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On Legendre's Conjecture

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Abstract: This paper considers some aspects of Legendre's conjecture as a conjecture and estimates the number of primes in some intervals in order to portray a compelling picture of some of the computational issues generated by the conjecture.

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1 Introduction

At the 1912 International Congress of Mathematicians, in Cambridge UK, Edmund Landau of Georg-August-Universität Göttingen in a paper entitled "Geloste und ungelöste Problème aus der Theorie der Primzahlverteilung und der Riemann'schen Zetafunktion" [1] listed four basic problems about primes. They are now known as Landau's problems. They are as follows (in slightly different terminology from that of Landau):

- Goldbach conjecture: Can every even integer greater than 2 be written as the sum of two primes? [2]
- Twin prime conjecture: Are there infinitely many primes p such that $p + 2$ is prime? [3]
- Legendre's conjecture: Does there always exist at least one prime between consecutive perfect squares? [4, 5]
- Landau question: Are there infinitely many primes of the form $n^2 + 1$? [6]

These are among many other problems that are unsolved because of our limited knowledge of infinity, let alone gaps between consecutive primes and the need to attack them with asymptotic proofs [7].

A table of maximal prime gaps shows that Legendre's conjecture holds to 4×10^{18} [8]. A counterexample near 10^{18} would require a prime gap fifty million times the size of the average gap. The prime number theorem implies that the actual number of primes between n^2 and $(n + 1)^2$ is asymptotic to $n / \ln(n)$. Since this number is large for large n, this lends credence to Legendre's conjecture, as does the work of Chen [9] who showed that a number P which is either a prime or semi-prime (now called Chen primes) does always satisfy this inequality. There is also always a prime between $n - n^{\theta}$ and n, where $\theta = 23 / 42$ [10: 415].

This paper is partly expository and partly suggests a characteristic function to explore Legendre's Conjecture further.

2 Related conjectures

Ingham [11], Iwaniec [12] and Lemke Oliver [13] have, in different approaches, considered gaps between primes or between squares of primes and near-square primes. Thus, Brocard's conjecture [14] pertains to the squares of prime numbers, namely that, with the exception of 4, there are always at least four primes between the square of a prime and the square of the next prime. In terms of the prime counting function, this would mean that

$$\pi(p_{n+1}^2) - \pi(p_n^2) > 3, \forall n > 1 .$$

A strengthening of this conjecture would be that there are always at least four primes between n^2 and $(n + 2)^2$ for $n \geq 1$. Some authors have also used arithmetic progressions and Carmichael numbers in their systematic approaches to the these problems [*cf.*, 15, 16].

Iwaniec and Chen also utilise "almost-primes" and "semi-primes". A number n with prime factorization $n = \prod_{i=1}^{r} p_i^{a_i}$ is called k-almost prime if it has a sum of exponents $\sum_{i=1}^{r} a_i = k$, that is, when the prime factor (multi-primality) function $\Omega(n) = k$. The set of k-almost primes is denoted P_k. The primes which correspond to the "1-almost prime" numbers are prime numbers, and the 2-almost prime numbers are called semi-primes. Conway *et al.* [17] proposed calling these numbers primes, biprimes, triprimes, and so on.

These add to modifications and refinements of the conjectures mentioned here, which like many more in the literature, have been checked computationally to huge powers of 10 by "Titans" [18]: a Titan, as defined by Samuel Yates (1919–1991), is anyone who has found a titanic prime. In 1984 he began the list of "Largest Known Primes" and coined the name titanic prime for any prime with 1,000 or more decimal digits. He also called those who proved their primality "titans". Whether these conjectures will be proven analytically or numerically it is clear that new approaches need to be continually generated rather than follow the same methods and expect different results.

Thus, Cramér's Conjecture, formulated in 1936 by the Swedish mathematician Harald Cramér (1893–1985), is an estimate for the size of gaps between consecutive prime numbers [19]: intuitively, that gaps between consecutive primes are always small, and the conjecture quantifies asymptotically just how small they must be.

More graphically, the Ulam spiral is a simple method of visualizing the prime numbers that reveals the apparent tendency of certain quadratic polynomials to generate unusually large numbers of primes. It was discovered in 1963 by the mathematician Stanislaw Ulam (1909–1984), while he was doodling during the presentation of a “long and very boring paper” [20, 21].

Hardy and Littlewood [22] stated a series of conjectures, one of which, if true, would explain some of the striking features of the Ulam spiral which Hardy and Littlewood called “Conjecture F” and which is a special case of the Bateman–Horn conjecture [23]: an assertion of an asymptotic formula for the number of primes of the form $ax^2 + bx + c$. Ulam wrote down a regular rectangular grid of numbers as in Figure 1, starting with 1 at the center, and spiralling out. He then circled all the prime numbers in Figure 1 to get the spiral which appears in Figure 2 with the prime numbers almost in a pattern tending to line up along diagonals [24].

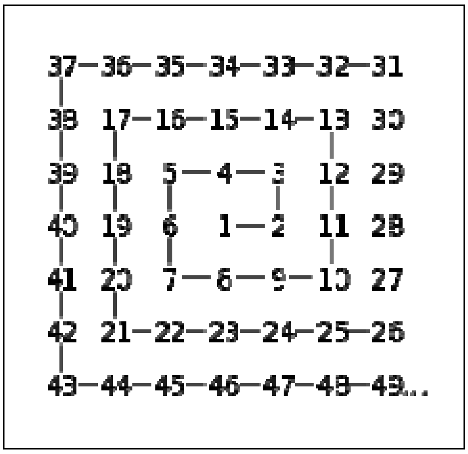

Figure 1: Ulam Spiral

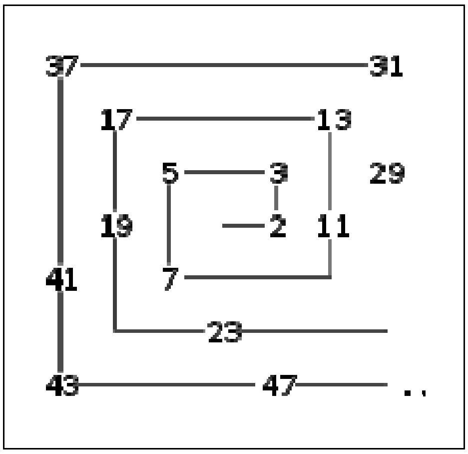

Figure 2: Ulam Spiral Primes

3 A characteristic function

Papers by Atanassov [25], Ribenboim [26], and Vassilev-Missana [27] have considered alternative views of checking for primality. In that spirit we define two arithmetic functions: the well-known divisor function $\delta(m, s)$ and a prime characteristic function $\rho(n, s)$, respectively, by

$$\delta(m,s) = \begin{cases} 1, & m \mid s, \\ 0, & m \nmid s, \end{cases} \tag{3.1}$$

and

$$\rho(n,s) = \begin{cases} 0, & \mathit{if}\ \exists j : j \mid (n,s), 1 < j < n, \\ 1, & \mathit{otherwise}. \end{cases} \tag{3.2}$$

Thus $\rho(n, s) = 1$ for $n = s = 1$, by default. Examples of $\rho(n, s)$ are set out in Table 1. The primes head the columns which lack zeros. It can be seen that the table also illustrates the sieve of Eratosthenes.

$n \to$ $s\downarrow$	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**
1	1	1	1	1	1	1	1	1	1	1
2	1	1	0	1	0	1	0	1	0	1
3	1	1	1	1	0	1	1	0	1	1
4	1	1	0	1	0	1	0	1	0	1
5	1	1	1	1	1	1	1	1	0	1
6	1	1	0	1	0	1	0	0	0	1
7	1	1	1	1	1	1	1	1	1	1
8	1	1	0	1	0	1	0	1	0	1

Table 1. $\rho(n$, s) for n, s = 1, 2, …, 8

These functions can be related to classical functions in the theory of numbers, such as.

$$\sum_{i=1}^{n}\delta(m,i)=\left\lfloor\frac{n}{m}\right\rfloor;$$

For example,

$$\sum_{i=1}^{6}\delta(2,i)=0+1+0+1+0+1=3=\left\lfloor\frac{6}{2}\right\rfloor.$$

Or Menon's identity [28] can be re-written simply as

$$\sum_{i=1}^{n}(\rho(i,n))(i,n)=\sigma_0(n)\varphi(n),$$

in which (a, b) represents the highest common factor of a and b, $\varphi(n)$ is Euler's totient function, and $\sigma_0(n)$ represents the number of divisors of n. For instance,

$$\begin{aligned}\sum_{i=1}^{6}(\rho(i,6))(i,6) &= (\rho(1,6))(1,6)+(\rho(2,6))(2,6)+(\rho(3,6))(3,6)+(\rho(4,6))(4,6)+\\ &\quad (\rho(5,6))(5,6)+(\rho(6,6))(6,6)\\ &=1+0+0+0+1+6\\ &=4\times 2\\ &=\sigma_0(6)\varphi(6).\end{aligned}$$

For notational convenience we simplify (3.2) when $m = s = n$ by means of

$$\rho_n \equiv \rho(n,n);$$

that is,

$$\rho_n=\begin{cases}1, & \textit{if } n \textit{ is } 1 \textit{ or prime},\\ 0, & \textit{otherwise}.\end{cases}$$

For example, $\rho_3 = 1$ and $\rho_4 = 0$. In other words, ρ_n is then the characteristic function of of the set of positive integers, n, which are either prime or unity. Therefore,

$$\sum_{j=1}^{k} \rho_j = 1 + \pi(k)$$

where $\pi(k)$ is the number of primes $\leq k$ so that we have Table 2, for example,

k	1	2	3	4	5	6	7	8
1+p(k)	1	2	3	3	4	4	5	5

Table 2. $1 + \pi(k)$, $k = 1, 2, ..., 8$

In what follows we pick out the n^{th} prime as the $(n + 1)^{\text{st}}$ number in the support of ρ_n (the set of arguments for which the function is not zero). The n^{th} prime, p_n, is then given by the characteristic function (which is not a formula):

$$p_n = 2 + \sum_{k=2}^{2^n} \left(1 - \delta\left(n, \left\lfloor \frac{n}{\sum_{j=1}^{k} \rho_j} \right\rfloor \right) \right). \tag{3.3}$$

In order to demonstrate this, we need

$$1 - \delta\left(n, \left\lfloor \frac{n}{\sum_{j=1}^{k} \rho_j} \right\rfloor \right) = \begin{cases} 1, & 1 < k < p_n, \\ 0, & 2 < p_n \leq k \end{cases} \tag{3.4}$$

***Proof of (3.4)*:** Let

$$K = \sum_{j=1}^{k} \rho_j .$$

If $1 < k < p_n$, then $1 < K \leq n$, since

$$\min_{1<k<p_n} \{K\} = \sum_{j=1}^{2} \rho_j = \rho_1 + \rho_2 = 2,$$

and

$$\max_{1<k<p_n} \{K\} = \sum_{j=1}^{p_n - 1} \rho_j = \pi(p_n - 1) + 1 = n.$$

- If $1 < K \leq n$, then $0 < \left\lfloor \frac{n}{K} \right\rfloor < n$, and $n \nmid \left\lfloor \frac{n}{K} \right\rfloor$, and $\delta\left(n, \left\lfloor \frac{n}{K} \right\rfloor \right) = 0.$
- If $2 < p_n < n$, then $1 < n < K$, since $n < 1$, and $K \geq \pi(p_n) + 1 = n + 1 > n.$
- If $1 < n < K$, then $\left\lfloor \frac{n}{K} \right\rfloor = 0$, and $n \mid \left\lfloor \frac{n}{K} \right\rfloor$, and $\delta\left(n, \left\lfloor \frac{n}{K} \right\rfloor \right) = 1,$

which completes the proof of (3.4).

Proof of (3.3): The right hand side of (3.3) is equal to

$$2+\left(1-\delta\left(n,\left\lfloor\frac{n}{\rho_1+\rho_2}\right\rfloor\right)\right)+\ldots+\left(1-\delta\left(n,\left\lfloor\frac{n}{\sum_{j=1}^{p_n-1}\rho_j}\right\rfloor\right)\right)+\left(1-\delta\left(n,\left\lfloor\frac{n}{\sum_{j=2}^{p_n}\rho_j}\right\rfloor\right)\right)+\ldots+$$

$$\left(1-\delta\left(n,\left\lfloor\frac{n}{\sum_{j=1}^{k}\rho_j}\right\rfloor\right)\right)=2+\underbrace{1+1+\ldots+1}_{p_n-2\,times}+0+\ldots+0$$

$$=p_n,$$

as required.

The main characteristic of the result is that one knows when p_n has been reached by the first zero in the summation. At this point the process stops and the higher upper limit will not be needed. This result can also be expressed as a recurrence relation:

$$p_n=p_{n-1}+\sum_{k=p_{n-1}+1}^{2^n}\left(1-\delta\left(n,\left\lfloor\frac{n}{\sum_{j=p_{n-1}+1}^{k}\rho_j}\right\rfloor\right)\right),\ n>1. \tag{3.5}$$

It seems at first glance that the formulas require the knowledge of the first 2^n prime numbers, but the process stops when the first zero appears in the summation. For instance,

$$p_1=2+(1-\delta(1,0))=2,$$

$$p_2=2+\left(1-\delta\left(2,\left\lfloor\frac{2}{2}\right\rfloor\right)\right)+\left(1-\delta\left(2,\left\lfloor\frac{2}{3}\right\rfloor\right)\right)=2+1+0=3.$$

4 Concluding comments

Can Legendre's Conjecture be reformulated in terms of ($\rho_{(n+1)^2}-\rho_{n^2}$) by searching for a prime in the region $2n+1\left(=(n+1)^2-n^2\right)>n^2$? That is, will there be primes in the region ($2n$ + 1) after n^2? The gap increases in size as n increases. Structural studies [29] indicate that the numbers of primes, while decreasing rapidly at very high integer values, becomes constant. An extension of Euler's prime generating function in particular shows this [30]. This is consistent with the prime gap reaching a limit of 70 million [31]. The largest prime, discovered relatively recently by Dr Curtis Cooper, the Editor of *The Fibonacci Quarterly* at the University of Central Missouri has 22 million digits. There is hardly a dearth of primes in this region if the gap has only 8 digits.

For small to moderate n it is true that there is a prime in this region since the prime gap is always smaller than $2n$ + 1. For larger n, for example,

$$n = 80{,}873{,}624{,}627{,}234{,}849$$

the prime gap is 1,220, which is smaller than $2n + 1$. When large gaps of, for example, 2,254,930 occur the smallest prime in the region has 81,853 digits, so that $2n + 1$ is larger than the gap [*cf.* 32]. All the evidence so far shows that the prime gaps are always smaller than the numbers in the region [8]. For instance, in the region of the prime

$$1{,}125{,}406{,}185{,}245{,}561$$

the gaps range from 800 to 900 with one around 1200. For the prime region

$$1{,}425{,}172{,}824{,}437{,}699{,}411$$

the gap is around 1476. The limiting prime gap is approximately 70 million (8 digits), but the integers in such regions have hundreds of digits. Hence one expects primes between two adjacent squares on the grounds that the numbers are larger than the prime gaps; that is, the basis of asymptotic proof approaches that when n approaches infinity $2n$ should contain primes.

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