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Fast converging series for zeta numbers in terms of polynomial representations of Bernoulli numbers

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Abstract: In this paper we introduce a new polynomial representation of the Bernoulli numbers in terms of polynomial sums allowing on the one hand a more detailed understanding of their mathematical structure and on the other hand provides a computation of B_{2n} as a function of B_{2n-2} only. Furthermore, we show that a direct computation of the Riemann zeta-function and their derivatives at $k \in \mathbb{Z}$ is possible in terms of these polynomial representation. As an explicit example, our polynomial Bernoulli number representation is applied to fast approximate computations of $\zeta(3), \zeta(5)$ and $\zeta(7)$.

Keywords: Bernoulli numbers, Bendersky's L-numbers, Riemann zeta function.
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1 Introduction

Much was written on Bernoulli numbers [1, 2, 3, 4, 5] since Jacob Bernoulli has discovered these fascinating quantities [6]. Their importance is widely documented as they appear in a variety of scientific fields of importance [7, 3, 8, 9]. In particular, their close relationship with the Riemann zeta-function [10, 11, 12], shows the need for a better understanding of their mathematical structure. Bernoulli numbers are rational, their fractional part is known by the results of Karl von Staudt and Thomas Clausen, often cited as the Staudt-Clausen theorem [13, 14]. Furthermore, several direct representations exist [2, 15, 16, 17], all show up as complicated expressions in form of double-sums providing not much information on the numerator as well as on the denominator of these numbers. No simple rules to compute Bernoulli numbers had been established so far and in consequence explicit formulas are more or less of pure academic interest [3]. On the other hand simpler representations could be of great practical importance, for example, in the field of analytic number theory. A direct computation of the Riemann zeta-function and their derivatives for all $k \in \mathbb{Z}$ in terms of Bernoulli numbers is possible. The famous Euler relation is such an example [23, 24]

$$\zeta(2n) = (-)^{n+1} B_{2n} \frac{(2\pi)^{2n}}{2(2n)!} \,, \tag{1.1}$$

or for negative integers it follows [12]

$$\zeta(-n) = -\frac{B_{n+1}}{n+1} \,. \tag{1.2}$$

More delicate is the direct computation of $\zeta(2n+1)$ and of $\zeta'(2n)$. Explicit relations were found by us [11] and by Adamchik [12] and Lima [25]. Some remarks on the corresponding computational scheme will be given with an explicit example later on, where we demonstrate the direct application of our polynomial Bernoulli number representation to $\zeta(3)$. At first, we found

$$\zeta(2n+1) = (-)^{n+1} L_{2n} \frac{2(2\pi)^{2n}}{(2n)!} \,, \tag{1.3}$$

and

$$\zeta'(2n) = (-)^n \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \left(\frac{2n}{B_{2n}} L_{2n-1} - \gamma_E - \ln(2\pi) \right) \,, \tag{1.4}$$

where γ_E denotes the Euler constant. The L-numbers were first introduced by Bendersky in context with the logarithmic Gamma-function [26, 31]. The first derivative on the odd numbers $\zeta'(2n+1)$ may be computed as follows

$$\zeta'(3) = 2\pi^2 \zeta''(-2) + \left(\ln(2\pi) - \frac{3}{2} + \gamma_E \right) \zeta(3) \,, \tag{1.5}$$

where

$$\zeta''(-2) = 3L_2 - \frac{17}{108} + 4 \sum_{n=2}^{\infty} \frac{B_{2n+2} H_{2n-2}}{(2n-1)(2n)(2n+1)(2n+2)} \tag{1.6}$$

can be written in terms of a Dirichlet series with the Bernoulli numbers and the harmonic numbers [32] involved. Furthermore, we recall one of our earlier results for $\zeta(3)$ published in [11] where we have introduced a fast converging series representation based on Bendersky's L-numbers [26]

$$\zeta(3) \;=\; \frac{\pi^2}{8} - \frac{\pi^2}{12}\ln\left(\frac{\pi}{3}\right) + \frac{\pi^2}{3}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n(2n+1)(2n+2)}\left(\frac{1}{6}\right)^{2n}, \tag{1.7}$$

which was also discussed by Srivastava [27], although not explicitly shown but in principle resulting from a combination of Eq. (2.10) on page 389 and Eq. (3.9) on page 391. Using the well-known Taylor–McLaurin series representation for $\zeta(s)$ [33]

$$\zeta(s) = \frac{1}{s-1} + \frac{1}{2} + \sum_{n=2}^{\infty}\frac{B_k}{k}\begin{pmatrix} s+k \\ k \end{pmatrix}, \tag{1.8}$$

with $s \in -N$, one finds for the first derivative at $s = -2$

$$\zeta'(-2) = -\frac{1}{36} + 2\sum_{n=4}^{\infty}\frac{B_{2n}}{(2n-3)(2n-2)(2n-1)2n}\,. \tag{1.9}$$

Following Bendersky [26] the logarithmic gamma function $\ln\Gamma_n(x)$ of order n may be written as follows

$$\lambda_n(x+1) = x^n\ln(x) - L_n + \ln\Gamma_n(x) \tag{1.10}$$

where, for example, $\lambda_2(x+1)$ is defined as [26]

$$\begin{aligned}\lambda_2(x+1) = &\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 + \frac{1}{6}x\right)\ln(x) \\ &- \frac{1}{9}x^3 + \frac{1}{12}x^2 + 2\sum_{n=2}^{\infty}\frac{B_{2n+2}\,x^{-2n+1}}{(2n-1)(2n)(2n+1)(2n+2)}\,.\end{aligned} \tag{1.11}$$

At $x = 1$ it follows

$$\lambda_2(2) \;=\; \zeta'(-2) \;=\; L_2\,. \tag{1.12}$$

The L-numbers had been computed by Bendersky for all integer numbers n [26]. For $n = 2$ we have

$$L_2 = \frac{1}{48}\left(\frac{3}{2} - \ln\left(\frac{\pi}{3}\right) + 4\sum_{n=1}^{\infty}\frac{\zeta(2n)}{(2n)(2n+1)(2n+2)}\left(\frac{1}{6}\right)^{2n}\right)\,. \tag{1.13}$$

This procedure allows a direct computation also of $\zeta'(-n)$ as shown by us before [11].

$$\zeta'(-n) = -L_n + \frac{B_{n+1}}{n+1}\sum_{q=1}^{n}\frac{1}{q}\,. \tag{1.14}$$

Within this procedure it is possible to find fast converging series in terms of the L-numbers for all $\zeta(2n-1)$ values. For example, for $n=3$ and $n=5$ one gets [11].

$$\begin{aligned}\zeta(5) &= \frac{3\pi^2}{29}\zeta(3) - \frac{25\pi^4}{12528} + \frac{\pi^4}{1044}\ln\left(\frac{\pi}{3}\right) \\ &- \frac{4\pi^4}{87}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n(2n+1)(2n+2)(2n+3)(2n+4)}\left(\frac{1}{6}\right)^{2n} .\end{aligned} \tag{1.15}$$

and

$$\begin{aligned}\zeta(7) &= \frac{72\pi^2}{659}\zeta(5) - \frac{2\pi^4}{1977}\zeta(3) + \frac{49\pi^6}{5337900} + \frac{\pi^6}{266895}\ln\left(\frac{\pi}{3}\right) \\ &- \frac{32\pi^6}{5931}\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)}\left(\frac{1}{6}\right)^{2n}\end{aligned} \tag{1.16}$$

These formulas had also been discussed by Srivastava [27], although not explicitly shown but in principle resulting from a combination of Eq. (2.10) on page 389 and Eq. (3.9) on page 391. For related constants like U_2 or U_4 we found:

$$U_2 = \sum_{n=1}^{\infty}(-)^{n+1}\frac{1}{(2n-1)^2} = 1 \; - \; 2\sum_{n=1}^{\infty}(-)^{n+1}\frac{nL_{2n}}{(2n)!}\left(\frac{\pi}{2}\right)^{2n} , \tag{1.17}$$

and

$$\begin{aligned}U_4 &= \sum_{n=1}^{\infty}(-)^{n+1}\frac{1}{(2n-1)^4} \\ &= \frac{9}{2} \; - \; 2\ln(2) \; - \; \frac{7}{6}U_2 \; - \; \frac{7}{8}\zeta(3) \; + \; \frac{4}{3}\sum_{n=1}^{\infty}(-)^{n+1}\frac{n^3L_{2n+2}}{(2n+2)!}\left(\frac{\pi}{2}\right)^{2n+2} .\end{aligned}$$

These examples show that not only zeta values can be expressed by Bendersky's L-numbers and as a consequence by the even Bernoulli numbers. With this work we introduce a new polynomial representation of the even Bernoulli numbers which permits a fast computation of these constants, which is comparable with the computation by use of available BBP-like formulas. Furthermore, our method permits the use of polylogarithmic identities on which BBP formulas are typically based.

The paper is organized as follows. In the next section we introduce a new formula for the even Bernoulli numbers which serves as a basis for our polynomial representation. We present a proof of this formula discuss it in context with other well-known series representations of Bernoulli numbers, with a special emphasize on the numerical effort which is needed to compute B_{2n}. In section 4 and 5 we present first applications of our formalism to the computation of $\zeta(3), \zeta(5)$ and $\zeta(7)$ and compare this with corresponding BBP-type formulas concerning the numerical effort. The final section 6 gives a summary of the results and a short outlook. In the appendix section we present as a further application of our polynomial representation the computation of B_{2n} as a function of B_{2n-2} only. Based on this formula an even faster computation of zeta numbers by one order of magnitude is possible.

2 An alternative formula for the even Bernoulli numbers

In this section we present the basic equation from which the Bernoulli numbers B_{2n} can be computed.

Theorem 2.1. *We found for* B_{2n}

$$B_{2n} = \frac{2^{2n+1}(2n)!}{2^{2n}-2} \sum_{k=1}^{n} \frac{(-)^{k+1}}{(2n+k)!} \binom{n+1}{k+1} * \sum_{l=0}^{[\frac{k}{2}]} (-)^l \binom{k}{l} \left(\frac{k}{2}-l\right)^{k+2n} . \tag{2.1}$$

The formula shown above represents an alternative computational scheme for the Bernoulli numbers, as only the even numbers are calculated, where the outer summation index runs up to n only. Similar expressions were found by Chang et al. [28] and recently by Qi [29, 30]. All other formulas available from literature [2, 15, 16, 17] including the representation which was introduced in 2009 [34] compute the Bernoulli numbers B_n by a summation up to n. A direct computation of the Bernoulli numbers by use of Eq. (2.1) is possible, where a runtime comparison with a variety of other existing recurrence formula reveals that our formula permits the computation of B_{2n} numbers which is typically faster by factors ranging between 1.5 and 5.4. Corresponding numerical results are presented in Table 1:

	[18]	[19]	[20]	[15]	[21]	This work
B_{10}	2.5	1.5	1.5	3.0	2.0	1.0
B_{20}	5.0	1.8	2.0	5.2	3.0	1.0
B_{30}	4.8	1.6	1.7	5.3	3.1	1.0
B_{40}	4.8	1.6	1.8	5.4	3.1	1.0

Table 1: Runtime comparison between different recurrence relations for Bernoulli numbers with our present formula. Shown is the factor which results from an explicit calculation of the even Bernoulli numbers for $n = 10, 20, 30$ and 40.

Furthermore, and this is the most important point, our double-sum like representation permits a direct link to the so called central factorial numbers described in detail by Riordan [22], which finally offers the possibility to introduce a polynomial representation of the Bernoulli numbers not known before. We show later on that this polynomial representation permits a fast computation of all odd zeta-numbers $\zeta(2n+1)$ and related constants discussed in the introduction.

To prove the theorem we first list some helpful identities.

Lemma 2.1.

Lemma 2.1.1.

$$\frac{k-2l}{i+1}\binom{i+1}{k-l}\binom{k-l}{l} = \binom{i}{k-l-1}\binom{k-l-1}{l}. \tag{2.2}$$

Lemma 2.1.2.

$$\frac{n+1}{n+1-k+2l}\binom{n}{k-l}\binom{k-l}{l} = \binom{n+1}{k-2l}\binom{n-k+2l}{l}. \tag{2.3}$$

These identities are simply proved by a direct calculation of both sides.

Lemma 2.1.3.

$$\sum_{i=0}^{n-1}\left(\begin{array}{c} k-2l \\ i \end{array}\right)\left(\begin{array}{c} n \\ i \end{array}\right)^{-1} = \frac{\left(n+1-\left(\begin{array}{c} k-2l \\ n \end{array}\right)\right)}{n+1-k+2l} . \tag{2.4}$$

Identity (2.4) is found in [22, 35]. Now we start our proof with the well-known recursive formula [5]:

$$B_{2n} \ = \ \frac{1}{2} - \frac{1}{2n+1}\sum_{l=0}^{n-1}\left(\begin{array}{c} 2n+1 \\ 2l \end{array}\right) B_{2l} \ . \tag{2.5}$$

This recursive relation can be explicitly iterated to compute B_{2n}. The first iteration step, which gives the correct result for B_2 and B_4 follows to

$$\begin{aligned} \frac{2B_{2n}}{(2n)!} \ &= \ \frac{1}{(2n)!} - \frac{2}{(2n+1)!}\left(1+\frac{1}{2}\sum_{l=1}^{n-1}\left(\begin{array}{c} 2n+1 \\ 2l \end{array}\right)\right) \\ &+ \ \frac{2}{(2n+2)!}\sum_{l=1}^{n-1}\left(\begin{array}{c} 2n+2 \\ 2l+1 \end{array}\right)\left(1+\frac{1}{2}\sum_{k=1}^{l-1}\left(\begin{array}{c} 2k+1 \\ 2l \end{array}\right)\right) \ \mp ... \end{aligned} \tag{2.6}$$

Introducing the abbreviations

$$S_1(n) \ = \ 1+\frac{1}{2}\sum_{l=1}^{n-1}\left(\begin{array}{c} 2n+1 \\ 2l \end{array}\right) \ = \ \frac{1}{2}(2^{2n}-2n) \ . \tag{2.7}$$

and

$$\begin{aligned} S_2(n) \ &= \ \sum_{l=1}^{n-1}\left(\begin{array}{c} 2n+2 \\ 2l+1 \end{array}\right)\left(1+\frac{1}{2}\sum_{k=1}^{l-1}\left(\begin{array}{c} 2l+1 \\ 2k \end{array}\right)\right) \\ &= \ \frac{9}{8}3^{2n} - (2n+1)2^{2n} + 2n^2 + n - \frac{9}{8} \ , \end{aligned} \tag{2.8}$$

we have

$$\frac{2B_{2n}}{2n} \ = \ \frac{1}{2n} - \frac{2S_1(n)}{2n(2n+1)} + \frac{2S_2(n)}{2n(2n+1)(2n+2)} \pm \ . \tag{2.9}$$

Lemma 2.2. *$S_i(n)$ for $i > 1$ is recursively defined by the following relation*

$$S_{i+1}(n) \ = \ \sum_{l=i}^{n-1}\left(\begin{array}{c} 2n+i+1 \\ 2l+i \end{array}\right) \ S_i(l) \ . \tag{2.10}$$

This relation simply follows from continued iteration in (2.6), where $S_1(n) = \frac{1}{2}(2^{2n} - 2n)$ is given explicitly for all $n \in \mathbb{N}$. Then the Bernoulli numbers B_{2n} result from

$$B_{2n} = \frac{1}{2} - \sum_{l=1}^{n} \frac{(-)^{l+1}}{l!} \left(\begin{array}{c} 2n+l \\ l \end{array} \right)^{-1} S_l(n) \; . \tag{2.11}$$

For a direct computation of (2.11) we convert in a first step all partial sums $S_i(n),\ i = 1, 2, 3, ... \in \mathbb{N}$ to a more appropriate form. For $i = 1$, 2 and 3 it follows explicitly

$$S_1(n) \; = \; \frac{1}{4}2^{2n+1} - \frac{1}{2}(2n)1^{2n+1} \; , \tag{2.12}$$

$$\begin{aligned} S_2(n) &= \frac{1}{8}3^{2n+2} + \frac{1}{4}2^{2n+2} - \frac{1}{8}1^{2n+2} - 2(2n+2)S_1(n) \\ &- \frac{1}{2}(2n+1)(2n+2) \; , \end{aligned}$$

$$\begin{aligned} S_3(n) = & \; \frac{1}{16}4^{2n+3} + \frac{1}{8}3^{2n+3} - \frac{1}{8}2^{2n+3} - \frac{3}{8}1^{2n+3} - 3(2n+3)S_2(n) \\ - & \; 3(2n+2)(2n+3)S_1(n) - \frac{1}{2}(2n+1)(2n+2)(2n+3) \; . \end{aligned}$$

Formula (2.14) represents a ternary sum because of the repeated use of the recursion formula (2.10), and as a consequence $S_n(n)$ denotes a n-fold sum. At a first glance, this fact sounds very discouraging but fortunately most of the terms cancel if all partial sums $S_i(n)$ with index $i = 1, ..., n-1$ will be inserted in the corresponding n-fold partial sum $S_n(n)$ with index n. To illustrate this procedure we compute the partial sum with index $i = 3$ by using explicitly the corresponding expressions obtained for $S_1(n)$ (2.12) and $S_2(n)$ (2.13)

Example 2.1.

$$\begin{aligned} S_3(n) &= \frac{1}{16}4^{2n+3} + \frac{1}{8}3^{2n+3} - \frac{1}{8}2^{2n+3} - \frac{3}{8}1^{2n+3} \\ &- 3(2n+3)\left(\frac{1}{8}3^{2n+2} + \frac{1}{4}2^{2n+2} - \frac{1}{8}1^{2n+2}\right) \\ &+ 3(2n+2)(2n+3)\left(\frac{1}{4}2^{2n+1} - \frac{1}{2}(2n)1^{2n+1}\right) \\ &- \frac{1}{2}(2n+1)(2n+2)(2n+3) \; . \end{aligned} \tag{2.13}$$

It follows that the term in $S_2(n)$ which contains $S_1(n)$ as a factor cancels in the summation for $S_3(n)$. If this procedure is applied to $S_4(n)$ all terms in the corresponding summation which contain $S_2(n)$ or $S_3(n)$ as factors cancel. This way we have calculated the $S_i(n)$ for $i = 1, ..., 5$, which is sufficient for an explicit analysis of the coefficient structure for all $S_i(n),\ i \in \mathbb{N}$. We found:

$$\begin{aligned} S_i(n) &= \frac{1}{2^{i+1}} \sum_{k=0}^{i} (-)^k (i+1-k)^{2n+i} \sum_{l=0}^{k} h_i(l,k)\; (k-l)! \\ &* \left(\frac{2}{i+2-k}\right)^{k-l} \left(\begin{array}{c} 2n+i \\ k-l \end{array} \right) \left(\begin{array}{c} i \\ i+l-k \end{array} \right) \end{aligned} \tag{2.14}$$

with

$$h_i(l,k) = \left(\cos\frac{\pi l}{2} - \sin\frac{\pi l}{2}\right)\begin{pmatrix} i+l-k \\ t_1(l) \end{pmatrix} - \left(\cos\frac{\pi l}{2} + \sin\frac{\pi l}{2}\right)\begin{pmatrix} i+l-k \\ t_2(l) \end{pmatrix}, \tag{2.15}$$

where $t_1(l) = \frac{l}{2} - \frac{1}{4} + \frac{(-)^l}{4})$ and $t_2(l) = \frac{l}{2} - \frac{3}{4} - \frac{(-)^l}{4})$. The index $\mathrm{t}_1(l)$ produces the numbers 0, 0, 1, 1, 2, 2, 3, 3, for $l = 0, 1, 2, 3, ..., k$. Thus has the same values for l even or odd. The second index $t_2(l)$ is slightly different as this sequences starts with -1 for $l = 0$. The corresponding analysis is presented in more detail in the appendix section.

As an example, we show in Table 2 the complete list of numbers which result for $S_i(n)$, where $i = 1, ..., 6$ and $n = 1, ..., 10$:

	$S_1(n)$	$S_2(n)$	$S_3(n)$	$S_4(n)$	$S_5(n)$	$S_6(n)$
$n=1$	1	0	0	0	0	0
$n=2$	6	20	0	0	0	0
$n=3$	29	392	1680	0	0	0
$n=4$	124	5112	73920	369600	0	0
$n=5$	507	55220	2000856	30270240	16816800	0
$n=6$	2024	544700	43099056	1462581120	2.287^{+10}	1.372^{+11}
$n=7$	8185	5135184	821292576	5.475^{+10}	1.788^{+12}	2.868^{+13}
$n=8$	32760	47313584	1.459^{+10}	1.772^{+12}	1.062^{+14}	3.384^{+15}
$n=9$	131063	430867484	2.489^{+11}	5.250^{+13}	5.365^{+15}	2.991^{+17}
$n=10$	524278	3900612564	4.140^{+12}	1.470^{+15}	2.444^{+17}	2.218^{+19}

Table 2: Computed numbers for $S_i(n)$ with i running from 1 to 6 and n running from 1 to 10.

Proof. Using now (2.11) one ends up with a first expression for B_{2n} in form of a ternary sum only

$$B_{2n} = 2^{2n}(2n)! \sum_{i=0}^{n}\sum_{k=0}^{i}\sum_{l=0}^{t_1(k)}(-)^{k+i+l}\,\frac{2n+i-2k+4l+1}{i-k+l+1} * \begin{pmatrix} i \\ k-2l \end{pmatrix}\begin{pmatrix} i-k+2l \\ l \end{pmatrix}\frac{\left(\frac{i+1-k}{2}\right)^{2n+i-k+2l+1}}{(2n+i-k+2l+1)!}, \tag{2.16}$$

Making use of Lemma (2.1) the different terms in (2.18) can be rearranged, leading to a simpler formula for B_{2n}

$$B_{2n} = \frac{2^{2n+1}(2n)!}{2^{2n}-2} \sum_{i=1}^{n} \sum_{k=0}^{i} \sum_{l=0}^{t_1(k)} (-)^{k+i+l} * \binom{i}{k-l+1} \binom{k-l-1}{l} \frac{\left(\frac{i+1-k}{2}\right)^{2n+i-k+2l+1}}{(2n+i-k+2l+1)!} . \tag{2.17}$$

A further rearrangement results in

$$B_{2n} = -\frac{2^{2n}(2n)!}{2^{2n-1}-1} \sum_{k=0}^{n-1} \sum_{l=0}^{t_1(k)} (-)^{n-k+l} * \binom{n}{k-l} \binom{k-l}{l} \frac{\left(\frac{n-k}{2}\right)^{3n-k+2l}}{(3n-k+2l)!} \left[\sum_{i=0}^{n-1} \binom{k-2l}{i} \binom{n}{i}^{-1} \right] , \tag{2.18}$$

where we have interchanged the sum over i with the other two sums. As the variable i appears in the inner sum only this offers the possibility to reduce the ternary sum to a binary one. This is done by use of Lemma (2.2) [22, 35].

As the contribution from the Binomial coefficient $\binom{k-2l}{n}$ is zero it remains for B_{2n}

$$B_{2n} = (-)^{n+1} \frac{2^{2n}(2n)!}{2^{2n-1}-1} \sum_{k=0}^{n-1} \sum_{l=0}^{t_1(k)} (-)^{l-k} * \binom{n+1}{k-2l} \binom{n-k+2l}{l} \frac{\left(\frac{n-k}{2}\right)^{3n-k+2l}}{(3n-k+2l)!} , \tag{2.19}$$

where we made use of Lemma (2.3). Next, we rewrite (2.19) as follows

$$B_{2n} = (-)^{n+1} \frac{2^{2n}(2n)!}{2^{2n-1}-1} \sum_{k=0}^{n} \frac{(-)^k}{(3n-k)!} \binom{n+1}{k} * \sum_{l=0}^{t_1(n-k)} (-)^l \binom{n-k}{l} \left(\frac{n-k}{2} - l\right)^{3n-k} , \tag{2.20}$$

and use the relation

$$\sum_{l=0}^{t_1(n-k)} (-)^l \binom{n-k}{l} (n-k-2l)^{3n-k} = \sum_{l=0}^{t_1(k)} (-)^l \binom{k}{l} (k-2l)^{2n+k} , \tag{2.21}$$

which is simply proved in replacing $n-k$ by k as the summation is symmetric in these indices. Finally we result in

$$B_{2n} = \frac{2^{2n}(2n)!}{2^{2n-1}-1} \sum_{k=0}^{n} (-)^{k+1} \binom{n+1}{n-k} \frac{(\frac{1}{2})^{2n+k}}{(2n+k)!} * \sum_{l=0}^{t_1(k)} (-)^l \binom{k}{l} (k-2l)^{2n+k} . \tag{2.22}$$

Finally the summation over $t_1(k)$ can be replaced by a summation over $[\frac{k}{2}]$ because the inner sum behaves symmetric with respect to the index $t_1(k)$. This last step completes the proof of the theorem as the binomial coefficients in (2.1) and (2.24) are identical. □

3 The α-function as a generator to compute Bernoulli numbers

For the product of the inner sum in (2.1) with the factor $\frac{2(2n)!}{(2n+k)!}$ represents a set of functions depending on the variable $k \in \mathbb{N}$ labelled by the index n. The $\alpha_l^{(n)}$, $l = 1, ..., n$, can be directly computed in a numerical sense by use of the definition shown below. For $k = 0, 1$ and 2 this product is known from literature [22, 35]. The generalization to $k \geq 3$ is straightforward, where explicit formulas for the coefficients $\alpha_l^{(n)}$ will be given at the end of this section. We then have

Definition 3.1.

$$\sum_{l=1}^{n} \alpha_l^{(n)}\, k^l \;=\; \frac{2(2n)!}{(2n+k)!} \sum_{j=0}^{[\frac{k}{2}]} (-)^j \binom{k}{j} (\frac{k}{2} - j)^{k+2n} \,. \tag{3.1}$$

Therefore, it follows for B_{2n}

$$B_{2n} = \frac{2^{2n}}{2^{2n}-2} \sum_{k=1}^{n} (-)^{k+1} \binom{n+1}{k+1} \sum_{l=1}^{n} \alpha_l^{(n)}\, k^l \,. \tag{3.2}$$

Next, we show that the set α-functions can be defined for negative integer numbers as arguments. Furthermore, it will be demonstrated that a computation of the α-functions for negative integers provides a new expression to define the Bernoulli numbers. We start with a recursive formula which the α-functions fulfills. It is

Lemma 3.1.

$$\begin{aligned} \sum_{l=1}^{n} \alpha_l^{(n)} k^l \;&=\; \frac{k(k-1)}{(2n+k)(2n+k-1)} \sum_{l=1}^{n} \alpha_l^{(n)} (k-2)^l \\ &+\; \frac{1}{4} k^2 \frac{2n(2n-1)}{(2n+k)(2n+k-1)} \sum_{l=0}^{n-1} \alpha_l^{(n-1)} k^l \end{aligned} \tag{3.3}$$

with $\alpha_0^0 = 1$.

To prove the above lemma let us first recall the following recursive formula obtained by Riordan [22], for the so called central factorial numbers[1]

$$T(n,k) \;=\; T(n-2,k-2) \;+\; \frac{1}{4} k^2 T(n-2,k) \,, \tag{3.4}$$

[1]Riordan discusses these numbers in context with the so called central difference operator δ in chapter V of his book "Combinatorial identities".

where

$$T(n,k) \;=\; \frac{1}{k!}\sum_{l=0}^{k}(-)^l \left(\begin{array}{c} k \\ l \end{array}\right)\left(\frac{k}{2}-l\right)^n \; . \tag{3.5}$$

From this it follows immediately

$$T(2n+k,k) \;=\; \left(\begin{array}{c} 2n+k \\ k \end{array}\right)\sum_{l=1}^{n}\alpha_l^{(n)}k^l \; . \tag{3.6}$$

Now it is easy to complete the proof. First we write

$$T(2n+k,k) \;=\; T(2n+k-2,k-2) \;+\; \frac{1}{4}k^2 T(2n+k-2,k) \; . \tag{3.7}$$

Substituting here (3.6) the recursive formula for the α-functions follows directly for $k \in \mathbb{N}$. Furthermore, we have shown that the central factorial numbers are are essentially the same as the α-functions, which can be presumed as a key quantity in computing the Bernoulli numbers.

Using the notation

Definition 3.2.

$$A^{(n)}(k) = \sum_{l=1}^{n}\alpha_l^{(n)}k^l \; , \tag{3.8}$$

it follows for example for $k = 1$

Example 3.1.

$$A^{(n)}(1) = \frac{1}{2^{2n}\,(2n+1)}. \tag{3.9}$$

This way the sum $A^{(n)}(k)$ may be calculated for all $k \in \mathbb{N}$. To extend the calculational scheme to negative integers we introduce the following relation

Lemma 3.2.

$$A^{(n)}(-k) \;=\; -\,k\left(\begin{array}{c} n+k \\ n \end{array}\right)\sum_{l=1}^{n}\frac{(-)^{l+1}}{l+k}\left(\begin{array}{c} n \\ l \end{array}\right)A^{(n)}(l) \; . \tag{3.10}$$

To prove the above relation we use the following identity [22, 35]

Proposition 3.1.

$$\sum_{l=1}^{n}\frac{(-)^l}{k+l}\left(\begin{array}{c} n \\ l \end{array}\right)l^m \;=\; (-)^m\,k^{m-1}\left(\begin{array}{c} n+k \\ k \end{array}\right)^{-1} \; . \tag{3.11}$$

Proof. Writing now

$$\begin{aligned} A^{(n)}(-k) &= -k \begin{pmatrix} n+k \\ k \end{pmatrix} \sum_{l=1}^{n} \frac{(-)^{l+1}}{l+k} \sum_{m=1}^{n} \alpha_m^{(n)} l^m \\ &= k \begin{pmatrix} n+k \\ k \end{pmatrix} \sum_{m=1}^{n} \alpha_m^{(n)} \sum_{l=1}^{n} \frac{(-)^{l}}{k+l} \begin{pmatrix} n \\ l \end{pmatrix} l^m \\ &= -k \begin{pmatrix} n+k \\ n \end{pmatrix} \sum_{l=1}^{n} \frac{(-)^{l+1}}{l+k} \begin{pmatrix} n \\ l \end{pmatrix} A^{(n)}(l) \, . \end{aligned} \tag{3.12}$$

□

As a consequence we find the result that $A^{(n)}(-1)$ is "proportional" to B_{2n}.

Example 3.2.

$$A^{(n)}(-1) = -\frac{2^{2n}-2}{2^{2n}} B_{2n} \, . \tag{3.13}$$

Using (3.3) it follows further

$$A^{(n)}(-2) = (2n-1) B_{2n} \, , \tag{3.14}$$

and for example

$$\begin{aligned} A^{(n)}(-3) &= \frac{1}{2}(2n)(2n-1)\frac{2^{2n}-2}{2^{2n}} B_{2n} \\ &- \frac{1}{4}(2n-1)(2n-2)\frac{2^{2n-2}-2}{2^{2n-2}} B_{2n-2} \, , \end{aligned} \tag{3.15}$$

where in the expression for $A^{(n)}(-3)$ both B_{2n} and B_{2n-2} appear. More detailed information about the properties of the $A^{(n)}(k)$ can be obtained by changing to an alternative representation, which was first introduced by Riordan [22], again in context with the so called central factorial numbers.

Next we introduce the coefficients $a_{n,l}$ which are defined by the recursive formula presented below [22]

Definition 3.3.

$$a_{n,l} = \sum_{i=l-1}^{n-1} a_{i,l-1} \begin{pmatrix} 2n+l-1 \\ 2n-2i \end{pmatrix} , \tag{3.16}$$

with $a_{n,0} = \delta_{n,0}$ and $a_{n,1}=1$.

The use of these coefficients allows a reformulation of the $A^{(n)}(k)$ functions

Lemma 3.3.

$$A^{(n)}(k) = \frac{1}{2^{2n}} \begin{pmatrix} 2n+k \\ 2n \end{pmatrix}^{-1} \sum_{l=0}^{n} a_{n,l} \begin{pmatrix} 2n+k \\ 2n+l \end{pmatrix} , \tag{3.17}$$

Proof. Following Riordan [22] we first have

$$2^{2n}T(2n+k,k) \;=\; \sum_{l=0}^{n} a_{n,l} \begin{pmatrix} 2n+k \\ 2n+l \end{pmatrix} . \tag{3.18}$$

Using (3.6) it follows

$$\begin{aligned} T(2n+k,k) &= \frac{1}{2^{2n}} \sum_{l=0}^{k} a_{n,l} \begin{pmatrix} 2n+k \\ 2n+l \end{pmatrix} \\ &= \begin{pmatrix} 2n+k \\ 2n \end{pmatrix} A^{(n)}(k) \end{aligned} \tag{3.19}$$

and this completes the proof. □

It should be mentioned here that a connection between the central factorial numbers and the Bernoulli numbers exists [28]. Although the Bernoulli numbers were not discussed by Riordan in context with the central factorial numbers. This is because the recursive relation (3.19), which allows to compute the central factorial numbers is linked by (3.16) only to the α-functions but not directly to the Bernoulli numbers.

Example 3.3.

$$\begin{aligned} A^{(2)}(k) &= \alpha_1^{(2)}k + \alpha_2^{(2)}k^2 = \frac{1}{2^4}\begin{pmatrix} k+4 \\ k \end{pmatrix}^{-1} \left[\begin{pmatrix} k+4 \\ 5 \end{pmatrix} + 10 \begin{pmatrix} k+4 \\ 6 \end{pmatrix} \right] \\ &= \frac{1}{48}k^2 + \frac{1}{4}B_4 k , \end{aligned} \tag{3.20}$$

where the Bernoulli number B_4 appears as the coefficient of the lowest order polynomial term.

Computing (3.16) for $k=-1$ we find for B_{2n}

$$B_{2n} \;=\; \frac{1}{2^{2n}-2} \sum_{l=1}^{n} (-)^{l+1}\, a_{n,l} \begin{pmatrix} 2n+l \\ 2n \end{pmatrix}^{-1} . \tag{3.21}$$

In a next step we substitute for $a_{n,l}$

Definition 3.4.

$$a_{n,l} \;=:\; \frac{(2n)!}{6^n} l \begin{pmatrix} 2n+l \\ 2n \end{pmatrix} P^{(n+1-l)}(n) \tag{3.22}$$

by introducing a new set of functions $P^{(n+1-l)}(n)$.

As for the $a_{n,l}$ coefficients we can define a recursive relation for the P-functions, which results from substituting (3.22) in (3.16). We get:

$$P^{(n-l+1)} \;=\; 6^n \frac{l-1}{2n-l} \sum_{i=l-1}^{n-1} \frac{P^{i-l+2}(i)}{6^i(2n-2i)} \quad , l > 1 \; . \tag{3.23}$$

Example 3.4. As an example we solve this recursion relation for $l = n$ and for $l = n - 1$. It follows first:

$$P^{(1)}(n) \;=\; \frac{n-1}{n} P^{(1)}(n-1)\,. \tag{3.24}$$

This is a homogenous difference equation of first order with the solution P$^{(1)}$(n) = $\frac{a}{n}$, where the constant a is fixed to $a = 1$ through the boundary condition that P$^{(1)}$(1) insertet in (3.25) must give B_2. For $l = n - 1$ we get:

$$P^{(2)}(n) \;-\; \frac{3(n-2)}{3n-1} P^{(2)}(n-1) \;=\; \frac{3}{2(3n-1)}\,. \tag{3.25}$$

Again this is a difference equation of first order with a non-constant coefficient, but of inhomogenous type. The special solution of this inhomogenous equation results to $P^{(2)}(n) = \frac{3}{10}$. This shows that for all l values $l = n-2, n-3, ...$ an inhomogenous differention equation results from the recursion relation, where the special solution is always a polynomial function in n, but the degree of the polynom increases where the highest exponent is given by $n - l - 1$.

Inserting (3.22) in (3.21) we are able to define the B$_{2n}$ in terms of this polynomials

$$B_{2n} \;=\; \frac{(2n)!}{(2^{2n}-2)6^n} \sum_{l=1}^{n} (-)^{l+1}\, l\, P^{(n+1-l)}(n)\,. \tag{3.26}$$

The P-polynomials introduced in (3.22) are related to the α-functions

Lemma 3.4.

$$P^{(n+1-l)}(n) \;=\; \frac{2^{2n}6^n(-)^l}{l(2n)!} \sum_{k=1}^{l} (-)^k \begin{pmatrix} l \\ k \end{pmatrix} A^{(n)}(k)\,, \tag{3.27}$$

and vice versa for $A^{(n)}(k)$

$$A^{(n)}(k) \;=\; \frac{(2n)!}{2^{2n}6^n} \sum_{l=1}^{n} l \begin{pmatrix} k \\ l \end{pmatrix} P^{(n+1-l)}(n)\,. \tag{3.28}$$

Proof. First, the relation for $A^{(n)}(k)$ results by substituting (3.22) in (3.16) and by reformulating the product of the three binomial coefficients. The above formula for the P-polynomials is then proved by substituting (3.24) in (3.23). This gives

$$B_{2n} \;=\; \frac{2^{2n}}{2^{2n}-2} \sum_{l=1}^{n} \sum_{k=1}^{l} (-)^{k+1} \begin{pmatrix} l \\ k \end{pmatrix} A^{(n)}(k)\,. \tag{3.29}$$

Replacing now l by $n + l - 1$ it follows

$$B_{2n} \;=\; \frac{2^{2n}}{2^{2n}-2} \sum_{l=1}^{n} \sum_{k=1}^{n+1-l} (-)^{k+1} \begin{pmatrix} n+1-l \\ k \end{pmatrix} A^{(n)}(k)\,. \tag{3.30}$$

The double sum can be written as

$$\sum_{k=1}^{n}(-)^{k+1}\begin{pmatrix} n \\ k \end{pmatrix} A^{(n)}(k) + ... + \sum_{k=1}^{n}(-)^{k+1}\begin{pmatrix} 1 \\ k \end{pmatrix} A^{(n)}(k)$$
$$= \sum_{k=1}^{n}(-)^{k+1}\begin{pmatrix} n+1 \\ k+1 \end{pmatrix} A^{(n)}(k)\ , \qquad (3.31)$$

which finally gives (3.2). □

As an example, we have for the first three P-polynomials

Example 3.5.

$$P^{(1)}(n) \ = \ \frac{1}{n}, \quad P^{(2)}(n) \ = \ \frac{3}{10}, \quad P^{(3)}(n) \ = \ \frac{3(21n-43)}{2^3\, 5^2\, 7}\ . \qquad (3.32)$$

All P-polynomials can be regarded as ordinary polynomials, apart from the first one.

It was shown by (3.10) that the α-functions can be defined for negative integers. The corresponding formula computed as a function of the P-polynomials is given by

Lemma 3.5.

$$A^{(n)}(-k) \ = \ (-)^n \frac{(2n)!}{6^n 2^{2n}}\ k\ \sum_{l=1}^{n}(-)^{l+1}\begin{pmatrix} n+k-l \\ k \end{pmatrix} P^{(l)}(n)\ . \qquad (3.33)$$

Proof. Computing equation (3.25) for negative k values instead for positive k values gives

$$A^{(n)}(-k) \ = \ \frac{(2n)!}{6^n 2^{2n}} \sum_{l=1}^{n}(-)^l\, l\ \begin{pmatrix} l+k-1 \\ l \end{pmatrix} P^{(n+1-l)}(n)\ . \qquad (3.34)$$

Replacing again l by $n+l-1$ it follows

$$A^{(n)}(-k) \ = \ \frac{(2n)!}{6^n 2^{2n}} \sum_{l=1}^{n}(-)^{n+1-l}(n+1-l)\begin{pmatrix} n+k-l \\ n+1-l \end{pmatrix} P^{(l)}(n)$$
$$= \ (-)^n \frac{(2n)!}{6^n 2^{2n}} \sum_{l=1}^{n}(-)^{l+1}(n+k-l)\begin{pmatrix} n+k-l-1 \\ k-1 \end{pmatrix} P^{(l)}(n)\ . \qquad (3.35)$$

□

The relation (3.32) allows an explicit definition of the $\alpha_l^{(n)}$-functions by comparing directly the corresponding coefficients in the polynomials on both sides. For $\alpha_1^{(n)}$ we found the interesting result

Corollary 3.1.

$$\frac{B_{2n}}{2n} \ = \ \frac{(2n)!}{6^n 2^{2n}}\ \sum_{l=1}^{n}(-)^{l+1}\ \frac{s(l,1)}{(l-1)!}\ P^{(n+1-l)}(n)$$
$$= \ \frac{(2n)!}{6^n 2^{2n}}\ \sum_{l=1}^{n}(-)^{l+1}\ P^{(n+1-l)}(n) \ = \ \alpha_1^{(n)}. \qquad (3.36)$$

The coefficients for $l \geq 2$ become

$$\alpha_2^{(n)} = \frac{(2n)!}{6^n 2^{2n}} \sum_{l=2}^{n} (-)^{l+2} \frac{s(l,2)}{(l-1)!} P^{(n+1-l)}(n) \, , \tag{3.37}$$

$$\begin{aligned} \alpha_n^{(n)} &= \frac{(2n)!}{6^n 2^{2n}} \sum_{l=n}^{n} (-)^{l+n} \frac{s(l,n)}{(l-1)!} P^{(n+1-l)}(n) \\ &= \frac{(2n)!}{6^n 2^{2n}} \frac{s(n,n)}{(n-1)!} P^{(1)}(n) = \frac{(2n)!}{6^n 2^{2n} n!} \, , \end{aligned} \tag{3.38}$$

where $s(l,k)$ denote the Stirling numbers of the first kind [36, 37, 38].

Formula (3.32) can be computed for different k-values. As an example we present here the computation for $k = 0$. The corresponding calculation for $k = 1$ and 2 is shown in the appendix section.

Example 3.6. We find for $k = 0$:

$$\sum_{l=1}^{n} (-)^{l+1} P^{(n+1-l)}(n) = \frac{2^{2n} 6^n}{2n(2n!)} B_{2n}, \tag{3.39}$$

or

$$\sum_{l=1}^{n} (-)^{l+1} P^{(l)}(n) = (-)^{n+1} \frac{2^{2n} 6^n}{2n(2n!)} B_{2n} \, , \tag{3.40}$$

where

$$\sum_{l=1}^{n} (-)^{l+1} P^{(n+1-l)}(n) = (-)^{n+1} \sum_{l=1}^{n} (-)^{l+1} P^{(l)}(n) \, . \tag{3.41}$$

4 Application to a series representation of $\zeta(3)$

As mentioned in the introduction our polynomial representation of the Bernoulli numbers can be used to compute rather fast converging sequences, for example, for $\zeta(3)$. Using (1.1) and (3.39) it follows

$$\zeta(2n) = \frac{1}{2}(2n)\zeta(2)^n \sum_{l=1}^{n} (-)^{l+1} P^{(l)}(n). \tag{4.1}$$

In other words this special P-polynomial sum converges rather fast as n increases

$$\lim_{n \to \infty} \sum_{l=1}^{n} (-)^{l+1} P^{(l)}(n) = 0 \, . \tag{4.2}$$

This is essential because the prefactor in (3.39) growths much faster with n as the Bernoulli numbers do. As a consequence the P-polynomial sum must compensate this behavior so that the product results in B_{2n}. Also, a representation of $\zeta(2n)$ in terms of $\zeta(2)$ is possible with the help of the polynomial representation. Furthermore, by use of (5.1) a direct representation of the $\ln\ \sin(x)$ function is possible. We find

$$\begin{aligned} \ln\ \sin(\pi x) &= \ln(\pi x) - 2\sum_{n=1}^{\infty}\frac{\zeta(2n)}{2n} \\ &= \ln(\pi x) - \sum_{n=1}^{\infty}\zeta(2)^n\sum_{l=1}^{n}(-)^{l+1}P^{(l)}(n)\,x^{2n}\,. \end{aligned} \tag{4.3}$$

Following Bendersky [26] we have

$$\zeta(3) = -6\pi^2\int_0^{\frac{1}{6}}\int_0^x \ln\ \sin(\pi y)dy - \frac{\pi^2}{12}\ln(2)\,. \tag{4.4}$$

By integrating (5.3) twice and combining the result with (5.4) it follows

$$\begin{aligned} \zeta(3) = \frac{\pi^2}{8} - \frac{\pi^2}{12}\ln\left(\frac{\pi}{3}\right) + \frac{\pi^2}{3}\sum_{n=1}^{\infty}\frac{1}{2n(2n+1)(2n+2)}\left(\frac{\pi}{6\sqrt{6}}\right)^{2n} \\ - \frac{\pi^2}{20}\sum_{n=2}^{\infty}\frac{1}{(2n+1)(2n+2)}\left(\frac{\pi}{6\sqrt{6}}\right)^{2n} + R\left(\frac{\pi}{6\sqrt{6}}\right). \end{aligned} \tag{4.5}$$

If we consider the first only the first two terms in the series $\zeta(3)$ follows with an error $\delta < 10^{-7}$. The convergence is very fast, as higher order sums according to the polynomials $P^{(i)}(n)$ with index $i = 3, 4, ...$ start with a summation index $n = i$ instead of $n = 1$.

In the next section we present alternative series for $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$ which converge even faster using Eq. (3.33) for $k = 2$.

5 Computation of $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$

As the main application we use now an alternative polynomial representation to compute $\zeta(3)$, $\zeta(5)$ and $\zeta(7)$. With Eq. (3.33) it follows:

Proposition 5.1.

$$\ln\ \sin(\pi x) = \ln(\pi x) - \sum_{n=1}^{\infty}\frac{2\zeta(2)^n}{(2n-1)2n}\left(\sum_{l=1}^{n}(-)^{l+1}\binom{n+2-l}{2}P^{(l)}(n)\right)x^{2n}\,. \tag{5.1}$$

As a consequence we find for $\zeta(3)$ by integration

$$\zeta(3) = \frac{\pi^2}{8} - \frac{\pi^2}{12}\ln\left(\frac{\pi}{3}\right) + 36\sum_{n=1}^{\infty}(-)^{n+1}c_n\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.2}$$

with

$$c_1 = \sum_{n=1}^{\infty}\frac{n(n+1)P^{(1)}(n)}{(2n-1)2n(2n+1)(2n+2)}\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.3}$$

$$c_2 \; = \; \sum_{n=1}^{\infty} \frac{n(n+1)P^{(2)}(n+1)}{(2n+1)(2n+2)(2n+3)(2n+4)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n} \; , \tag{5.4}$$

$$c_3 \; = \; \sum_{n=1}^{\infty} \frac{n(n+1)P^{(3)}(n+2)}{(2n+3)(2n+4)(2n+5)(2n+6)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n} \; , \tag{5.5}$$

$$c_4 \; = \; \sum_{n=1}^{\infty} \frac{n(n+1)P^{(4)}(n+3)}{(2n+5)(2n+6)(2n+7)(2n+8)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n} \; , \tag{5.6}$$

and this way for higher coefficients c_n. Using these first four coefficients only we find the following approximate value for $\zeta(3)$:

$$\zeta(3) \; > \; 1.2020569031595738... \, , \tag{5.7}$$

with an error $\delta < 0.2*10^{-13}$. The convergence is about 3 orders of magnitude with the summation index n. This is about 1 order of magnitude faster compared to the corresponding numerical values which result from the series representation (1.7). Furthermore, this new series shows up with an alternating sign which has some benefit in estimating its the convergence properties.

A further advantage of this type of summation is that all infinite sums appearing can be expressed in terms of elementary functions based on logarithmic expressions. This allows a more detailed insight on $\zeta(3)$ and in consequence on all other zeta-values evaluated at odd integer numbers.

Example 5.1.

$$\sum_{n=1}^{2} (-)^{n+1} c_n \left(\frac{\pi}{6\sqrt{6}}\right)^{2n} = \left(2x - \frac{1}{20}x^2\right) \ln\left(\frac{1+x}{1-x}\right) + \left(\frac{1}{2}x^2 - \frac{1}{20}\right) \ln\left(1-x^2\right) - \frac{15}{16}x^2 \tag{5.8}$$

with $x = \frac{\pi}{6\sqrt{6}}$.

The explicit comparison between other series representations and our formula is shown in Table 3: where the numerical comparison has been performed between our approach and the BBP-formalism by use of the following formula [39]:

$$\begin{aligned} \frac{5\pi^2 \ln(3)}{104} - \frac{3\ln(3)^3}{104} = \frac{1}{1053} \sum_{n=1}^{\infty} \left(\frac{1}{729}\right)^n \Big(& \frac{729}{(12n+1)^3} + \frac{243}{(12n+2)^3} - \frac{81}{(12n+4)^3} \\ & - \frac{81}{(12n+5)^3} - \frac{54}{(12n+6)^3} - \frac{27}{(12n+7)^3} \\ & - \frac{9}{(12n+8)^3} + \frac{3}{(12n+10)^3} + \frac{3}{(12n+11)^3} + \frac{2}{(12n+12)^3} \Big) \end{aligned} \tag{5.9}$$

Using (1.15) we analogously we find for $\zeta(5)$:

$$\zeta(5) \; = \; \frac{3\pi^2}{29}\zeta(3) - \frac{25\pi^4}{12528} + \frac{\pi^4}{1044} \ln\left(\frac{\pi}{3}\right) \; - \; \frac{144\pi^2}{29} \sum_{n=1}^{\infty} (-)^{n+1} c_n \left(\frac{\pi}{6\sqrt{6}}\right)^{2n} \; , \tag{5.10}$$

$\zeta(3)$	$\zeta(3)$-Zeta series	$\zeta(3)$-(Zeta series+BP)	$\zeta(3)$-BBP formula [39]
1st order n=1	δ=0.2*10^{-04}	δ=0.2*10^{-05}	δ=0.5*10^{-06}
2nd order n=2	δ=0.2*10^{-06}	δ=0.1*10^{-08}	δ=0.9*10^{-10}
3rd order n=3	δ=0.3*10^{-08}	δ=0.8*10^{-12}	δ=0.4*10^{-13}
4th order n=4	δ=0.4*10^{-10}	δ=0.2*10^{-13}	δ=0.2*10^{-16}

Table 3: Approximate computation of $\zeta(3)$ as a function of the summation index n by use of Eq. (1.7) without and with use of the polynomial representation. The numerical errors are compared to the BBP-type formula [39]

with

$$c_1 = \sum_{n=1}^{\infty} \frac{n(n+1)P^{(1)}(n)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.11}$$

$$c_2 = \sum_{n=1}^{\infty} \frac{n(n+1)P^{(2)}(n+1)}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.12}$$

$$c_3 = \sum_{n=1}^{\infty} \frac{n(n+1)P^{(3)}(n+2)}{(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)(2n+8)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.13}$$

$$c_4 = \sum_{n=1}^{\infty} \frac{n(n+1)P^{(4)}(n+3)}{(2n+5)(2n+6)(2n+7)(2n+8)(2n+9)(2n+10)} \left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \tag{5.14}$$

Table 4 shows the corresponding computational results: where the numerical comparison has been performed between our approach and the BBP-formalism by use of the following formula [40, 41]:

$$\begin{aligned} \frac{31}{32}\zeta(5) &- \frac{343}{99360}\pi^4 \ln(2) + \frac{5}{2484}\pi^2(\ln(2))^3 - \frac{2}{1035}(\ln(2))^5 \\ &= \frac{128}{69}\sum_{n=1}^{\infty}\left(\frac{1}{\sqrt{2}}\right)^n \frac{\cos(\frac{n\pi}{4})}{n^5} - \frac{20}{69}\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n \frac{1}{n^5} \end{aligned} \tag{5.15}$$

$\zeta(5)$	$\zeta(5)$-Zeta series	$\zeta(5)$-(Zeta series+BP)	$\zeta(5)$-BBP formula [41]
1st order $n=1$	$\delta=0.6*10^{-06}$	$\delta=0.1*10^{-06}$	$\delta=0.7*10^{-06}$
2nd order $n=2$	$\delta=0.4*10^{-08}$	$\delta=0.1*10^{-09}$	$\delta=0.1*10^{-08}$
3rd order $n=3$	$\delta=0.3*10^{-10}$	$\delta=0.1*10^{-12}$	$\delta=0.8*10^{-11}$
4th order $n=4$	$\delta=0.3*10^{-12}$	$\delta=0.1*10^{-15}$	$\delta=0.1*10^{-12}$

Table 4: Approximate computation of $\zeta(5)$ as a function of the summation index n by use of Eq. (1.15) without and with use of the polynomial representation. The numerical errors are compared to the BBP-type formula [41]

For $\zeta(7)$ we found:

$$\begin{aligned}\zeta(7) = &\frac{72\pi^2}{659}\zeta(5) - \frac{2\pi^4}{1977}\zeta(3) + \frac{49\pi^6}{5337900} + \frac{\pi^6}{266895}\ln\left(\frac{\pi}{3}\right) \\ &+ \frac{3456\pi^4}{5931}\sum_{n=1}^{\infty}(-)^{n+1}c_n\left(\frac{\pi}{6\sqrt{6}}\right)^{2n} ,\end{aligned} \tag{5.16}$$

with

$$c_1 = \sum_{n=1}^{\infty}\frac{n(n+1)P^{(1)}(n)\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)} \tag{5.17}$$

$$c_2 = \sum_{n=1}^{\infty}\frac{n(n+1)P^{(2)}(n+1)\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)(2n+8)} \tag{5.18}$$

$$c_3 = \sum_{n=1}^{\infty}\frac{n(n+1)P^{(3)}(n+2)\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}}{(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)(2n+8)(2n+9)(2n+10)} \tag{5.19}$$

$$c_4 = \sum_{n=1}^{\infty}\frac{n(n+1)P^{(4)}(n+3)\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}}{(2n+5)(2n+6)(2n+7)(2n+8)(2n+9)(2n+10)(2n+11)(2n+12)} \tag{5.20}$$

Table 5 shows the corresponding computational results: where no BBP-type formula is available for $\zeta(7)$.

$\zeta(7)$	$\zeta(7)$-Zeta series	$\zeta(7)$-(Zeta series+BP)	$\zeta(7)$-BBP formula
1st order $n=1$	$\delta = 0.7 * 10^{-08}$	$\delta = 0.2 * 10^{-08}$	
2nd order $n=2$	$\delta = 0.3 * 10^{-10}$	$\delta = 0.1 * 10^{-11}$	
3rd order $n=3$	$\delta = 0.2 * 10^{-12}$	$\delta = 0.8 * 10^{-15}$	
4th order $n=4$	$\delta = 0.2 * 10^{-14}$	$\delta = 0.6 * 10^{-18}$	

Table 5: Approximate computation of $\zeta(7)$ as a function of the summation index n by use of Eq. (1.16) without and with use of the polynomial representation. For $\zeta(7)$ no BBP-type formula is available for reasons of comparison.

6 Approximations of zeta numbers using B_{2n} and B_{2n-2}

We present a formula which allows to compute the Bernoulli number B_{2n} as a function of the Bernoulli number B_{2n-2} only. This formula decouples B_{2n} from all other Bernoulli numbers. We simply compute (3.32) for $k = 4$ with the help of (3.3). It follows:

$$\frac{B_{2n}}{2n} = \frac{B_{2n-2}}{2n-2} + (-)^{n+1}\frac{(2n)!}{2^{2n}6^n}\begin{pmatrix} 2n \\ 4 \end{pmatrix}^{-1}\sum_{l=1}^{n}(-)^{l+1}\begin{pmatrix} n+4-l \\ 4 \end{pmatrix}P^{(l)}(n)\,. \qquad (6.1)$$

This formula allows the computation of B_{2n} as a function of B_{2n-2} only by use of the corresponding Bernoulli-type polynomials $P^{(l)}(n)$. Furthermore, it permits an approximation of the odd zeta numbers which works faster by about one order of magnitude when compared to (5.2). It follows first:

$$\zeta(2n) = \frac{(\zeta(2))^n}{2(2n-1)}\sum_{l=1}^{n+1}(-)^{l+1}\begin{pmatrix} n+5-l \\ 4 \end{pmatrix}P^{(l)}(n+1) - \frac{2n(2n+1)}{4\pi^2}\zeta(2n+2)\,. \qquad (6.2)$$

With this we can write:

$$\zeta(3) = \frac{\pi^4}{15552} + \frac{\pi^2}{8} - \frac{\pi^2}{12}\ln\left(\frac{\pi}{3}\right) + \frac{\pi^2}{6}\sum_{n=1}^{\infty}(-)^{n+1}c_n\left(\frac{\pi}{6\sqrt{6}}\right)\,, \qquad (6.3)$$

with

$$c_1 = \frac{1}{24}\Big[\sum_{n=1}^{\infty}\frac{(n+1)(n+2)(n+3)(n+4)P^{(1)}(n+1)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)} - \frac{24n(n+1)P^{(1)}(n)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)}\Big]\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}\,, \qquad (6.4)$$

$$c_2 = \frac{1}{24}\Big[\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)(n+3)P^{(2)}(n+1)}{(2n-1)2n(2n+1)(2n+2)(2n+3)(2n+4)}$$
$$- \frac{24n(n+1)P^{(2)}(n+1)\left(\frac{\pi}{6\sqrt{6}}\right)^2}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)}\Big]\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \quad (6.5)$$

$$c_3 = \frac{1}{24}\Big[\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)(n+3)P^{(3)}(n+2)}{(2n+1)(2n+2)(2n+3)(2n+4)(2n+5)(2n+6)}$$
$$- \frac{24n(n+1)P^{(3)}(n+2)\left(\frac{\pi}{6\sqrt{6}}\right)^2}{(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)(2n+8)}\Big]\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \quad (6.6)$$

$$c_4 = \frac{1}{24}\Big[\sum_{n=1}^{\infty}\frac{n(n+1)(n+2)(n+3)P^{(4)}(n+3)}{(2n+3)(2n+4)(2n+5)(2n+6)(2n+7)(2n+8)}$$
$$- \frac{24n(n+1)P^{(4)}(n+3)\left(\frac{\pi}{6\sqrt{6}}\right)^2}{(2n+5)(2n+6)(2n+7)(2n+8)(2n+9)(2n+10)}\Big]\left(\frac{\pi}{6\sqrt{6}}\right)^{2n}, \quad (6.7)$$

and this way for higher coefficients c_n. Using these first four coefficients only we find an approximate value for $\zeta(3)$ with an error $\delta < 0.4*10^{-14}$. The convergence is about 4 orders of magnitude with the summation index n. This procedure permits a systematic improvement of the convergence behavior if higher values $k = 6$ or $k = 8$ are used in (3.2). As an example we present the corresponding formula for $k = 6$:

$$\binom{2n}{6}\left(\frac{B_{2n}}{2n} - 5\frac{B_{2n-2}}{2n-2} + 4\frac{B_{2n-4}}{2n-4}\right) = (-)^{n+1}\frac{(2n)!}{2^{2n}6^n}\sum_{l=1}^{n}(-)^{l+1}\binom{n+6-l}{6}P^{(l)}(n) \quad (6.8)$$

Our results establish a new and very fast option to compute zeta and related numbers. As an outlook, it could be of great interest to combine our formalism with the BBP approach by use of corresponding polylogarithmic identities to further improve the convergence properties in explicit computations.

7 Summary

In summary, we have presented a unique computational scheme for the explicit calculation of the Riemann ζ function and its first derivatives at all positive and negative integer values. This way we have shown that all these numbers are directly attributed to Bernoulli numbers, but with an increasing level of complexity when going, for example, from $\zeta(2)$ to a related sub-sum like U_4. The computational scheme is based on a new polynomial representation of the Bernoulli numbers in connection with Bendersky's L-numbers, which appear in context with the logarithmic Gamma function. As a first application we performed approximate calculations of $\zeta(3), \zeta(5)$ and $\zeta(7)$ in terms our polynomial representation, where this computational procedure is applicable to all ζ-values with integer arguments, as well as to related numbers like Catalan's constant. Finally, we have shown that a computation of B_{2n} as a function of B_{2n-2} or, for example, of B_{2n-2} and B_{2n-4} only is possible by use of the polynomial representation. The result is a further improved approximate computation of these numbers.

8 Appendix A

Here we present the explicit coefficient analysis for $S_i(n), i = 1, ..., 5$ which allows the determination of $S_i(n), i \in \mathbb{N}$. As a non-trivial example we found by an explicit computation for $S_4(n)$:

$$
\begin{aligned}
S_4(n) = {} & \binom{4}{0} 5^{2n+4} - \bigg[\frac{2^1}{4^1}\binom{4}{3}\binom{3}{0}(2n+4) - \frac{2^0}{4^0}\left(\binom{4}{0}+\binom{4}{0}\right)\bigg] 4^{2n+4} \\
& + \bigg[\frac{2^2}{3^2}\binom{4}{2}\binom{2}{0}(2n+3)(2n+4) - \frac{2^1}{3^1}\binom{4}{3}\left(\binom{3}{0}+\binom{3}{0}\right)(2n+4) \\
& - \frac{2^0}{3^0}\binom{4}{4}\left(\binom{4}{1}-\binom{4}{0}\right)\bigg] 3^{2n+4} - \bigg[\frac{2^3}{2^3}\binom{4}{2}\binom{1}{0}(2n+2)(2n+3)(2n+4) \\
& - \frac{2^2}{2^2}\binom{4}{2}\left(\binom{2}{0}+\binom{2}{0}\right)(2n+3)(2n+4) \\
& - \frac{2^1}{2^1}\binom{4}{3}\left(\binom{3}{1}-\binom{3}{0}\right)(2n+4) + \frac{2^0}{2^0}\binom{4}{4}\left(\binom{4}{1}+\binom{4}{1}\right)\bigg] 2^{2n+4} \\
& + \bigg[\frac{2^4}{1^4}\binom{0}{0}\binom{4}{0}(2n+1)(2n+2)(2n+3)(2n+4) \\
& - \frac{2^3}{1^3}\binom{4}{1}\left(\binom{1}{0}+\binom{1}{0}\right)(2n+2)(2n+3)(2n+4) \\
& - \frac{2^2}{1^2}\binom{4}{2}\left(\binom{2}{1}+\binom{2}{1}\right)(2n+3)(2n+4) + \frac{2^1}{1^1}\binom{4}{3}\left(\binom{3}{1}+\binom{3}{1}\right)(2n+4) \\
& + \frac{2^0}{1^0}\binom{4}{4}\left(\binom{4}{2}+\binom{4}{1}\right) 1^{2n+4}\bigg]\bigg]\bigg\}
\end{aligned}
\tag{8.1}
$$

This expression can be written in a more compact form:

$$
S_4(n) = \frac{1}{2^5}\sum_{k=0}^{4}(-)^k(5-k)^{2n+4}\sum_{l=0}^{k}(k-l)!\left(\frac{2}{5-k}\right)^{k-l}\binom{2n+4}{k-l}\binom{4}{4-k+l}h_4(l,k), \tag{8.2}
$$

where the $h_i(l,k)$ have been defined in (2.16). Analogously we found for $S_2(n)$, $S_3(n)$ and $S_5(n)$ ($S_1(n)$ is trivial)

$$
S_2(n) = \frac{1}{2^3}\sum_{k=0}^{2}(-)^k(3-k)^{2n+2}\sum_{l=0}^{k}(k-l)!\left(\frac{2}{3-k}\right)^{k-l}\binom{2n+2}{k-l}\binom{2}{2-k+l}h_2(l,k), \tag{8.3}
$$

$$
S_3(n) = \frac{1}{2^4}\sum_{k=0}^{3}(-)^k(4-k)^{2n+3}\sum_{l=0}^{k}(k-l)!\left(\frac{2}{4-k}\right)^{k-l}\binom{2n+3}{k-l}\binom{3}{3-k+l}h_3(l,k), \tag{8.4}
$$

$$S_5(n)=\frac{1}{2^6}\sum_{k=0}^{5}(-)^k(6-k)^{2n+5}\sum_{l=0}^{k}(k-l)!\left(\frac{2}{6-k}\right)^{k-l}\binom{2n+5}{k-l}\binom{5}{5-k+l}h_5(l,k). \quad (8.5)$$

From the explicit calculation of $S_i(n)$, $i=1,...,5$ equation (2.16) follows immediately.

9 Appendix B

In analogy to example (3.5) for $k=0$ it results for $k=1$ and $k=2$:

$$\sum_{l=1}^{n}(-)^{l+1}\, l\, P^{(n+1-l)}(n) \;=\; \frac{(2^{2n}-2)6^n}{2n(2n!)}B_{2n}, \quad (9.1)$$

or

$$\sum_{l=1}^{n}(-)^{l+1}\, l\, P^{(l)}(n) \;=\; (-)^{n+1}\frac{(4n-(n-1)2^{2n})6^n}{2n(2n!)}B_{2n}\;, \quad (9.2)$$

with

$$\sum_{l=1}^{n}(-)^{l+1}\, l\, P^{(n+1-l)}(n) \;= \quad (9.3)$$
$$(-)^{n+1}\frac{2n(2^{2n}-2)}{4n-(n-1)2^{2n}}\sum_{l=1}^{n}(-)^{l+1}\, l\, P^{(l)}(n)\;.$$

For $k=2$ we find

$$\sum_{l=1}^{n}(-)^{l+1}\, l^2\, P^{(n+1-l)}(n) \;=\; \frac{(2+(2n-2)2^{2n})6^n}{(2n!)}B_{2n}, \quad (9.4)$$

or

$$\sum_{l=1}^{n}(-)^{l+1}\, l^2\, P^{(l)}(n) \;= \quad (9.5)$$
$$(-)^{n+1}\frac{(4n(2n+3)+(n^2-6n+1)2^{2n})6^n}{(2n!)2n}B_{2n},$$

with

$$\sum_{l=1}^{n}(-)^{l+1}l^2\, P^{(n-l+1)}(n) \;= \quad (9.6)$$
$$(-)^{n+1}\;\frac{2n(2+(2n-2)2^{2n})}{4n(2n+3)+(n^2-6n+1)2^{2n}}\sum_{l=1}^{n}(-)^{l+1}l^2\, P^{(l)}(n)\;,$$

where the simple symmetry property

$$\sum_{l=1}^{n}(-)^{l+1}l^m\, P^{(l)}(n) \;\;=$$
$$(-)^{n+1}\sum_{l=1}^{n}(-)^{l+1}(n+1-l)^m\, P^{(n+1-l)}(n)\;, \quad (9.7)$$

which we have used before in the case of $m=1$ has been applied here again for $m=0,1$ and 2.

10 Appendix C

Here we show the P-polynomials up to $n = 6$. It follows:

$$P^{(1)}(n) = \frac{1}{n} \tag{10.1}$$

$$P^{(2)}(n) = \frac{3}{2*5} \tag{10.2}$$

$$P^{(3)}(n) = \frac{3(21n-43)}{2^3*5^2*7} \tag{10.3}$$

$$P^{(4)}(n) = \frac{63n^2-387n+590}{2^4*5^3*7} \tag{10.4}$$

$$P^{(5)}(n) = \frac{3(4851n^3-59598n^2+242737n+327210)}{2^7 5 *^4 *7^2*11} \tag{10.5}$$

$$P^{(6)}(n) = \frac{3(189189n^4-3873870n^3+29616015n^2-100104550n+126087736)}{2^8*5^6*7^2*11*13} \tag{10.6}$$

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