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Non-split domination subdivision critical graphs

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Abstract: A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with v adjacent to u. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$. A *dominating set* D of a graph $G = (V, E)$ is a *non-split dominating set* if the induced graph $\langle V - D \rangle$ is connected. The *non-split domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *non-split domination set*. The purpose of this paper is to initiate the investigation of those graphs which are critical in the following sense: A graph G is called *vertex domination critical* if $\gamma(G - v) < \gamma(G)$ for every vertex v in G. A graph G is called *vertex non-split critical* if $\gamma_{ns}(G - v) < \gamma_{ns}(G)$ for every vertex v in G. Thus, G is k-γ_{ns}-*critical* if $\gamma_{ns}(G) = k$, for each vertex $v \in V(G)$, $\gamma_{ns}(G - v) < k$. A graph G is called *edge domination critical* if $\gamma(G + e) < \gamma(G)$ for every edge e in $\overline{G}$. A graph G is called *edge non-split critical* if $\gamma_{ns}(G + e) < \gamma_{ns}(G)$ for every edge $e \in \overline{G}$. Thus, G is k-γ_{ns}-*critical* if $\gamma_{ns}(G) = k$, for each edge $e \in \overline{G}$, $\gamma_{ns}(G + e) < k$. First we have constructed a bound for a non-split domination number of a subdivision graph $S(G)$ of some particular classes of graph in terms of vertices and edges of a graph G. Then we discuss whether these particular classes of subdivision graph $S(G)$ are γ_{ns}-critical or not with respect to vertex removal and edge addition.

Keywords: Domination number, Non-split domination, Non-split domination number, Critical graph, Subdivision graph, Vertex critical, Edge critical.
AMS Classification: 05C69.

1 Introduction

In this paper all our graphs will be finite, connected , undirected and without loops or multiple edges such that $G - v$ should not a null graph for vertex removal. Terminology not defined here will conform to that in [2]. Let $P_n, C_n, K_{1,n}, K_n, K_{m,n}$ denote the *path, cycle, star, complete and bipartite graph.*

An *end vertex* in a graph G is a vertex of *deg 1* and *support vertex* is a vertex which is adjacent to an *end vertex*.

A *subdivision* of an edge $e = uv$ of a graph G is the replacement of an edge e by a path (u, v, w) where $w \notin V(G)$. The graph obtained from G by subdividing each edge of G exactly once is called the *subdivision graph* of G and it is denoted by $S(G)$. The neighborhood of a vertex in the graph G is the set of vertices adjacent to v. The neighborhood is denoted by $N(v)$.

A set of vertices S is said to *dominate* the graph G if for each $v \notin S$, there is a vertex $u \in S$ with v adjacent to u. The minimum cardinality of any *dominating set* is called the *domination number* of G and is denoted by $\gamma(G)$.

The concept of non-split domination has been studied by V. R. Kulli and B. Janakiram [3]. A *dominating set* D of a graph $G = (V, E)$ is a *non-split dominating set* if the induced graph $\langle V - D \rangle$ is *connected*. The *non-split domination number* $\gamma_{ns}(G)$ is the minimum cardinality of a *non-split domination set*. The concept of γ-*critical* graphs has been studied by Sumner and Blitch [1] and Sumner [6].

In this paper, we study the *non-split domination critical* graphs. A graph G is called *vertex non-split critical* if $\gamma_{ns}(G - v) < \gamma_{ns}(G)$ for every vertex v in G. Thus, G is k-γ_{ns}-*critical* if $\gamma_{ns}(G) = k$, for each vertex $v \in V(G)$, $\gamma_{ns}(G - v) < k$. A graph G is called *edge domination critical* if $\gamma(G + e) < \gamma(G)$ for every edge $e \in \overline{G}$. A graph G is called *edge non-split critical* if $\gamma_{ns}(G + e) < \gamma_{ns}(G)$ for every edge $e \in \overline{G}$. Thus, G is k-γ_{ns}-*critical* if $\gamma_{ns}(G) = k$, for each edge $e \in \overline{G}$, $\gamma_{ns}(G + e) < k$.

First we have obtained a γ_{ns} set for some particular classes of subdivision graph $S(G)$ in terms of vertices and edges of a graph G. Then we have discuss whether these particular classes of subdivision graph $S(G)$ are γ_{ns} critical or not with respect to vertex removal and edge addition.

2 Construction of γ_{ns} set for a particular classes of graph

2.1 Construction of γ_{ns} set of a subdivision of a complete graph $S(K_n)$

STEP 1: To cover all the vertices that subdivides $E(K_n)$, we require minimum $n - 1$ vertices of K_n which will not cover n^{th} vertex of K_n.

STEP 2: Removal of these $n - 1$ vertices from $S(K_n)$ makes the graph $S(K_n)$ disconnected in which $\frac{(n-1)(n-2)}{2}$ components are K_1 and another component $K_{1,n-1}$. Therefore in γ_{ns} set contains $\frac{(n-1)(n-2)}{2}$ vertices from each of $\frac{(n-1)(n-2)}{2}$ components of K_1.

STEP 3: Now to cover $K_{1,n-1}$ vertex, we need a vertex of $(V(S(K_n)) - V(K_n)) \cap V(K_{1,n-1})$.

Therefore, $\gamma_{ns}(S(K_n)) = (n-1) + \frac{(n-1)(n-2)}{2} + 1.$
$= (n-1)(1 + \frac{n-2}{2}) + 1.$
$= (n-1)(\frac{2+n-2}{2}) + 1.$
$= \frac{(n)(n-1)}{2} + 1.$

2.2 Construction of γ_{ns} set of a subdivision of a bipartite graph $S(K_{m,n}), m \geq n$

STEP 1: To cover all the vertices that subdivides $E(K_{m,n})$, we require m number of vertices of $K_{m,n}$.

STEP 2: Removal of these m vertices from $S(K_{m,n})$ makes the graph $S(K_{m,n})$ disconnected into n number of components of $K_{1,m}$ say $G_1, G_2.G_3, \ldots, G_n$.

STEP 3: Therefore γ_{ns} set contains $V(G_1) \cup V(G_2) \cup V(G_3) \cup \cdots \cup V(G_{n-1})$.

STEP 4: Now to cover n^{th} component $K_{1,m}$, we need one vertex of $V(G_n)$ such that $K_{1,m}$ is connected.

Therefore, $\gamma_{ns}(S(K_{m,n})) = m + (m+1)(n-1) + 1.$
$= m + mn - m + n - 1 + 1.$
$= n(1+m).$

2.3 Construction of γ_{ns} set of a subdivision of a wheel graph $S(W_n)$

STEP 1: To cover all the vertices that subdivides $E(W_{m,n})$, we require minimum of $n-1$ vertices of W_n not containing the vertex of degree $n-1$.

STEP 2: Removal of these $n-1$ vertices from $S(W_n)$ makes the graph $S(W_n)$ disconnected in which $n-1$ components are of K_1 and one component of $K_{1,n-1}$.

STEP 3: Therefore in γ_{ns} set contains $n-1$ vertices of K_1.

STEP 4: Now to cover $K_{1,n-1}$, we need one vertex of $(V(S(K_n)) - V(K_n)) \cap V(K_{1,n-1})$.

Therefore, $\gamma_{ns}(S(W_n)) = (n-1) + (n-1) + 1.$
$= 2(n-1) + 1.$
$= 2n - 1.$

3 Non-split vertex domination of a subdivision critical graph

Theorem 3.1. *The graph $S(K_n)$ is non-split vertex critical $n \geq 3$.*

Proof. Let D be the γ_{ns} set of $S(K_n)$ and let $|V(S(K_n)| = \frac{(n)(n-1)}{2} + n$. Let $A = V(S(K_n)) - V(K_n)$ and $B = \{v_r/v_r \in V(S(K_n)) - D\}$. we consider the following cases:

Case 1: Let $v \in V(K_n)$ and $v \notin N(B)$, then $\gamma_{ns}(S(K_n) - v) = |D| - |v| = \gamma_{ns}(S(K_n)) - 1$.

Otherwise $v \in N(B)$ then,

$\gamma_{ns}(S(K_n) - v) = |V(S(K_n))| - |v_j| - |K| + |v_s| - |v|$, where $v_j \in V(K_n), v_j \neq v, K = \{v_m \in A/v_m \in N(v_j)\}$ with $|K| = n - 1$ and $v_s \in K \cap N(v)$.

$$=(\frac{(n)(n-1)}{2} + n) - 1 - (n-1) + 1 - 1.$$

$$=\frac{(n)(n-1)}{2} + 1 + (n-1) - (n-1) - 1.$$

$$=\gamma_{ns}(S(K_n)) - 1.$$

Case 2: Let $v \in A$ and $v \notin N(B)$ then, $\gamma_{ns}(S(K_n) - v) = |D| - |v|$.

$$=(\frac{(n)(n-1)}{2} + 1) - 1.$$

$$=\gamma_{ns}(S(K_n)) - 1.$$

Otherwise $v \in N(B)$ then,

$\gamma_{ns}(S(K_n) - v) = |V(S(K_n))| - |v_j| - |K| + |v_s| - |v|$, where $v_j \in V(K_n), v_j \neq N(v), K = \{v_m \in A/v_m \in N(v_j)\}$ with $|K| = n - 1$ and $v_s \in K$.

$$=(\frac{(n)(n-1)}{2} + n) - 1 - (n-1) + 1 - 1.$$

$$=\frac{(n)(n-1)}{2} + 1 + (n-1) - (n-1) - 1.$$

$$=\gamma_{ns}(S(K_n)) - 1.$$

From Case(1) and Case(2), we have $\gamma_{ns}(S(K_n) - v) < \gamma_{ns}(S(K_n))$, therefore $S(K_n)$ is vertex non-split critical $n \geq 3$. □

Lemma 3.2. *The graph $S(C_n)$ is non-split vertex critical for $n \geq 3$.*

Lemma 3.3. *The graph $S(P_n)$ is not non-split vertex critical for $n \geq 5$ and not nonsplit vertex critical for $n < 5$.*

Lemma 3.4. *The graph $S(T)$,$T \neq P_n$ is not a non-split vertex critical for $n \geq 3$.*

Theorem 3.5. *The graph $S(K_{m,n})$ is non-split vertex critical for $m \geq n$, $m, n \geq 2$.*

Proof. Let $V(K_{m,n}) = V_1 \cup V_2$ where $|V_1| = m$, $|V_2| = n$. Let D be the γ_{ns} set of $S(K_{m,n})$,$A = V(S(K_{m,n})) - V(K_{m,n})$ and $B =< V(S(K_{m,n}) - D >$. We consider the following cases.

Case 1: Let $v \in V_2$ and if $v \in N(B)$, then $\gamma_{ns}(S(K_{m,n}) - v) = |D| - |v|$.

$$= \gamma_{ns}(S(K_{m,n})) - 1.$$

Otherwise $v \notin N(B)$ then,

$\gamma_{ns}(S(K_{m,n}) - v) = |V(S(K_{m,n})| - |v_j| - |K| + |v_r| - |v|$, where $v_j \in V_1, K = \{v_p \in A/v_p \in N(v_j)\}$ with $|K| = m, v_r \in K \cup N(v)$.

$$=m + n + mn - 1 - m + 1 - 1.$$

$$=n(1 + m) - 1.$$

$$=\gamma_{ns}(S(K_{m,n})) - 1.$$

Case 2: Let $v \in V_1 \cap D$ then, $\gamma_{ns}(S(K_{m,n}) - v) = |D| - |v|$.

$$= \gamma_{ns}(S(K_{m,n})) - 1.$$

Otherwise $v \in V_1, v \notin D$ then,
$\gamma_{ns}(S(K_{m,n}) - v) = |V(S(K_{m,n}))| - |v_j| - |K| + |v_r| - |v|$, where $v_j \neq v, v_j \in V_1, K = \{v_p \in A/v_p \in N(v_j)\}$ with $|K| = m, v_r \in K$.

$$=m + n + mn - 1 - m + 1 - 1.$$

$$=n(1 + m) - 1.$$

$$=\gamma_{ns}(S(K_{m,n})) - 1.$$

Case 3: Let $v \in A$ and $v \notin N(B)$ then, $\gamma_{ns}(S(K_{m,n}) - v) = |D| - |v|$.

$$= \gamma_{ns}(S(K_{m,n})) - 1.$$

Otherwise $v \in N(B)$ then,
$\gamma_{ns}(S(K_{m,n}) - v) = |V(S(K_{m,n}))| - |v_j| - |K| + |v_r| - |v|$, where $v_j \notin N(v), v_j \in V_1, K = \{v_p \in A/v_p \in N(v_j)\}$ with $|K| = m, v_r \in K$.

$$=m + n + mn - 1 - m + 1 - 1.$$

$$=n(1 + m) - 1.$$

$$=\gamma_{ns}(S(K_{m,n})) - 1.$$

From all the above cases, we have $\gamma_{ns}(S(K_{m,n}) - v) < \gamma_{ns}(S(K_{m,n}))$, therefore $S(K_{m,n})$ is vertex non-split critical. □

Theorem 3.6. *The graph $S(W_n)$ in not vertex non-split critical for $n \geq 5$ and vertex non-split critical for $n = 4$.*

Proof. Let D be the γ_{ns} set of $S(W_n)$ and $|V(S(W_n))| = 3n - 2$. Let $v_k \in V(W_n)$,$deg(v_k) = n-1$. Let $B = \{v_i/v_i \in V(S(W_n)) - v_k\}$ and $C = V(S(W_n)) - V(W_n)$ and $F =< V(S(K_n)) - D >$. we consider the following cases:

Case 1: Let $v \in B \cap D$ and $v \notin N(F)$ then $\gamma_{ns}(S(W_n) - v) = |D| - |v|$.

$$=\gamma_{ns}(S(W_n)) - 1.$$

Otherwise $v \in N(F)$ and $n \geq 5$ then, $\gamma_{ns}(S(W_n) - v) = |D| - |v| + |v_r| - |v_s|$. Where $v_s \in N(v_k) \cap D$ and covers $v_k, v_r \in N(v) \cap N(v_k)$.

$$=\gamma_{ns}(S(W_n)) - 1.$$

Otherwise for $n = 4$,$\gamma_{ns}(S(W_n) - v) = |D| - |v| + |v_r| - |v_s|$. Where $v_s \in N(F) \cap C$, $v_r \in N(v) \cap F$.

$$=\gamma_{ns}(S(W_n)) - 1.$$

For $n = 4$, $v \in B, v \notin D$ then, $\gamma_{ns}(S(W_4) - v) = |V(S(W_4))| - |v| - |v_j| - |K| + |v_s|$. Where $v_j \in \{B\} - \{v\}, K = \{v_r/v_r \in N(v_j)\}$ with $|K| = 3, v_s \in K \cap N(v)$.

$$=3n - 2 - 1 - 1 - 3 + 1.$$

$$=3n - 6.$$

Case 2: Let $v = v_k$ then, $\gamma_{ns}(S(W_n) - v) = |V(S(W_n))| - |v| - |v_j| - |K| + |v_m|$. Where $v_j \in B, K = \{v_l / v_l \in N(v_j)\}$ with $|K| = 3, v_m \in K \cap N(v_k)$.

$$=3n - 2 - 1 - 1 - 3 + 1.$$

$$=3n - 6.$$

Case 3: Let $v \in C, v \notin N(v_k)$ and if $v \notin N(F)$ then, $\gamma_{ns}(S(W_n) - v) = |D| - |v|$.

$$=\gamma_{ns}(S(W_n)) - 1.$$

Otherwise $v \in N(F)$ and $n = 4$ then,

$\gamma_{ns}(S(W_n) - v) = |V(S(W_n))| - |v| - |v_j| - |K| + |v_m|$. Where $v_j \in B, v_j \neq N(v), K = \{v_l / v_l \in N(v_j)\}$ with $|K| = 3, v_m \in K$.

$$=3n - 2 - 1 - 1 - 3 + 1.$$

$$=3n - 6.$$

Otherwise $v \in N(v_k)$, then $\gamma_{ns}(S(W_n) - v) = |V(S(W_n))| - |v| - |v_j| - |K| + |v_m|$. Where $v_j \in B, v_j \notin N(v), K = \{v_l / v_l \in N(v_j)\}$ with $|K| = 3, v_m \in K$.

$$=3n - 2 - 1 - 1 - 3 + 1.$$

$$=3n - 6.$$

From Case 1,Case 2 and Case 3:

(i) If $n = 4$, $\gamma_{ns}(S(W_n) - v) = 3n - 6 < 2n - 1 = \gamma_{ns}(S_(W_n))$.

(ii) If $n \geq 5$, $\gamma_{ns}(S(W_n) - v) = 3n - 6 \geq 2n - 1 = \gamma_{ns}(S_(W_n))$.

Hence the proof. □

4 Non-split edge domination of a subdivision critical graph

Theorem 4.1. *The graph $S(K_n)$ is edge non-split critical for $n \geq 3$.*

Proof. Let $C = V(S(K_n) - V(K_n)$. We consider the following cases.

Case 1: Let $e = v_1v_2 \in E(\overline{S(K_n)}), \{v_1, v_2\} \in V(K_n)$ then,
$\gamma_{ns}(S(K_n) + e) = |V(S(K_n))| - |v_1| - |K|$, where $K = \{v_s \in C / v_s \in N(v_1)\}$ with $|K| = n - 1$.

$$=\tfrac{(n)(n-1)}{2} + n - 1 - (n - 1).$$

$$=\tfrac{(n)(n-1)}{2} + 1 - 1.$$

$$=\gamma_{ns}(S(K_n)) - 1.$$

Case 2: Let $e = v_1v_2 \in E(\overline{S(K_n)}), \{v_1, v_2\} \in C, \{v_1, v_2\} \in N(v_k), v_k \in V(K_n)$ then,
$\gamma_{ns}(S(K_n) + e) = |V(S(K_n))| - |v_2| - |\{v_k, v_l\}| - |K| - |R| + |v_m|$. Where $\{v_k, v_l\} \in V(K_n) \cap N(v_2), v_k \in N(v_1), K = \{v_s \in C / v_s \neq (v_2, v_1), v_s \in N(v_k)\}, R = \{v_r \in C / v_r \neq v_2, v_r \in N(v_l)\}$, with $|K| = n - 3, |R| = n - 2, v_m \in R$.

$$=\tfrac{(n)(n-1)}{2} + n - 1 - 2 - (n - 3) - (n - 2) + 1.$$

$=\frac{(n)(n-1)}{2} - n + 3.$

$=\gamma_{ns}(S(K_n)) - (n-2).$

Case 3: Let $e = v_1v_2 \in E(\overline{S(K_n)}), v_1 \in V(K_n), v_2 \in C, \{v_1, v_2\} \notin N(v_k), v_k \in V(K_n)$ then, $\gamma_{ns}(S(K_n)+e) = |V(S(K_n))| - |v_2| - |\{v_k, v_l\}| - |K| - |R| + |\{v_m, v_n\}|$. Where $\{v_k, v_l\} \in V(K_{m,n}) \cap N(v_2), K = \{v_s \in C/v_s \neq v_2, v_s \in N(v_k)\}, R = \{v_r \in C/v_r \neq v_2, v_r \in N(v_l)\}$, with $|K| = n-2, |R| = n-2, v_m \in K, v_n \in R$.

$=\frac{(n)(n-1)}{2} + n - 1 - 2 - (n-2) - (n-2) + 2.$

$=\frac{(n)(n-1)}{2} - n + 3.$

$=\gamma_{ns}(S(K_n)) - (n-2).$

The result follows from Case(1),Case(2) and Case(3). □

Lemma 4.2. *The graph $S(C_n)$ is non-split edge critical for $n \geq 3$.*

Lemma 4.3. *The graph $S(P_n)$ is not non-split edge critical for $n \geq 3$.*

Lemma 4.4. *The graph $S(T)$ is not non-split edge critical,if $T \neq K_{1,n}$.*

Theorem 4.5. *The graph $S(K_{m,n})$ is not edge critical for $m > n$ and edge critical for $m = n$ where $m, n \geq 2$.*

Proof. Let $V(K_{m,n}) = V_1 \cup V_2$ where $|V_1| = m$, $|V_2| = n$. Let D be the γ_{ns} set of the graph $S(K_{m,n})$ and $C = V(S(K_{m,n})) - V(K_{m,n})$. We consider the following cases:

Case 1: Let $e = v_1v_2 \in E(\overline{S(K_{m,n})}), v_1 \in D, v_2 \notin D$ then, $\gamma_{ns}(S(K_{m,n})+e) = |D| - |v_r|, v_r \in N(v_2) \cap D.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

Case 2: Let $e = v_1v_2 \in E(\overline{S(K_{m,n})}), \{v_1, v_2\} \in V_2$ or $v_1 \in C, v_2 \in V_2$ and suppose $m > n$ then, $\gamma_{ns}(S(K_{m,n})+e) = |V(S(K_{m,n}))| - |v_j| - |K| + |v_m|, v_j \in V_1, K = \{v_s \in C/v_s \in N(v_j)\}$, with $|K| = m, v_m \in K$.

$= mn + m + n - 1 - m + 1.$

$= n(m+1).$

$= \gamma_{ns}(S(K_{m,n})).$

Suppose $m = n$ then,
$\gamma_{ns}(S(K_{m,n})+e) = |V(S(K_{m,n}))| - |v_2| - |K|, K = \{v_s \in C/v_s \in N(v_2)\}$ with $|K| = n$.

$= mn + m + n - 1 - n.$

$= mn + m + n - 1 - m$(since $m = n$).

$= n(m+1) - 1.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

Case 3: Let $e = v_1v_2 \in E(\overline{S(K_{m,n})}), \{v_1, v_2\} \in V_1$ then,
$\gamma_{ns}(S(K_{m,n}) + e) = |V(S(K_{m,n}))| - |v_1| - |K|, K = \{v_s \in C/v_s \in N(v_1)\}$ with $|K| = m$.

$= mn + m + n - 1 - m.$

$= n(m + 1) - 1.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

Case 4: Let $e = v_1v_2 \in E(\overline{S(K_{m,n})}), v_1 \in V_2, v_2 \in V_1$ or $v_1 \in C, v_2 \in V_1$ then,
$\gamma_{ns}(S(K_{m,n}) + e) = |V(S(K_{m,n}))| - |v_2| - |K|, K = \{v_s \in C/v_s \in N(v_2)\}$ with $|K| = m$.

$= mn + m + n - 1 - m.$

$= n(m + 1) - 1.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

Case 5: Let $e = v_1v_2 \in E(\overline{S(K_{m,n})}), \{v_1, v_2\} \in C$ and $\{v_1, v_2\} \in N(v_r), v_r \in V_1$ then,
$\gamma_{ns}(S(K_{m,n})+e) = |V(S(K_{m,n}))|-|v_r|-|K|+|v_1|-|v_m|, K = \{v_s \in C/v_s \in N(v_r)\}, v_m \in N(v_2) \cap V_2$ with $|K| = m$.

$= mn + m + n - 1 - m + 1 - 1.$

$= n(m + 1) - 1.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

Otherwise $v_1 \in N(v_r), v_2 \in N(v_s), v_r \neq v_s, (v_r, v_s) \in V_1$ then,
$\gamma_{ns}(S(K_{m,n})+e) = |V(S(K_{m,n}))|-|v_r|-|K|-|v_m|+|v_s|, K = \{v_s \in C/v_s \in N(v_r)\}, v_m \in N(v_1) \cap V_2, v_s \in K \neq v_1$ with $|K| = m$.

$= mn + m + n - 1 - m - 1 + 1.$

$= n(m + 1) - 1.$

$= \gamma_{ns}(S(K_{m,n})) - 1.$

The result follows from the above cases. □

Theorem 4.6. *The graph $S(W_n)$ in not edge non-split critical for $n \geq 5$ and edge non-split critical for $n = 4$.*

Proof. Let $|V(S(W_n))| = 3n - 2$ and $B = \{v_i/v_i \in V(W_n) - v_k\}$, where v_k is the vertex of degree $n - 1$ and $C = V(S(W_n)) - V(W_n)$. We consider the following cases:

Case 1: Let $e = v_1v_k \in E(\overline{S(W_n)}), v_1 \in B \cup C$ then,
$\gamma_{ns}(S(W_n) + e) = |V(S(W_n))| - |v_k| - |K|$ with $|K| = n - 1$

$=3n - 2 - 1 - (n - 1) = 2n - 2.$

Since $2n - 2 < 2n - 1$. Therefore $\gamma_{ns}(S(W_n) - v) < \gamma_{ns}(S(W_n))$.

Case 2: Let $e = v_1v_2 \in E(\overline{S(W_n)}), v_1 \in B \cup C, v_2 \in C, \{v_1, v_2\} \notin N(v_k)$ then,
$\gamma_{ns}(S(W_n) + e) = |V(S(W_n)) - |v_2| - |\{v_i, v_j\}| - |\{v_r, v_s\}|$. Where $\{v_i, v_j\} \in N(v_2) \cap B, (v_r, v_s) \in N(v_k) \cap N(v_i, v_j)$.

$=3n-2-1-2-2.$

$=3n-7.$

(i) If $n=4,5$ then $3n-7<2n-1$, therefore $\gamma_{ns}(S(W_n)+e)<\gamma_{ns}(S(W_n))$.

(ii) If $n\geq 6$ then $3n-7\geq 2n-1$, therefore $\gamma_{ns}(S(W_n)+e)\geq\gamma_{ns}(S(W_n))$.

Case 3: Let $e=v_1v_2\in E(\overline{S(W_n)}), \{v_1,v_2\}\in B$ and $n\geq 5$ then,
$\gamma_{ns}(S(W_n)+e)=|V(S(W_n))|-|v_k|-|K|+|v_m|$. Where $K=\{v_s\in C/v_s\in N(v_k)\}, v_m\in K$ with $|K|=n-1$.

$=3n-2-1-(n-1)+1.$

$=2n-1.$

$=\gamma_{ns}(S(W_n)).$

Otherwise for$n=4$ then, $\gamma_{ns}(S(W_n)+e)=|V(S(W_n))|-|v_1|-|K|$. Where $K=\{v_m\in C/v_m\in N(v_1)\}$.

$=3n-2-1-(n-1).$

$=3n-6.$

Since $3n-6<2n-1$ for $n=4$, therefore $\gamma_{ns}(S(W_n)+e)<\gamma_{ns}(S(W_n))$.

Case 4: Let $e=v_1v_2\in E(\overline{S(W_n)}), \{v_1,v_2\}\in B\cup C, v_1\notin N(v_k), v_2\in N(v_k)$ then,
$\gamma_{ns}(S(W_n)+e)=|V(S(W_n))|-|v_k|-|K|-|v_r|-|v_s|+|v_m|$, Where $K=\{v_p\in C/v_p\in N(v_k)\cap C\}, v_r\in B\cap N(v_2), v_s\in N(v_r)\cap C, v_s\neq(v_1,v_2), v_m\neq v_2, v_m\in K$ with $|K|=n-1$.

$=3n-2-1-(n-1)-1-1+1.$

$=2n-3.$ Otherwise $v_1\in C\cap N(v_k)$ then,

$\gamma_{ns}(S(W_n)+e)=|V(S(W_n))|-|v_k|-|K|-|v_r|-|v_s|+|v_1|$, Where $K=\{v_p\in C/v_p\in N(v_k)\cap C\}, v_r\in B\cap N(v_2), v_s\in N(v_r)\cap C, v_s\neq v_2$ with $|K|=n-1$.

$=3n-2-1-(n-1)-1-1+1.$

$=2n-3.$

Since $2n-3<2n-1$, therefore $\gamma_{ns}(S(W_n)-v)<\gamma_{ns}(S(W_n))$.

The result follows from the above cases. □

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