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On quasiperfect numbers

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Abstract: A natural number N is said to be *quasiperfect* if $\sigma(N) = 2N + 1$ where $\sigma(N)$ is the sum of the positive divisors of N. No quasiperfect number is known. If a quasiperfect number N exists and if $\omega(N)$ is the number of distinct prime factors of N then G. L. Cohen has proved $\omega(N) \geq 7$ while H. L. Abbott et al. have shown $\omega(N) \geq 10$ if $(N, 15) = 1$. In this paper we first prove that every quasiperfect numbers N has an odd number of *special factors* (see Definition 2.3 below) and use it to show that $\omega(N) \geq 15$ if $(N, 15) = 1$ which refines the result of Abbott et al. Also we provide an alternate proof of Cohen's result when $(N, 15) = 5$.
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1 Introduction

For any natural number N, let $\sigma(N)$ denote the sum of its positive divisors. A number N is called *abundant*, *perfect* or *deficient* if $\sigma(N) > 2N$, $\sigma(N) = 2N$ or $\sigma(N) < 2N$ respectively. It is well known that there are infinitely many abundant numbers and infinitely many deficient numbers. In [7] Sierpinski asks whether there is at least one abundant number satisfying

$$\sigma(N) = 2N + 1, \tag{1.1}$$

for which there is no definite answer till date. P. Cattaneo [2] called any N satisfying (1.1) *quasiperfect*, and initiated a study of such numbers. Later H. L. Abbott, C. E. Aull, Ezra Brown and D. Suryanarayana [1] continued investigations on quasiperfect numbers and proved the following: If a quasiperfect number N exists and if $\omega(N)$ is the number of distinct prime factors of N, then

$$\omega(N) \geq 5 \quad \text{([1] , Theorem 2)} \tag{1.2}$$

and

$$\omega(N) \geq 10 \text{ if } (N, 15) = 1 \quad \text{([1] , Theorem 5)} \tag{1.3}$$

M. Kishore [5] improved (1.2) to $\omega(N)) \geq 6$ while G. L. Cohen and Peter Hagis Jr. [3] have obtained a refinement to it as

$$\omega(N) \geq 7 \tag{1.4}$$

Further details of research on quasiperfect numbers can be seen in the book by J. Sándor and B. Crstici ([6], pp. 38–39).

In this paper we first prove that if a quasiperfect number N exists then it has an odd number of *special factors* (defined in Section 2) and use it to refine (1.3) as

$$\omega(N) \geq 15 \text{ if } (N, 15) = 1 \tag{1.5}$$

and also provide an alternate proof of (1.4) in case $(N, 15) = 5$.

2 Preliminaries

P. Cattaneo [2] has proved the following:

Theorem 2.1. *If N is a quasiperfect number then it is of the form $N = p_1^{2e_1} p_2^{2e_2} ... p_t^{2e_t}$, where p_i are distinct odd primes. Also $e_i \equiv 0$ or $1 \pmod 4$ if $p_i \equiv 1 \pmod 8$; $e_i \equiv 0 \pmod 2$ if $p_i \equiv 3 \pmod 8$; $e_i \equiv 0$ or $-1 \pmod 4$ if $p_i \equiv 5 \pmod 8$ and $e_i \geq 1$ if $p_i \equiv 7 \pmod 8$. Further if M is a natural number for which $\sigma(M) \geq 2M$ then no non-trivial multiple of M is quasiperfect.*

Remark 2.2. It follows from the theorem that every quasiperfect number is the square of an odd integer while the last part of it shows that every quasiperfect number is *primitive abundant*, in the sense that it is an abundant number having no non-deficient number as a divisor.

In the canonical representation of a quasiperfect number each factor is of the form $p_i^{2e_i}$ where p_i is an odd prime, of which we consider the following special type of factors.

Definition 2.3. If p is an odd prime and $e \geq 1$ is an integer such that either $p \equiv 1 \pmod 8$ and $e \equiv 1 \pmod 4$ or $p \equiv 5 \pmod 8$ and $e \equiv -1 \pmod 4)$ then p^{2e} will be called a *special factor*.

For example, $5^6, 17^{10}$ and 13^{14} are special factors. Precisely the set of all special factors is given by $S = \{p^{2e} : [p \equiv 1 \pmod 8, e \equiv 1 \pmod 4] or [p \equiv 5 \pmod 8, e \equiv -1 \pmod 4]\}$.

Observe that $5^4, 17^8$ and 13^{16} are not special factors. Also if 3 divides a quasiperfect number then 3^{2e} is among its non-special factors; while if 5^{2e} is a factor of N then it is a special factor or a non-special factor according as $e \equiv 1 \pmod 4$ or $e \equiv 0 \pmod 4$. Further any factor p^{2e} of a quasiperfect number N is either a special factor or a non-special factor but not both.

3 Main results

First we prove the following

Definition 3.1. If a quasiperfect number N exists, then it has an odd number of special factors.

Proof. Suppose N is a quasiperfect number of the form

$$N = p_1^{2e_1} p_2^{2e_2} ... p_t^{2e_t}, \text{ where } p_i \text{ are odd primes.} \tag{3.2}$$

Then $p_i^2 \equiv 1 \pmod 8$ for $1 \leq i \leq t$.
Therefore

$$2N + 1 = 2.(p_1^2)^{e_1}(p_2^2)^{e_2}...(p_t^2)^{e_t} + 1 \equiv 2.1 + 1 \pmod 8 \equiv 3 \pmod 8. \tag{3.3}$$

Also for any i,

$$\begin{aligned} \sigma(p_i^{2e_i}) &= 1 + p_i + p_i^2 + ... + p_i^{2e_i} \\ &= (1 + p_i) + p_i^2(1 + p_i) + ... + p_i^{2(e_i-1)}(1 + p_i) + p_i^{2e_i} \\ &= (1 + p_i)(1 + p_i^2 + ... + p_i^{2(e_i-1)}) + p_i^{2e_i} \\ &\equiv (1 + p_i)e_i + 1 \pmod 8 \end{aligned}$$

so that

$$\sigma(p_i^{2e_i}) \equiv \begin{cases} 3 \pmod 8 & \text{if } p_i^{2e_i} \text{ is a special factor} \\ 1 \pmod 8 & \text{otherwise} \end{cases} \tag{3.4}$$

For example, if $p_i \equiv 5 \pmod 8$ and $e_i \equiv 3 \pmod 4$, say $p_i = 8u_i + 5$ and $e_i = 4v_i + 3$ then $(1 + p_i)e_i + 1 = (8u_i + 6)(4v_i + 3) + 1 \equiv 3 \pmod 8$. Also if $p_i \equiv 3 \pmod 8$ and $e_i \equiv 0 \pmod 2$ then $(1 + p_i)e_i + 1 = (8u_i' + 4)(2v_i') + 1 \equiv 1 \pmod 8$.
Hence

$$\sigma(N) = \prod_{i=1}^{t} \sigma(p_i^{2e_i}) \equiv 3^k \pmod 8, \tag{3.5}$$

where k is the number of special factors of N.

Now (3.3) and (3.5) give $3^k \equiv 3 \pmod 8$, which holds only if k is odd, thus proving the theorem. □

Remark 3.6. If N is a quasiperfect number of the form (3.2) then it follows from the theorem that not all e_i can be even showing that N cannot be the fourth power of a natural number. That is, no number of the form m^4 is quasiperfect. This result has been proved in [3] by a slightly different method.

Using Theorem 3.1 we now improve (1.3) as below:

Theorem 3.7. *If N is a quasiperfect number with $(N, 15) = 1$ then*

$$\omega(N) \geq 15.$$

Proof. Suppose N is the square of an odd integer of the form

$$N=\prod_{i=1}^{s}P_i^{2e_i}.\prod_{i=1}^{r}Q_j^{2f_j}, \tag{3.8}$$

where $P_i^{2e_i}$ are special factors, $Q_j^{2f_j}$ are non-special factors,

$$(P_i,Q_j)=1, P_1<P_2<...<P_s \text{ and } Q_1<Q_2<...<Q_r.$$

It is easy to see that

$$\frac{\sigma(N)}{N}=\prod_{i=1}^{s}\frac{\sigma(P_i^{2e_i})}{P_i^{2e_i}}.\prod_{i=1}^{r}\frac{\sigma(Q_j^{2f_j})}{Q_j^{2f_j}}<\pi^*.\pi^{**} \tag{3.9}$$

where $\pi^*=\prod_{i=1}^{s}\frac{P_i}{P_i-1}$ and $\pi^{**}=\prod_{j=1}^{r}\frac{Q_j}{Q_j-1}$

Now we introduce a notation: For any integer $k\geq 1$, if the k-tuples $(a_1,a_2,\ldots,a_k)$ and $(b_1,b_2,\ldots,b_k)$ of primes are such that $a_i\geq b_i$ for $i=1,2,\ldots,k$ then we write $(a_1,a_2,\ldots,a_k)\geq(b_1,b_2,\ldots,b_k)$. Clearly

$$\prod_{i=1}^{k}\frac{a_i}{a_i-1}\leq\prod_{i=1}^{k}\frac{b_i}{b_i-1} \text{ if } (a_1,a_2,\ldots,a_k)\geq(b_1,b_2,\ldots,b_k), \tag{3.10}$$

since$\frac{x}{x-1}$ is a decreasing function for $x>1$.

If N is of the form (3.8) with $(N,15)=1$, s odd and $\omega(N)\leq 14$ then we will prove that N is deficient and hence cannot be quasiperfect, so that the theorem follows.

It is enough to prove in the case $\omega(N)=14$. That is, $s+r=14$, with s odd and $(N,15)=1$. The set E of ordered pairs (s,r) of positive integers with the above properties is given by $E=\{(1,13),(3,11),(5,9),(7,7),(9,5),(11,3),(13,1)\}$. For each $(s,r)\in E$, the primes dividing N is a 14-tuple of the form $(P_1,P_2,\ldots,P_s,Q_1,Q_2,\ldots,Q_r)$ and we can find a 14-tuple of distinct primes $(p_1,p_2,...,p_{14})$ such that

$$(P_1,P_2,\ldots,P_s,Q_1,Q_2,\ldots,Q_r)\geq(p_1,p_2,\ldots,p_{14}),$$

where $p_i\equiv 1 \textit{ or } 5 \pmod 8$ for $i=1,2,\ldots,s$ and p_j is any prime for $j=s+1,\ldots,14$. Then, by (3.10)

$$\pi^*.\pi^{**}\leq\prod_{k=1}^{14}\frac{p_k}{p_k-1}. \tag{3.11}$$

Table A below gives the 14-tuple $(p_1,p_2,\ldots,p_{14})$ for each $(s,r)\in E$ and the corresponding value of $\prod_{k=1}^{14}\frac{p_k}{p_k-1}$. Here each $p_i\geq 7$ since $(N,15)=1$. As each entry in the last column is less than 2, it follows from (3.9) and (3.11) that N is deficient. Thus $\omega(N)\leq 14$ is not possible for any quasiperfect number N with $(N,15)=1$, proving $\omega(N)\geq 15$.

I	II	III
(s,r)	$(p_1,p_2,\dots,p_{14})$	$\prod_{k=1}^{14}\frac{p_k}{p_k-1}$
(1,13)	(13,7,11,17,19,23,29,31,37,41,43,47,53,59)	1.99331532
(3,11)	(13,17,29,7,11,19,23,31,37.41,43,47,53,59)	1.99331532
(5,9)	(13,17,29,37,41,7,11,19,23,31,43,47,53,59)	1.99331532
(7,7)	(13,17,29,37,41,53,61,7,11,19,23,31,43,47)	1.99218916
(9,5)	(13,17,29,37,41,53,61,73,89,7,11,19,23,31)	1.95285089
(11,3)	(13,17, 29,37,41,53,61,73,89,97,101,7,11,19)	1.84478333
(13,1)	(13,17,29,37,41,53,61,73,89,97,101,109,113,7)	1.61783693

Table A

□

Theorem 3.12. *If N is a quasiperfect number with $(N,15)=5$ then*

$$\omega(N)\geq 7.$$

Proof. Suppose N is the square of an odd integer of the form (3.8) with $(N,15)=5$, s odd and $\omega(N)\leq 6$. We will show, as in the proof of Theorem 3.1, that N is deficient and hence cannot be quasiperfect. As before it suffices to prove the case $\omega(N)=6$. That is, $s+r=6$, s odd and $(N,15)=5$.

Unlike in the previous theorem, here 5 divides N so that the factor 5^{2e} may or may not be a special factor for N.

The set F of ordered pairs (s,r) of positive integers with the stated conditions is $F=\{(1,5),(3,3),(5,1)\}$. Now for each $(s,r)\in F$ and for the 6-tuple of primes $(P_1,P_2,...,P_s,Q_1,Q_2,...,Q_r)$ dividing N, we find two 6-tuples $(p_1,p_2,...,p_6)$ and $(p'_1,p'_2,...,p'_6)$ of primes such that $(P_1,P_2,...,P_s,Q_1,Q_2,...,Q_r)\geq(p_1,p_2,...,p_6)$ or $(p'_1,p'_2,...,p'_6)$ according as 5^{2e} is a special factor or not for N.

Table B gives the 6-tuples $(p_1,p_2,...,p_6)$ and $(p'_1,p'_2,...,p'_6)$ and the corresponding products $\prod_{i=1}^{6}\frac{p_i}{p_i-1}$ and $\prod_{i=1}^{6}\frac{p'_i}{p'_i-1}$ for any given $(s,r)\in F$. Since each entry in columns III and V is less than 2, it follows N is deficient. Thus $\omega(N)\leq 6$ is not possible for a quasiperfect number with $(N,15)=5$. Hence the theorem holds.

I	II	III	IV	V
(s,r)	$(p_1,p_2,...,p_6)$	$\prod_{i=1}^{6}\frac{p_i}{p_i-1}$	$(p'_1,p'_2,...,p'_6)$	$\prod_{i=1}^{6}\frac{p'_i}{p'_i-1}$
(1,5)	(5,7,11,13,17,19)	1.94904394	(13,5,7,11,17,19)	1.94904394
(3,3)	(5,13,17,7,11,19)	1.94904394	(13,17,29,5,7,11)	1.91240778
(5,1)	(5,13,17,29,37,7)	1.78684565	(13,17,29,37,41,5)	1.56987153

Table B

□

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References

[1] Abbott, H. L., C. E. Aull, Brown, E., & D.Suryanarayana (1973) Quasiperfect numbers, *Acta Arithmetica*, XXII, 439-447; correction to the paper, *Acta Arithmetica*, XXIX (1976), 636–637.

[2] Cattaneo, P. (1951), Sui numeri quasiperfetti, *Boll. Un. Mat. Ital.*, 6(3), 59–62.

[3] Cohen, G. L. (1982) The non-existence of quasiperfect numbers of certain form, *Fib. Quart.*, 20(1), 81–84.

[4] Cohen, G. L. & Peter Hagis Jr. (1982) Some results concerning quasiperfect numbers, *J.Austral.Math.Soc.(Ser.A)*, 33, 275–286.

[5] Kishore, M. (1975) Quasiperfect numbers are divisible by at least six distinct divisors, *Notices. AMS*, 22, A441.

[6] Sándor , J. & Crstici, B. (2004) *Hand book of Number Theory II*, Kluwer Academic Publishers, Dordrecht/ Boston/ London.

[7] Sierpinski, W. *A Selection of problems in the Theory of Numbers*, New York, (page 110).