Notes on Number Theory and Discrete Mathematics
Print ISSN 1310-5132, Online ISSN 2367-8275
Vol. 23, 2017, No. 2, 81–83

Extensions of D'Aurizio's trigonometric inequality

József Sándor

Babes-Bolyai University, Department of Mathematics
Cluj-Napoca, Romania
e-mail: jsandor@math.ubbcluj.ro

Received: 15 October 2016 **Accepted:** 31 March 2017

Abstract: We offer extensions of D'Aurizio's trigonometric inequality, as well to its counterpart, proved in [1] and [2].
Keywords: Inequalities, Trigonometric functions, D'Aurizio's inequality.
AMS Classification: 26D15, 26D99.

1 Introduction

D'Aurizio's trigonometric inequality states that (see [1, 3]) for any $x \in (0, \frac{\pi}{2})$ one has

$$\frac{1 - \frac{\cos x}{\cos(\frac{x}{2})}}{x^2} < \frac{4}{\pi^2}. \tag{1}$$

D'Aurizio's proof was based on certain infinite product expansions, as well as inequalities on infinite series and Riemann's zeta function. The author ([3]) has obtained a new proof, by applying known trigonometric inequalities and an auxiliary function. The method implied also the following counterpart to (1):

$$\frac{3}{8} < \frac{1 - \frac{\cos x}{\cos(\frac{x}{2})}}{x^2} \tag{2}$$

for any $x \in (0, \frac{\pi}{2})$.

Our aim in what follows is to extend inequalities (1) and (2) for any positive integer n , in place of 2.

For example, for $n = 3$ we have the following double inequality:

$$\frac{4}{\pi^2} < \frac{1 - \frac{\cos x}{\cos(\frac{x}{3})}}{x^2} < \frac{4}{9}. \tag{3}$$

2 Main results

The following extensions of inequalities (1)–(3) will be proved.

Theorem 1. *For any $x \in (0, \frac{\pi}{2})$ and all positive integers $n \geq 3$, one has*

$$\frac{4}{\pi^2} < \frac{1 - \frac{\cos x}{\cos(\frac{x}{n})}}{x^2} < \frac{n^2 - 1}{2n^2}. \tag{4}$$

Proof. As in [3], let $x = nt$, where

$$f(x) = \frac{1 - \frac{\cos x}{\cos(\frac{x}{n})}}{x^2}.$$

Put

$$g(t) = \frac{\cos t - \cos nt}{t^2 \cos t},$$

so clearly

$$g(t) = n^2.f(nt).$$

In order to study the monotonicity of function $f(x)$ upon x, it will be sufficient to consider the monotonicity of $g(t)$ upon $t = \frac{x}{n} \in (0, \frac{\pi}{2n})$. It was shown in [3] that for $n = 2$, $g(t)$ is strictly increasing. Now let us see the case $n = 3$. In this case one has

$$g(t) = \frac{\cos t - 4(\cos t)^3 + 3\cos t}{t^2 \cos t} = 4\frac{1 - (\cos t)^2}{t^2} = 4\left(\frac{\sin t}{t}\right)^2.$$

As the function $s(t) = \frac{\sin t}{t}$ is known to be strictly decreasing on $(0, \frac{\pi}{2})$ (see e.g. [2]), it will be strictly decreasing also on $(0, \frac{\pi}{4})$, so we get that for $n = 3$, the function $g(t)$ introduced above is strictly decreasing, contrary to the case $n = 2$. This implies immediately inequalities (3).

Now, we shall prove that, in the general case, for any $n \geq 3$, $g(t)$ is strictly decreasing function. First computing the derivative of function $g(t)$, one obtains

$$(t^3).g'(t) = \frac{2\cos nt.\cos t - 2(\cos t)^2 + nt.\sin nt.\cos t - t\cos nt.\sin t}{(\cos t)^2} = h(t). \tag{5}$$

In order to prove that $g'(t) < 0$, it will be sufficient to show that $h(t) < 0$. One has $h(0) = 0$, so it will be enough to prove that $h'(t) < 0$. By using (5), and the classical addition formulae

$$\sin(a + b) = \sin a.\cos b + \cos a.\sin b,$$

$$\sin(a - b) = \sin a.\cos b - \cos a.\sin b,$$

after some quite long and elementary computations (which we omit here), the following can be deduced: $h'(0) = 0$ and for the second derivative of h one has:

$$[8n^2.(\cos t)^4].h''(t)$$

$$= -[(n+1)^3 . \sin(n-3)t + (n-1)^3 . \sin(n+3)t + A(n). \sin(n-1)t + B(n). \sin(n+1)t], \quad (6)$$

where

$$A(n) = 3n^3 + 3n^2 - 15n - 23$$

and

$$B(n) = 3n^3 - 3n^2 - 15n + 23.$$

Now, remark that $A(n) = 3n(n^2 - 5) + 3n^2 - 23 > 0$, as for $n \geq 3$ we get $n^2 - 5 \geq 4 > 0$ and $3n^3 - 23 \geq 4 > 0$. Similarly, $B(n) = 3n(n^2 - n - 5) + 23$, with $n^2 - n - 5 = n(n-1) - 5 \geq 6 - 5 = 1$, so we get $B(n) > 0$ again.

As $(n-1)t$ is in $(0, \frac{n-1}{n}\frac{\pi}{2})$, which is in $(0, \frac{\pi}{2})$, and similarly for $(n-3)t$ for $n > 3$ (for $n = 3$, one has $(n-3)t = 0$), by relation (6) we get that $h''(t) < 0$. This implies $h'(t) < h'(0) = 0$, so that $h(t) < h(0) = 0$, giving that $g(t)$ is strictly decreasing. Thus the function f is strictly decreasing, too. Finally, inequalities (4) are consequences of the monotonicity of f, implying: $f(\frac{\pi}{2}-) < f(x) < f(0+)$, and using the L'Hópital rule. □

Remark 1. *As it is well-known that* $\cos(nt) = T_n(\cos t)$*, where* T_n *are the classical Chebyshev polynomials, we get from the above proved results, that the fraction* $\cos t - \frac{T_n(\cos t)}{t^2 \cos t}$ *is a strictly decreasing function of t for any* $n \geq 3$*, while for* $n = 2$ *it is strictly increasing.*

Conjecture 1. *We conjecture that, the above function* $g(t)$ *is strictly decreasing for any real number* $n \geq 3$.

References

[1] D'Aurizio, J. (2014) Refinements of the Shafer–Fink inequality of arbitrary precision, *Math. Ineq. Appl.*, 17(4), 1487–1498.

[2] Sándor, J. (2013) On new refinements of Kober's and Jordan's inequalities, *Notes Number Theory Discr. Math.* 19(1), 73–83.

[3] Sándor, J. (2016) On D'Aurizio's trigonometric inequality, *J. Math. Ineq.*, 10(3), 885–888.