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On two arithmetic functions

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Abstract: Some properties of two new arithmetic functions are studied. Three conjectures are formulated.

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1 Introduction

In a recent paper [1], it was introduced the following arithmetic function: let p be a prime number, and let $\downarrow p$ denote the greatest prime smaller than p, for $p \geq 3$, and let $\downarrow p = 1$, if $p = 2$. If $n \geq 2$ has the prime factorization $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $k, \alpha_1, ..., \alpha_r, r \geq 1$ are natural numbers and $p_1, ..., p_r$ are different primes, then let us define

$$\downarrow n = \prod_{i=1}^{r} (\downarrow p_i)^{\alpha_i}. \tag{1}$$

Let $\downarrow 1 = 1$.

It is immediate that for distinct primes p and q one has $\downarrow (pq) = \downarrow p. \downarrow q$, so by (1) it is immediate that for $(n, m) = 1$ one has $\downarrow (nm) = \downarrow n. \downarrow m$, which means that this arithmetical function is multiplicative function.

In what follows we define the dual of this arithmetic function as follows. For a prime p, let $\uparrow p$ denote the least prime greater than p. For example, $\uparrow 2 = 3, \uparrow 3 = 5$, etc. Similarly to (1), we define for $n \geq 2$:

$$\uparrow n = \prod_{i=1}^{r} (\uparrow p_i)^{\alpha_i}. \tag{2}$$

Let $\uparrow 1 = 1$. Then this arithmetical function is multiplicative, too.

2 Main results

Lemma 1. *One has*

$$\downarrow p \leq p - 2 \textit{ for any } p \geq 5 \tag{3}$$

and

$$\uparrow p \geq p + 2 \textit{ for any } p \geq 3. \tag{4}$$

Proof. For p odd prime, $p - 1$ is even number, and this is not prime for $p - 1$ distinct from 2. Thus the greatest prime $q < p$ will be in the best possible case, the number $q = p - 2$. These are not valid for $p = 2, 3$, but for $p \geq 5$, are true, so (3) follows. The proof of (5) follows on the same lines.

Obviously, one has

$$\downarrow p = p - 1 \text{ for } p = 2 \text{ or } p = 3 \tag{5}$$

and

$$\uparrow p = p + 1 \text{ for } p = 2 \tag{6}$$

We see directly, that there is equality in (3) if and only if the pair $(p - 2, p)$ is a twin pair. Similarly, there is equality in (4) if and only if the pair $(p, p + 2)$ is a twin pair. □

It is not known, if there exist infinitely many such pairs, and this is one of the most notorious open problems of Number theory.

Lemma 2. *One has*

$$\uparrow p - \downarrow p \geq 6 \textit{ for any prime } p \geq 7. \tag{7}$$

Proof. By Lemma 1, one has $\uparrow p - \downarrow p \geq 4$ for any $p \geq 5$. We shall prove that for a prime $p \geq 7$, no all terms of the three numbers $p - 2, p, p + 2$ cannot be primes. This is true however for $p = 5$. It is well-known that any prime $p \geq 5$ can be written in one of the following forms: $p = 6k - 1$ or $p = 6k + 1$. In the first case, one has $p - 2 = 6k - 3$, divisible by 3, and so not prime for $k \geq 2$. In the second case $p + 2 = 6k + 3$ is divisible again by 3, and is not prime for $k \geq 1$. These prove essentially inequality (7).

Obviously, one has

$$\uparrow p - \downarrow p = 2 \text{ for } p = 2,$$

$$\uparrow p - \downarrow p = 3 \text{ for } p = 3,$$

$$\uparrow p - \downarrow p = 4 \text{ for } p = 4.$$

This means that one has

$$\uparrow p - \downarrow p \geq 2 \text{ for any } p \geq 2.$$

□

Lemma 3. *Let $\{x_i\}_{i=1}^r$ and $\{y_i\}_{i=1}^r$ be two sequences of positive real numbers. Then one has*

$$(x_1 + y_1) \dots (x_r + y_r) \geq x_1 \dots x_r + y_1 \dots y_r. \tag{7}$$

If $x_i - y_i \geq 1$, then

$$(x_1 - y_1) \dots (x_r - y_r) \leq x_1 \dots x_r - y_1 \dots y_r. \tag{8}$$

Proof. (8) is well-known, and follows, e.g., on induction upon r. For the proof of (9) apply (8) for $x_i - y_i$ instead of x_i and y_i for y_i. Then (8) becomes (9). □

Theorem 2.1. *One has*

$$\uparrow n \geq n + 2^{\Omega(n)} \text{ for any odd } n \geq 3 \tag{9}$$

and

$$\downarrow n \leq n - 2^{\Omega(n)} \text{ for any } n \text{ not divisible by } 6, \tag{10}$$

where $\Omega(n)$ denotes the total number of prime factors of n (i.e. for the prime factorization of n in the Introduction, $\Omega(n) = \sum_{i=1}^{r} a_r$).

Proof. By relation (4) of Lemma 1, and by Lemma 3 , relation (8), one can write

$$\uparrow n \geq (p_1 + 2)^{a_1} \dots (p_r + 2)^{a_r} \geq (p_1^{a_1} + 2^{a_1}) \dots (p_r^{a_r} + 2^{a_r}) \geq n + 2^{a_1 + \dots + a_r} = n + 2^{\Omega(n)},$$

which proves (10). The proof of (11) goes on the same lines, by using relation (5) of Lemma 1 and relation (9) of Lemma 3. □

Now, we extend relation (7).

Theorem 2.2. *One has*

$$\uparrow n - \downarrow n \geq 2^{\Omega(n)}, \text{ for any } n \geq 2. \tag{11}$$

Proof. Actually, we will prove a stronger result , by using the following inequality of Hölder. □

Lemma 4. *If $\{x_i\}_{i=1}^r$ and $\{y_i\}_{i=1}^r$ be two sequences of positive real numbers. Then one has*

$$((x_1 + y_1) \dots (x_r + y_r))^{1/r} \geq (x_1 \dots x_r)^{1/r} + (y_1 \dots y_r)^{1/r}. \tag{12}$$

Apply now inequality (7), and (13) for $x_i = 2^{a_i}, y_i = (\uparrow p_i)^{a_i}$. Then we get

$$\uparrow n \geq (2^{\Omega(n)/\omega(n)} + (\downarrow n)^{1/\omega(n)})^{\omega(n)}, \tag{13}$$

where $r = \omega(n)$ denotes the number of distinct prime factors of n.

It is immediate now that (14) is a refinement of (12), which follows at once from the inequality $(a+b)^r \geq a^r + b^r$, with a the first term, while b is the second term in the parentheses of (14). Therefore, (14) follows, even in the improved form (12).

The following limit properties are valid:

Theorem 2.3. *One has*

$$\lim_{p\to\infty} \frac{\downarrow p}{p} = \lim_{p\to\infty} \frac{\uparrow p}{p} = \lim_{p\to\infty} \frac{\downarrow p}{\uparrow p} = 1, \tag{14}$$

$$\liminf_{n\to\infty} \frac{\downarrow n}{n} = 0, \ \limsup_{n\to\infty} \frac{\downarrow n}{n} = 1, \tag{15}$$

$$\liminf_{n\to\infty} \frac{\uparrow n}{n} = 1, \ \limsup_{n\to\infty} \frac{\uparrow n}{n} = \infty. \tag{16}$$

Proof. Let $p_1 < p_2 < \cdots < p_{k-1} < p_k < p_{k+1} < \ldots$ be the increasing sequence of the consecutive primes, and suppose that $p = p_k$. Then one has $\downarrow p = p_{k-1}$ and $\uparrow p = p_{k+1}$.Therefore, relation (15) becomes

$$\lim_{k\to\infty} \frac{p_{k-1}}{p_k} = \lim_{k\to\infty} \frac{p_{k+1}}{p_k} = \lim_{k\to\infty} \frac{p_{k-1}}{p_{k+1}} = 1. \tag{17}$$

The first two relations are well-known (see e.g. [3]), and are in fact consequences of the prime number theorem written in the form

$$\lim_{k\to\infty} \frac{p_k}{k \log k} = 1. \tag{18}$$

The last relation of (18) follows by the identity

$$\frac{p_{k-1}}{p_{k+1}} = \frac{p_{k-1}}{p_k} . \frac{p_k}{p_{k+1}},$$

and the first two relations. For the proof of the first relation of (16) it is sufficient to consider the sequence of numbers $n = 3^k$. Then

$$\frac{\downarrow n}{n} = \left(\frac{2}{3}\right)^k,$$

which tends to zero, as k tends to infinity.

For the second relation of (16) remark that $\frac{\downarrow n}{n} \leq 1$, and for the particular case $n = p$ (prime), by (15) the limit is 1. Therefore the $\limsup$ should be equal to 1.

The first equality of (17) follows on the same lines, by remarking that $\frac{\downarrow n}{n} \geq 1$, and using again (15). For the second relation, let us take again $n = 3^k$, when

$$\frac{\downarrow n}{n} = \left(\frac{5}{3}\right)^k,$$

which tends to infinity for k tending to ∞.

It is immediate consequence of (15) and (17) that

$$\liminf_{n\to\infty} \frac{\uparrow n}{\downarrow n} = 1 \text{ and } \limsup_{n\to\infty} \frac{\uparrow n}{\downarrow n} = \infty. \qquad \square$$

By a particular case of a theorem of Maynard ([2]) one gets that

$$\lim_{n\to\infty}\inf\,(p_{k+1}-p_{k-1})\le C,$$

where $C>0$ is a constant. This implies immediately:

Theorem 2.4. *One has*

$$\lim_{p\to\infty}\inf\frac{\uparrow p - \downarrow p}{\log p}=0. \tag{19}$$

One has

$$\lim_{n\to\infty}\inf\frac{\uparrow n - \downarrow n}{\log n}=0 \textit{ and } \lim_{n\to\infty}\sup\frac{\uparrow n - \downarrow n}{\log n}=\infty. \tag{20}$$

The first relation of (20) is a consequence of (19), while the second relation follows by the remark that

$$\frac{p_{k+1}-p_{k-1}}{\log p_k}>\frac{p_{k+1}-p_k}{\log p_k}=w_k$$

and it is well-known by a result of Westzynthius (see, [3]) that $\lim\sup w_k=\infty$.

Suggested by Lemma 2, we formulate

Conjecture 1. *For each prime number p:*

$$\lim_{p\to\infty}\inf\,(\uparrow p - \downarrow p)=6.$$

For the prime number p let us define

$$\Delta(p)=\frac{\uparrow p + \downarrow p}{2},$$

$$E(p)=\frac{\uparrow p - \downarrow p}{2},$$

$$Z(p)=|p-\Delta(p)|.$$

Let us construct the following Table.

p	$\Delta(p)$	$E(p)$	$Z(p)$
5	5	2	0
7	8	3	1
11	10	3	1
13	14	3	1
17	16	3	1
19	20	3	1
23	24	5	1
29	27	4	2
31	33	4	2
37	36	5	1
41	40	3	1

p	$\Delta(p)$	$E(p)$	$Z(p)$
47	44	3	1
53	53	6	0
59	57	4	2
61	63	4	2
67	66	5	1
71	70	3	1
73	75	4	2
79	78	5	1
83	84	5	1
89	90	8	1
97	95	8	2

Conjecture 2. *For each prime number p:*

$$[\ln p] \leq \max_{q \leq p} E(q).$$

Conjecture 3. *For each prime number p:*

$$[\ln \ln p] \leq \max_{q \leq p} Z(q).$$

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