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The q-Lah numbers and the n-th q-derivative of $\exp_q\left(\frac{1}{x}\right)$

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Abstract: A recently reported nice and surprising property of the Lah numbers is shown to hold for the q-Lah numbers as well, i.e., they can be obtained by taking successive q-derivatives of $\exp_q\left(\frac{1}{x}\right)$, where $\exp_q(x)$ is the q-exponential.

Keywords: q-Lah numbers, q-exponential.
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1 Introduction

The Lah numbers are the transformation coefficients allowing the expression of a rising factorial as a linear combination of falling factorials, i.e.,

$$x(x+1)(x+2)\cdots(x+k-1) = \sum_{\ell=1}^{k} L(k,\ell)x(x-1)(x-2)\cdots(x-\ell+1)\,.$$

The Lah numbers are given by the simple explicit expression

$$L(k,\ell) = \binom{k}{\ell}\frac{(k-1)!}{(\ell-1)!}\,.$$

They are closely related to the Stirling numbers, satisfying

$$L(k,\ell) = \sum_{j=\ell}^{k} \begin{bmatrix} k \\ j \end{bmatrix} \begin{Bmatrix} j \\ \ell \end{Bmatrix}\,,$$

and they satisfy the recurrence relation

$$L(k+1,\ell) = (k+\ell)L(k,\ell) + L(k,\ell-1)\,,$$

with $L(1,1) = 1$.

It was recently shown by Daboul *et al.* [1] that the k-th derivative of $\exp\left(\frac{1}{x}\right)$ yields the Lah numbers, i.e.,

$$\frac{d^k}{dx^k}\left(\exp\left(\frac{1}{x}\right)\right) = (-1)^k \exp\left(\frac{1}{x}\right)\sum_{\ell=1}^{k}\frac{L(k,\ell)}{x^{k+\ell}}\,. \tag{1}$$

The q-Lah numbers

$$[n]_q[n+1]_q\cdots[n+k-1]_q = \sum_{\ell=1}^{k} L_q(k,\ell)[n]_q[n-1]_q\cdots[n-\ell+1]_q\,, \tag{2}$$

were introduced by Garsia and Remmel [2], who derived the recurrence relation

$$L_q(k+1,\ell) = [k+\ell]_q L_q(k,\ell) + q^{k+\ell-1}L_q(k,\ell-1) \tag{3}$$

and the explicit expression

$$L_q(k,\ell) = \binom{k}{\ell}_q \frac{[k-1]_q!}{[\ell-1]_q!}q^{\ell(\ell-1)}\,.$$

In the next section we derive the q-analogue of equation (1).

2 q-analogue of equation (1)

Recall the definitions of the q-exponential

$$\exp_q(x) = \sum_{i=0}^{\infty}\frac{x^i}{[i]_q!}\,,$$

of the q-derivative

$$Df(x) = \frac{f(qx)-f(x)}{x(q-1)}\,,$$

and the q-Leibniz rule for the q-derivative of a product

$$D\Big(f(x)g(x)\Big) = (Df(x))g(qx) + f(x)(Dg(x))\,.$$

The following relations are easily established

$$D\left(\frac{1}{(xq^{k-1})^{\ell}}\right) = -q^k\frac{[\ell]_q}{(xq^k)^{\ell+1}}\,,$$

and

$$D\exp_q\left(\frac{1}{xq^{k-1}}\right) = -\frac{q^k}{(xq^k)^2}\exp_q\left(\frac{1}{xq^k}\right)\,. \tag{4}$$

Hence,

$$D\left(\frac{1}{(xq^{k-1})^\ell}\exp_q\left(\frac{1}{xq^{k-1}}\right)\right) = -q^k\left(\frac{[\ell]_q}{(xq^k)^{\ell+1}} + \frac{q^\ell}{(xq^k)^{\ell+2}}\right)\exp_q\left(\frac{1}{xq^k}\right). \tag{5}$$

The expression for the q-derivative of $\exp_q\left(\frac{1}{x}\right)$ yields a finite sum involving the q-Lah numbers $L_q(k,\ell)$, defined by equation (2), as stated in

Theorem 1.

$$D^k\left(\exp_q\left(\frac{1}{x}\right)\right) = (-1)^k q^{\binom{k+1}{2}}\exp_q\left(\frac{1}{x_k}\right)\sum_{\ell=1}^{k}\frac{L_q(k,\ell)}{x_k^{k+\ell}} \tag{6}$$

where $x_k = xq^k$.

Proof. By induction. For $k = 1$ we obtain $D\left(\exp_q\left(\frac{1}{x}\right)\right) = -q\exp_q\left(\frac{1}{qx}\right)\frac{L_q(1,1)}{(xq)^2}$, which is consistent with equation (4), since $L_q(1,1) = 1$.

Now, assume that the theorem holds for k and take the q-derivatives of both sides of equation (6). The left-hand-side becomes

$$D^{k+1}\left(\exp_q\left(\frac{1}{x}\right)\right) = (-1)^{k+1}q^{\binom{k+2}{2}}\exp_q\left(\frac{1}{x_{k+1}}\right)\sum_{\ell=1}^{k+1}\frac{L_q(k+1,\ell)}{x_{k+1}^{k+\ell+1}}$$

and, using equation (5) followed by an appropriate shift of the summation index, the right-hand side becomes

$$(-1)^{k+1}q^{\binom{k+2}{2}}\exp_q\left(\frac{1}{x_{k+1}}\right)\sum_{\ell=1}^{k+1}\frac{1}{x_{k+1}^{k+\ell+1}}\Big([k+\ell]_q L_q(k,\ell) + q^{k+\ell-1}L_q(k,\ell-1)\Big).$$

One readily obtains the recurrence relation, equation (3). □

Different q-analogues of the Lah numbers have been considered by Lindsay, Mansour and Shattuck [3] and by Wagner [4]. Whether they can be produced by appropriately modified q-differentiations and q-exponentials remains to be seen.

References

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