q) = \sigma(q^k) < n$, so that we then have a further refinement of the following bounds (see the paper [8]):

$$\frac{\sigma(q)}{n} < 1 < I(q) \leq \frac{6}{5} < \left(\frac{5}{3}\right)^{\frac{\ln(4/3)}{\ln(13/9)}} < I(n) < 2 < \frac{\sigma(n)}{q}.$$

Again, by Lemma 1.1, if $k = 1$ then the ratio

$$\frac{\sigma(n)}{q^k} = \frac{\sigma(n)}{q}$$

is not bounded from above. This implies that the ratio

$$\frac{\sigma(q^k)}{n} = \frac{\sigma(q)}{n}$$

is not bounded from below. This means that we can take $\sigma(q)/n$ to be arbitrarily small, from which we conclude that q has to be vastly smaller than n.

These considerations beg answers to several (obvious) questions, which we leave for other researchers to investigate.

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