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On a Pillai's Conjecture and gaps between consecutive primes

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Abstract: In this article we show that the following Pillai's conjecture (p_n is the n-th prime number)

$$\lim_{x\to\infty} \frac{\left|\sum_{p_i\leq x}(-1)^i p_i\right|}{x} = \frac{1}{2}$$

can be established in terms of gaps between consecutive primes. We also study the general sequences that have this property. We call these sequences Pillai-sequences. We prove that the sequence of perfect powers is a Pillai-sequence.

Keywords: Pillai's conjecture, Gaps between consecutive primes, General sequences, Sequence of perfect powers.
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1 Introduction

Let us consider the sequence p_n of prime numbers. S. S. Pillai established the following conjecture (see [1], page 92),

$$\lim_{x\to\infty} \frac{\left|\sum_{p_i\leq x}(-1)^i p_i\right|}{x} = \frac{1}{2}. \tag{1}$$

This conjecture is equivalent to the following establishment,

$$\lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n}(-1)^i p_i\right|}{p_n} = \frac{1}{2}. \tag{2}$$

Clearly (1) implies (2) if $x = p_n$. On the other hand if $x \in [p_n, p_{n+1})$ we have

$$\frac{|\sum_{i=1}^{n}(-1)^i p_i|}{p_{n+1}} \leq \frac{\left|\sum_{p_i \leq x}(-1)^i p_i\right|}{x} \leq \frac{|\sum_{i=1}^{n}(-1)^i p_i|}{p_n}. \tag{3}$$

Now, limit (2) implies that both sides in (3) have limit $1/2$ since $p_{n+1} \sim p_n$. Consequently (2) implies (1).

In this article we study general sequences where limit (2) is fulfilled. We call these sequences Pillai-sequences. Consequently we can establish the Pillai's conjecture in the form: the sequence of prime numbers is a Pillai-sequence. We do not prove the Pillai's conjecture. However, we prove that the sequence of perfect powers is a Pillai-sequence.

2 Pillai-sequences

Now, we shall work with an arbitrary sequence $f(n)$ (in particular $f(n) = p_n$).

Theorem 2.1. *Suppose than $f(n)$ $(n \geq 1)$ is a sequence such that:*
a) $f(n)$ is nonnegative for every n.
b) $f(n)$ is strictly increasing, that is, if $n_1 < n_2$ then $f(n_1) < f(n_2)$.
c) We have the following limit

$$\lim_{n\to\infty} f(n) = \infty. \tag{4}$$

d) We have the following limit

$$\lim_{n\to\infty} \frac{f(n+1)}{f(n)} = 1. \tag{5}$$

Let us consider the following sequence

$$S(n) = \frac{\left|\sum_{k=1}^{n}(-1)^k f(k)\right|}{f(n)}. \tag{6}$$

Suppose there exists $\lim_{n\to\infty} S(n)$. Then

$$\lim_{n\to\infty} S(n) = \frac{1}{2}. \tag{7}$$

Proof. If n is even, then

$$\left|\sum_{k=1}^{n}(-1)^k f(k)\right| = \sum_{k=1}^{n}(-1)^k f(k). \tag{8}$$

On the other hand, if n is odd, then

$$\left|\sum_{k=1}^{n}(-1)^k f(k)\right| = -\sum_{k=1}^{n}(-1)^k f(k). \tag{9}$$

Equations (8) and (9) give

$$\left|\sum_{k=1}^{n}(-1)^k f(k)\right| = (-1)^n \sum_{k=1}^{n}(-1)^k f(k) = \sum_{k=1}^{n}(-1)^{n-k} f(k).$$

Consequently (6) can be written in the form

$$S(n) = \sum_{k=1}^{n} (-1)^{n-k} \frac{f(k)}{f(n)}. \tag{10}$$

That is,

$$S(n) = (-1)^{n-1}\frac{f(1)}{f(n)} + (-1)^{n-2}\frac{f(2)}{f(n)} + \ldots - \frac{f(n-1)}{f(n)} + 1. \tag{11}$$

Note that (see (10))

$$S(1) = 1. \tag{12}$$

Besides, (11) gives

$$S(n+1) = -\frac{f(n)}{f(n+1)} S(n) + 1 \qquad (n \geq 1). \tag{13}$$

Consequently (12) and (13) are a recursive definition of $S(n)$.

From (12) and (13) can be proved without difficulty using mathematical induction that $0 < S(n) \leq 1$. That is, $S(n)$ is bounded.

Equation (13) gives

$$S(n+1) - S(n) = -\frac{f(n)}{f(n+1)} S(n) - S(n) + 1 = -\left(1 + \frac{f(n)}{f(n+1)}\right) S(n) + 1.$$

That is

$$S(n) = \frac{1 - (S(n+1) - S(n))}{\left(1 + \frac{f(n)}{f(n+1)}\right)}. \tag{14}$$

If $S(n)$ has limit then $(S(n+1) - S(n)) \to 0$ and consequently (14) and (5) imply that

$$S(n) \to \frac{1}{2}.$$

□

Definition 2.2. *Suppose that $f(n)$ satisfies $a), b), c)$ and $d)$ (see Theorem 2.1). We shall say that $f(n)$ is a Pillai-sequence if and only if $S(n)$ has limit $1/2$.*

Consequently we can establish the Pillai's conjecture in the form: the sequence of prime numbers p_n is a Pillai-sequence.

Theorem 2.3. *If $f(n)$ is a Pillai-sequence then $af(n) + b$ $(a > 0)$ is also a Pillai-sequence.*

Proof. Clearly $af(n) + b$ satisfies the conditions $a), b), c)$ and $d)$ (see Theorem 2.1). Now, we shall prove that $S(n) \to 1/2$. We have (see (11)),

$$\begin{aligned} S(n) &= (-1)^{n-1}\frac{af(1)+b}{af(n)+b} + (-1)^{n-2}\frac{af(2)+b}{af(n)+b} + \ldots - \frac{af(n-1)+b}{af(n)+b} + 1 \\ &= \left(\left((-1)^{n-1}\frac{f(1)}{f(n)} + (-1)^{n-2}\frac{f(2)}{f(n)} + \ldots - \frac{f(n-1)}{f(n)} + 1\right) \frac{af(n)}{af(n)+b}\right) + 1 \\ &- \frac{af(n)}{af(n)+b} + \frac{O(1)}{af(n)+b} \to 1/2. \end{aligned}$$

□

3 Pillai-sequences and gaps

In the following theorem and corollary we describe a Pillai-sequence in terms of the gaps between consecutive numbers in the sequence. Let us consider the first n gaps in the sequence, this theorem establish that the sum of the gaps in position even is asymptotically equal to the sum of the gaps in position odd.

Theorem 3.1. *Suppose that $f(n)$ satisfies $a), b), c)$ and $d)$ (see Theorem 2.1). Then $S(n)$ has limit $1/2$ (that is, $f(n)$ is a Pillai-sequence) if and only if the following sequence $A(n)$ has limit 1.*

$$A(n) = \frac{\sum_{k=1,3,5,\ldots,(n-1)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\ldots,(n-2)}(f(k+1)-f(k))} \quad (n \text{ even}).$$

$$A(n) = \frac{\sum_{k=1,3,5,\ldots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\ldots,(n-1)}(f(k+1)-f(k))} \quad (n \text{ odd}).$$

Remark 3.2. *Note that*

$$A(3) = \frac{f(2)-f(1)}{f(3)-f(2)},$$

$$A(4) = \frac{(f(2)-f(1))+(f(4)-f(3))}{f(3)-f(2)},$$

$$A(5) = \frac{(f(2)-f(1))+(f(4)-f(3))}{(f(3)-f(2))+(f(5)-f(4))},$$

$$A(6) = \frac{(f(2)-f(1))+(f(4)-f(3))+(f(6)-f(5))}{(f(3)-f(2))+(f(5)-f(4))},$$

$$\vdots$$

where in the numerator are the gaps in position odd and in the denominator are the gaps in position even. The first gap is $f(2)-f(1)$, the second gap is $f(3)-f(2)$, etc.

Proof. If n is even then

$$\begin{aligned} \left|\sum_{k=1}^{n}(-1)^k f(k)\right| &= \sum_{k=1,3,5,\ldots,(n-1)}(f(k+1)-f(k)) \\ &= (f(2)-f(1))+(f(4)-f(3))+\cdots+(f(n)-f(n-1)). \end{aligned}$$

If n is odd then

$$\begin{aligned} \left|\sum_{k=1}^{n}(-1)^k f(k)\right| &= f(1)+\sum_{k=2,4,6,\ldots,(n-1)}(f(k+1)-f(k)) \\ &= f(1)+(f(3)-f(2))+(f(5)-f(4))+\cdots+(f(n)-f(n-1)). \end{aligned}$$

Therefore, we can define $S(n)$ in the following equivalent form.

$$S(n) = \frac{\sum_{k=1,3,5,\dots,(n-1)}(f(k+1)-f(k))}{f(n)} \quad (n\ even). \tag{15}$$

$$S(n) = \frac{f(1)+\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))}{f(n)} \quad (n\ odd). \tag{16}$$

Therefore, $f(n)$ is a Pillai-sequence if and only if the subsequences (15) and (16) have limit $1/2$.

Now, (16) has limit $1/2$ if and only if the sequence

$$S'(n) = \frac{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))}{f(n)} \quad (n\ odd). \tag{17}$$

has limit $1/2$.

Therefore, $f(n)$ is a Pillai-sequence if and only if the sequences (15) and (17) have limit $1/2$.

If n is even note that

$$f(n)-f(1) = \sum_{k=1,3,5,\dots,(n-1)}(f(k+1)-f(k)) + \sum_{k=2,4,6,\dots,(n-2)}(f(k+1)-f(k)). \tag{18}$$

(18) and (15) give

$$\lim_{n\to\infty} \frac{\sum_{k=2,4,6,\dots,(n-2)}(f(k+1)-f(k))}{f(n)} = \frac{1}{2} \quad (n\ even). \tag{19}$$

If n is odd note that

$$f(n)-f(1) = \sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k)) + \sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k)). \tag{20}$$

(20) and (17) give

$$\lim_{n\to\infty} \frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{f(n)} = \frac{1}{2} \quad (n\ odd). \tag{21}$$

(15), (19), (17) and (21) give

$$A(n) = \frac{\sum_{k=1,3,5,\dots,(n-1)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-2)}(f(k+1)-f(k))} \to 1 \quad (n\ even). \tag{22}$$

$$A(n) = \frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))} \to 1 \quad (n\ odd). \tag{23}$$

That is, the sequence $A(n)$ has limit 1.

Now, suppose that (23) is true. (20) gives

$$\frac{f(n)-f(1)}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))} = \frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))} + 1.$$

That is

$$\frac{\frac{f(n)-f(1)}{f(n)}}{\frac{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))}{f(n)}} = \frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))} + 1. \tag{24}$$

Therefore, (24) and (23) give

$$\frac{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))}{f(n)} = \frac{\frac{f(n)-f(1)}{f(n)}}{\frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))}+1} \to 1/2.$$

That is (17). Analogously from (22) and (18) we obtain (15). Consequently if (22) and (23) are true then $f(n)$ is a Pillai-sequence. □

Corollary 3.3. *Suppose that $f(n)$ satisfies $a), b), c)$ and $d)$ (see Theorem 2.1). Then $S(n)$ has limit $1/2$ (that is, $f(n)$ is a Pillai-sequence) if and only if the sequences (15) and (17) have limit $1/2$.*

4 Sufficient Conditions

We have the following lemma (see [5], page 332)

Lemma 4.1. *Let $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$ be two series of positive terms such that $\lim_{i\to\infty} \frac{a_i}{b_i} = 1$. Then if $\sum_{i=1}^{\infty} b_i$ is divergent, the following limit holds*

$$\lim_{n\to\infty} \frac{\sum_{i=1}^{n} a_i}{\sum_{i=1}^{n} b_i} = 1.$$

Theorem 4.2. *Suppose that $f(n)$ satisfies $a), b), c)$ and $d)$ (see Theorem 2.1) and the following limit holds.*

$$\lim_{n\to\infty} \frac{f(n+2)-f(n+1)}{f(n+1)-f(n)} = 1. \tag{25}$$

Then $S(n)$ has limit $1/2$ (that is, $f(n)$ is a Pillai-sequence)

Proof. If n is odd then an immediate consequence of condition $c)$ (see Theorem 2.1), equation (20), equation (25) and Lemma 4.1 is that the subsequence (see theorem 3.1)

$$A(n) = \frac{\sum_{k=1,3,5,\dots,(n-2)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\dots,(n-1)}(f(k+1)-f(k))},$$

has limit 1.

If n is even we have (see Theorem 3.1)

$$\begin{aligned} A(n) &= \frac{\sum_{k=1,3,5,\ldots,(n-1)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\ldots,(n-2)}(f(k+1)-f(k))} \\ &= \frac{(f(2)-f(1))+\sum_{k=3,5,\ldots,(n-1)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\ldots,(n-2)}(f(k+1)-f(k))} \\ &= \frac{f(2)-f(1)}{\sum_{k=2,4,6,\ldots,(n-2)}(f(k+1)-f(k))} \\ &+ \frac{\sum_{k=3,5,\ldots,(n-1)}(f(k+1)-f(k))}{\sum_{k=2,4,6,\ldots,(n-2)}(f(k+1)-f(k))}. \end{aligned}$$

Therefore, (Lemma 4.1) this subsequence has also limit 1 and consequently the sequence $A(n)$ has limit 1. Theorem 3.1 implies that $f(n)$ is a Pillai-sequence. □

Example 4.3. *Let k be a positive integer. Let us consider the sequence of positive integers (see [2], example 19),*

$$f(n) = \sum_{i=1}^{n} p_i^k \sim \frac{n^{k+1}}{k+1} \log^k n.$$

Where p_i is the i-th prime. We have

$$\lim_{n\to\infty} \frac{f(n+2)-f(n+1)}{f(n+1)-f(n)} = \lim_{n\to\infty} \frac{p_{n+2}^k}{p_{n+1}^k} = 1.$$

Since $p_{n+1} \sim p_n$. Therefore, $f(n)$ is a Pillai-sequence.

Theorem 4.4. *If $f(n)$ satisfies the conditions of Theorem 4.2 (and consequently it is a Pillai-sequence) then $f(n)^\alpha$ $(\alpha > 0)$ also satisfies the conditions of Theorem 4.2 and consequently it is also a Pillai-sequence.*

Proof. Clearly $f(n)^\alpha$ satisfies the conditions $a), b), c)$ and $d)$ (see Theorem 4.2). Now, we shall prove that the condition (25) is also fulfilled. We have (mean value Theorem),

$$\frac{f(n+2)^\alpha - f(n+1)^\alpha}{f(n+1)^\alpha - f(n)^\alpha} = \frac{(f(n+2)-f(n+1))}{(f(n+1)-f(n))} \frac{\alpha\, a(n)^{\alpha-1}}{\alpha\, b(n)^{\alpha-1}}. \tag{26}$$

Where

$$f(n+1) < a(n) < f(n+2). \tag{27}$$

$$f(n) < b(n) < f(n+1). \tag{28}$$

(27) gives

$$1 < \frac{a(n)}{f(n+1)} < \frac{f(n+2)}{f(n+1)}.$$

Consequently

$$\frac{a(n)}{f(n+1)} \to 1. \tag{29}$$

(28) gives

$$1 < \frac{b(n)}{f(n)} < \frac{f(n+1)}{f(n)}.$$

Consequently

$$\frac{b(n)}{f(n)} \to 1. \tag{30}$$

(29) and (30) give

$$\frac{a(n)}{b(n)} \to 1. \tag{31}$$

Finally (26), (25) and (31) give

$$\frac{f(n+2)^\alpha - f(n+1)^\alpha}{f(n+1)^\alpha - f(n)^\alpha} \to 1.$$

□

Theorem 4.5. *Let us consider the function $f(x)$ on the interval $[1, \infty)$. Suppose that,*

a) $f(x) \geq 0$ on the interval $[1, \infty)$.

b) $f'(x) > 0$ on the interval $[1, \infty)$ and it is either strictly increasing or strictly decreasing on the interval $[a, \infty)$ $(a \geq 1)$.

c)We have the following limit

$$\lim_{x \to \infty} f(x) = \infty.$$

d) We have the following limit

$$\lim_{n \to \infty} \frac{f(n+1)}{f(n)} = 1.$$

e) We have the following limit

$$\lim_{n \to \infty} \frac{f'(n+1)}{f'(n)} = 1. \tag{32}$$

Then $f(n)$ is a Pillai-sequence (that is, $S(n) \to 1/2$).

Proof. We shall prove that (see Theorem 4.2)

$$\lim_{n \to \infty} \frac{f(n+2) - f(n+1)}{f(n+1) - f(n)} = 1. \tag{33}$$

We have (mean value Theorem)

$$\frac{f(n+2) - f(n+1)}{f(n+1) - f(n)} = \frac{f'(n+1+\epsilon_2(n))}{f'(n+\epsilon_1(n))},$$

where $0 < \epsilon_1(n) < 1$ and $0 < \epsilon_2(n) < 1$.

If $f'(x)$ is strictly increasing, we have,

$$1 \leq \frac{f'(n+1+\epsilon_2(n))}{f'(n+\epsilon_1(n))} \leq \frac{f'(n+2)}{f'(n)}.$$

If $f'(x)$ is strictly decreasing, we have,

$$\frac{f'(n+2)}{f'(n)} \leq \frac{f'(n+1+\epsilon_2(n))}{f'(n+\epsilon_1(n))} \leq 1.$$

Consequently

$$\lim_{n\to\infty} \frac{f(n+2)-f(n+1)}{f(n+1)-f(n)} = \lim_{n\to\infty} \frac{f'(n+1+\epsilon_2(n))}{f'(n+\epsilon_1(n))} = 1.$$

□

Example 4.6. *We now give some examples of Pillai-sequences. The sequence* $f(n) = n^\alpha$ $(\alpha > 0)$ *(in particular the sequence of positive integers* $f(n) = n^k$ *where* k *is a positive integer). The sequence more general* $f(n) = an^\alpha + b$ $(\alpha > 0)$ $(a > 0)$ $(b > 0)$*. The sequences* $f(n) = \log n$*,* $f(n) = n \log n$*, etc.*

5 Functions of slow increase. Pillai-sequences

We recall the definition of function of slow increase given in [2].

Definition 5.1. *Let* $f(x)$ *be a function defined on the interval* $[a, \infty)$ *such that* $f(x) \geq 0$*,* $\lim_{x\to\infty} f(x) = \infty$ *and with continuous derivative* $f'(x) > 0$ *. The function* $f(x)$ *is of slow increase if and only if the following condition holds.*

$$\lim_{x\to\infty} \frac{xf'(x)}{f(x)} = 0. \tag{34}$$

Typical functions of slow increase are $f(x) = \log x$, $f(x) = \log^2 x$, $f(x) = \log \log x$.

Theorem 5.2. *If* $f(x)$ *is a function of slow increase on the interval* $[1, \infty)$ *then the sequence* $F(n) = n^\alpha f(n)$ $(\alpha > 0)$ *is a Pillai-sequence.*

Proof. Let us consider the following function on the interval $[1, \infty)$.

$$F(x) = x^\alpha f(x).$$

We shall prove that the sequence $F(n) = n^\alpha f(n)$ $(\alpha > 0)$ satisfies the conditions of Theorem 4.2.

Clearly conditions $a), b)$ and $c)$ are fulfilled (see Definition 5.1).

Let us consider the condition $d)$. We have (see Theorem 8 in [2])

$$\lim_{x\to\infty} \frac{F(x+1)}{F(x)} = \lim_{x\to\infty} \frac{(x+1)^\alpha}{x^\alpha} \frac{f(x+1)}{f(x)} = 1.$$

Therefore, the condition $d)$ is fulfilled.

Now, we shall prove (25).

The derivative of the function $F(x) = x^\alpha f(x)$ is $F'(x) = \alpha x^{\alpha-1} f(x) + x^\alpha f'(x)$. Therefore, (mean value Theorem) we have

$$\begin{aligned} & F(x+1) - F(x) = (x+1)^\alpha f(x+1) - x^\alpha f(x) \\ = \ & \alpha(x+\epsilon(x))^{\alpha-1} f(x+\epsilon(x)) + (x+\epsilon(x))^\alpha f'(x+\epsilon(x)), \end{aligned}$$

where $0 < \epsilon(x) < 1$. Consequently

$$\begin{aligned} & \lim_{x\to\infty} \frac{F(x+1)-F(x)}{\alpha x^{\alpha-1} f(x)} = \lim_{x\to\infty} \frac{(x+1)^\alpha f(x+1) - x^\alpha f(x)}{\alpha x^{\alpha-1} f(x)} \\ = & \lim_{x\to\infty} \frac{\alpha(x+\epsilon(x))^{\alpha-1} f(x+\epsilon(x)) + (x+\epsilon(x))^\alpha f'(x+\epsilon(x))}{\alpha x^{\alpha-1} f(x)} \\ = & \lim_{x\to\infty} \left(\left(1 + \frac{1}{\alpha} \frac{(x+\epsilon(x)) f'(x+\epsilon(x))}{f(x+\epsilon(x))}\right) \frac{f(x+\epsilon(x))}{f(x)} \left(1 + \frac{\epsilon(x)}{x}\right)^{\alpha-1} \right) \\ = & \ 1. \end{aligned} \tag{35}$$

Since (34) gives

$$\lim_{x\to\infty} \frac{(x+\epsilon(x)) f'(x+\epsilon(x))}{f(x+\epsilon(x))} = 0.$$

On the other hand, we have

$$1 \leq \frac{f(x+\epsilon(x))}{f(x)} \leq \frac{f(x+1)}{f(x)}.$$

Now, both sides have limit 1 (see Theorem 8 in [2]).

Finally (35) and Theorem 8 in [2] give

$$\lim_{x\to\infty} \frac{F(x+2)-F(x+1)}{F(x+1)-F(x)} = \lim_{x\to\infty} \frac{(x+1)^{\alpha-1} f(x+1)}{x^{\alpha-1} f(x)} \frac{\frac{F(x+2)-F(x+1)}{\alpha(x+1)^{\alpha-1} f(x+1)}}{\frac{F(x+1)-F(x)}{\alpha x^{\alpha-1} f(x)}} = 1.$$

Therefore, equation (25) is proved. $\square$

Theorem 5.3. *Let $f(x)$ be a function of slow increase on the interval $[1,\infty)$. Let $A(n)$ be a strictly increasing sequence of positive integers such that,*

$$A(n) = n^s f(n) + O(n^{s-1}) \sim n^s f(n) \qquad (s \geq 1)$$

Then $A(n)$ is a Pillai-sequence.

Proof. We shall prove that the conditions of Theorem 4.2 are fulfilled.

Clearly conditions $a), b), c)$ and $d)$ are fulfilled.

Now, we shall prove (25). That is

$$\lim_{n\to\infty} \frac{A(n+2)-A(n+1)}{A(n+1)-A(n)} = 1. \tag{36}$$

We have

$$A(n+2) = (n+2)^s f(n+2) + h(n+2)(n+2)^{s-1},$$
$$A(n+1) = (n+1)^s f(n+1) + h(n+1)(n+1)^{s-1},$$
$$A(n) = n^s f(n) + h(n) n^{s-1},$$

where $|h(n)| < H$.

Therefore, (see (35)) we have,

$$\lim_{n\to\infty} \frac{A(n+2)-A(n+1)}{s(n+1)^{s-1} f(n+1)} = 1. \tag{37}$$

$$\lim_{n\to\infty}\frac{A(n+1)-A(n)}{sn^{s-1}f(n)}=1. \tag{38}$$

Finally, (37), (38) and Theorem 8 in [2] give (36). □

Example 5.4. *The sequence of positive integers*

$$A(n)=\left[n^2\log(n+1)\right]+2n\sim n^2\log(n+1)$$

is a Pillai-sequence.

6 The sequence of perfect powers

A natural number of the form m^n where m and $n\geq 2$ are positive integers is called a perfect power. The first few terms of the integer sequence of perfect powers are

$$1,4,8,9,16,25,27,32,36,49,64,81,100,121,125,128\ldots$$

In this section we shall prove that the sequence of perfect powers is a Pillai-sequence.

We shall need the following lemmas.

Lemma 6.1. *We have the following formulae*

$$2+4+6+\cdots+2k=k^2+k$$

$$1+3+5+\cdots+(2k-1)=k^2$$

Proof. Mathematical induction. □

Lemma 6.2. *Let P_n be the n-th perfect power. We have the following asymptotic formula, $P_n\sim n^2$.*

Proof. See [4]. □

Lemma 6.3. *We have the following inequality*

$$d_n=P_{n+1}-P_n\leq 2n.$$

Proof. See [3]. □

Lemma 6.4. *Let us consider the first n consecutive differences*

$$d_1=(P_2-P_1),d_2=(P_3-P_2),\ldots,d_n=(P_{n+1}-P_n).$$

Let $v(n)$ be the number of differences such that $(2-\epsilon)i<d_i\leq 2i$ $(\epsilon>0)$. The following limit holds

$$\lim_{n\to\infty}\frac{v(n)}{n}=1.$$

Proof. See [3]. □

Theorem 6.5. *The sequence P_n of perfect powers is a Pillai-sequence, that is*

$$\lim_{x\to\infty} \frac{\left|\sum_{P_i\le x}(-1)^i P_i\right|}{x} = \frac{1}{2}. \tag{39}$$

$$\lim_{n\to\infty} \frac{\left|\sum_{i=1}^{n}(-1)^i P_i\right|}{P_n} = \frac{1}{2}. \tag{40}$$

Proof. We have to prove that the sequence (see Corollary 3.3)

$$\frac{(P_2-P_1)+(P_4-P_3)+(P_6-P_5)+\cdots+(P_n-P_{n-1})}{P_n}, \tag{41}$$

where n is even, and the sequence

$$\frac{(P_3-P_2)+(P_5-P_4)+(P_7-P_6)+\cdots+(P_n-P_{n-1})}{P_n}, \tag{42}$$

where n is odd, have limit $1/2$.

Note that in the numerator of (41) we have $n/2$ terms. On the other hand, in the numerator of (42) we have $(n-1)/2$ terms.

Equation (41), Lemma 6.3, Lemma 6.1 and Lemma 6.2 give

$$\begin{aligned} & \frac{(P_2-P_1)+(P_4-P_3)+(P_6-P_5)+\cdots+(P_n-P_{n-1})}{P_n} \\ \le\ & \frac{2.1+2.3+2.5+\cdots+2(n-1)}{n^2}\frac{n^2}{P_n} = \frac{2(1+3+5+\cdots+(2(n/2)-1))}{n^2}\frac{n^2}{P_n} \\ =\ & \frac{2(n/2)^2}{n^2}\frac{n^2}{P_n} = \frac{1}{2}\frac{n^2}{P_n} \le \frac{1}{2}+\epsilon. \end{aligned} \tag{43}$$

Note that (Lemma 6.4) in the numerator of equation (41) there are n_1 differences $d_i = P_{i+1} - P_i$ such that $(2-\epsilon)i < d_i$ and $n_1 \sim n/2$. Therefore, we have

$$\begin{aligned} & \frac{(P_2-P_1)+(P_4-P_3)+(P_6-P_5)+\cdots+(P_n-P_{n-1})}{P_n} \\ \ge\ & \frac{(2-\epsilon).1+(2-\epsilon).3+(2-\epsilon).5+\cdots+(2-\epsilon)(2n_1-1)}{n^2}\frac{n^2}{P_n} \\ =\ & \frac{(2-\epsilon)n_1^2}{n^2}\frac{n^2}{P_n} = \left(\frac{1}{2}-\frac{\epsilon}{4}\right) f(n)\frac{n^2}{P_n} \ge \frac{1}{2}-\epsilon, \end{aligned} \tag{44}$$

where $f(n) \to 1$.

The inequalities (43) and (44) imply that (41) has limit $1/2$ since ϵ is arbitrarily small.

Equation (42), Lemma 6.3, Lemma 6.1 and Lemma 6.2 give

$$\begin{aligned} & \frac{(P_3-P_2)+(P_5-P_4)+(P_7-P_6)+\cdots+(P_n-P_{n-1})}{P_n} \\ \le\ & \frac{2.2+2.4+2.6+\cdots+2(n-1)}{n^2}\frac{n^2}{P_n} = \frac{2(2+4+6+\cdots+2((n-1)/2))}{n^2}\frac{n^2}{P_n} \\ =\ & \frac{2}{n^2}\left(\left(\frac{n-1}{2}\right)^2+\left(\frac{n-1}{2}\right)\right)\frac{n^2}{P_n} = \frac{1}{2}\frac{n^2-1}{n^2}\frac{n^2}{P_n} \le \frac{1}{2}+\epsilon. \end{aligned} \tag{45}$$

Note that (Lemma 6.4) in the numerator of equation (42) there are n_2 differences $d_i = P_{i+1} - P_i$ such that $(2-\epsilon)i < d_i$ and $n_2 \sim n/2$. Therefore, we have

$$\begin{aligned} & \frac{(P_3 - P_2) + (P_5 - P_4) + (P_7 - P_6) + \cdots + (P_n - P_{n-1})}{P_n} \\ \geq \ & \frac{(2-\epsilon).2 + (2-\epsilon).4 + (2-\epsilon).6 + \cdots + (2-\epsilon).2n_2}{n^2} \frac{n^2}{P_n} \\ = \ & (2-\epsilon)\frac{n_2^2 + n_2}{n^2}\frac{n^2}{P_n} = \left(\frac{1}{2} - \frac{\epsilon}{4}\right) g(n) \frac{n^2}{P_n} \geq \frac{1}{2} - \epsilon, \end{aligned} \quad (46)$$

where $g(n) \to 1$.

The inequalities (45) and (46) imply that (42) has limit $1/2$ since ϵ is arbitrarily small. $\square$

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