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On 2-powerfully perfect numbers in three quadratic rings

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Abstract: Using an extension of the abundancy index to imaginary quadratic rings with unique factorization, we define what we call n-powerfully perfect numbers in these rings. This definition serves to extend the concept of perfect numbers that have been defined and studied in the integers. We investigate the properties of 2-powerfully perfect numbers in the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, the three imaginary quadratic rings with unique factorization in which 2 is not a prime.
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1 Introduction

Throughout this paper, we will let $\mathbb{N}$ denote the set of positive integers, and we will let $\mathbb{N}_0$ denote the set of nonnegative integers.

The arithmetic functions σ_k are defined, for every integer k, by $\sigma_k(n) = \sum_{\substack{c \mid n \\ c>0}} c^k$. For each integer $k \neq 0$, σ_k is multiplicative and satisfies $\sigma_k(p^\alpha) = \dfrac{p^{k(\alpha+1)}-1}{p^k-1}$ for all (integer) primes p and positive integers α. The abundancy index of a positive integer n is defined by $I(n) = \dfrac{\sigma_1(n)}{n}$. A positive integer n is said to be t-perfect if $I(n) = t$ for a positive integer $t \geq 2$, and 2-perfect numbers are called perfect numbers.

For any square-free integer d, let $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ be the quadratic integer ring given by

$$\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \begin{cases} \mathbb{Z}[\frac{1+\sqrt{d}}{2}], & \text{if } d \equiv 1 \pmod 4; \\ \mathbb{Z}[\sqrt{d}], & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Throughout the remainder of this paper, we will work in the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for different values of d. We will use the symbol "$|$" to mean "divides" in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ in which we are working. Whenever we are working in a ring other than $\mathbb{Z}$, we will make sure to emphasize when we wish to state that one integer divides another in $\mathbb{Z}$. For example, if we are working in $\mathbb{Z}[i]$, the ring of Gaussian integers, we might say that $1+i|1+3i$ and that $2|6$ in $\mathbb{Z}$. We will also refer to primes in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ as "primes," whereas we will refer to (positive) primes in $\mathbb{Z}$ as "integer primes." For an integer prime p and a nonzero integer n, we will let $\upsilon_p(n)$ denote the largest integer k such that $p^k|n$ in $\mathbb{Z}$. For a prime π and a nonzero number $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we will let $\rho_\pi(x)$ denote the largest integer k such that $\pi^k|x$. Furthermore, we will henceforth focus exclusively on values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain and $d < 0$. In other words, $d \in K$, where we will define K to be the set $\{-163, -67, -43, -19, -11, -7, -3, -2, -1\}$. The set K is known to be the complete set of negative values of d for which $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a unique factorization domain [3].

For an element $a + b\sqrt{d} \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $a, b \in \mathbb{Q}$, we define the conjugate by $\overline{a + b\sqrt{d}} = a - b\sqrt{d}$. The norm and absolute value of an element z are defined, respectively, by $N(z) = z\overline{z}$ and $|z| = \sqrt{N(z)}$. For $x, y \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we say that x and y are associated, denoted $x \sim y$, if and only if $x = uy$ for some unit u in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Furthermore, we will make repeated use of the following well-known facts.

Fact 1.1. *Let $d \in K$. If p is an integer prime, then exactly one of the following is true.*

- *p is also a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say that p is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*
- *$p \sim \pi^2$ and $\pi \sim \overline{\pi}$ for some prime $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say p ramifies (or p is ramified) in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*
- *$p = \pi\overline{\pi}$ and $\pi \not\sim \overline{\pi}$ for some prime $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. In this case, we say p splits (or p is split) in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$.*

Fact 1.2. *Let $d \in K$. If $\pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ is a prime, then exactly one of the following is true.*

- *$\pi \sim q$ and $N(\pi) = q^2$ for some inert integer prime q.*
- *$\pi \sim \overline{\pi}$ and $N(\pi) = p$ for some ramified integer prime p.*
- *$\pi \not\sim \overline{\pi}$ and $N(\pi) = N(\overline{\pi}) = p$ for some split integer prime p.*

Fact 1.3. *If $d \in K$, q is an integer prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, and $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$, then $\upsilon_q(N(x))$ is even and $\rho_q(x) = \frac{1}{2}\upsilon_q(N(x))$.*

Fact 1.4. *Let p be an odd integer prime. Then p ramifies in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if and only if $p|d$ in $\mathbb{Z}$. If $p \nmid d$ in $\mathbb{Z}$, then p splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if and only if d is a quadratic residue modulo p. Note that this implies that p is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ if and only if $p \nmid d$ in $\mathbb{Z}$ and d is a quadratic nonresidue modulo p. Also, the integer prime 2 ramifies in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$ and $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, splits in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, and is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ for all $d \in K\backslash\{-1,-2,-7\}$.*

Fact 1.5. *Let $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})}$ be the set of units in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Then $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-1})} = \{\pm 1, \pm i\}$, $\mathcal{O}^*_{\mathbb{Q}(\sqrt{-3})} = \left\{\pm 1, \pm\frac{1+\sqrt{-3}}{2}, \pm\frac{1-\sqrt{-3}}{2}\right\}$, and $\mathcal{O}^*_{\mathbb{Q}(\sqrt{d})} = \{\pm 1\}$ whenever $d \in K\backslash\{-1,-3\}$.*

For a nonzero complex number z, let $\arg(z)$ denote the argument, or angle, of z. We convene to write $\arg(z) \in [0, 2\pi)$ for all $z \in \mathbb{C}$. For each $d \in K$, we define the set $A(d)$ by

$$A(d) = \begin{cases} \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \leq \arg(z) < \frac{\pi}{2}\}, & \text{if } d = -1; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \leq \arg(z) < \frac{\pi}{3}\}, & \text{if } d = -3; \\ \{z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} : 0 \leq \arg(z) < \pi\}, & \text{otherwise.} \end{cases}$$

Thus, every nonzero element of $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ can be written uniquely as a unit times a product of primes in $A(d)$. Also, every $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ is associated to a unique element of $A(d)$. The author has defined analogues of the arithmetic functions σ_k in quadratic rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$ [1], and we will state the important definitions and properties for the sake of completeness.

Definition 1.1. Let $d \in K$, and let $n \in \mathbb{Z}$. Define the function $\delta_n \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to [1, \infty)$ by

$$\delta_n(z) = \sum_{\substack{x|z \\ x \in A(d)}} |x|^n.$$

Remark 1.1. We note that, for each x in the summation in the above definition, we may cavalierly replace x with one of its associates. This is because associated numbers have the same absolute value. In other words, the only reason for the criterion $x \in A(d)$ in the summation that appears in Definition 1.1 is to forbid us from counting associated divisors as distinct terms in the summation, but we may choose to use any of the associated divisors as long as we only choose one. This should not be confused with how we count conjugate divisors (we treat $2+i$ and $2-i$ as distinct divisors of 5 in $\mathbb{Z}[i]$ because $2+i \not\sim 2-i$).

Remark 1.2. We mention that the function δ_n is different in each ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Perhaps it would be more accurate to write $\delta_n(z, d)$, but we will omit the latter component for convenience. We note that we will also use this convention with functions such as I_n (which we will define soon).

We will say that a function $f \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to \mathbb{R}$ is multiplicative if $f(xy) = f(x)f(y)$ whenever x and y are relatively prime (have no nonunit common divisors). The author has shown that, for any integer n, δ_n is multiplicative [1].

Definition 1.2. For each positive integer n, define the function $I_n \colon \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\} \to [1, \infty)$ by $I_n(z) = \dfrac{\delta_n(z)}{|z|^n}$. For a positive integer $t \geq 2$, we say that a number $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ is *n-powerfully t-perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$* if $I_n(z) = t$, and, if $t = 2$, we simply say that z is *n-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$*.

As an example, we will let $d = -1$ so that $\mathcal{O}_{\mathbb{Q}(\sqrt{d})} = \mathbb{Z}[i]$. Let us compute $I_2(9+3i)$. We have $9+3i = 3(1+i)(2-i)$, so $\delta_2(9+3i) = N(1)+N(3)+N(1+i)+N(2-i)+N(3(1+i))+N(3(2-i))+N((1+i)(2-i))+N(3(1+i)(2-i)) = 1+9+2+5+18+45+10+90 = 180$. Then $I_2(9+3i) = \dfrac{180}{N(3(1+i)(2-i))} = 2$, so $9+3i$ is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$.

Theorem 1.1. *Let $n \in \mathbb{N}$, $d \in K$, and $z_1, z_2, \pi \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$ with π a prime. Then, if we are working in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, the following statements are true.*

(a) The range of I_n is a subset of the interval $[1, \infty)$, and $I_n(z_1) = 1$ if and only if z_1 is a unit in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. If n is even, then $I_n(z_1) \in \mathbb{Q}$.

(b) I_n is multiplicative.

(c) $I_n(z_1) = \delta_{-n}(z_1)$.

(d) If $z_1|z_2$, then $I_n(z_1) \leq I_n(z_2)$, with equality if and only if $z_1 \sim z_2$.

We refer the reader to [1] for a proof of Theorem 1.1. The author has already investigated 1-powerfully t-perfect numbers in imaginary quadratic rings with unique factorization, and he has shown that, for any integers $n \geq 3$ and $t \geq 2$, no n-powerfully t-perfect numbers exist in these rings [2]. Hence, the remainder of this paper will focus on the interesting topic of 2-powerfully t-perfect numbers.

2 Investigating 2-powerfully t-perfect numbers

Trying to find 2-powerfully t-perfect numbers is quite a pleasant activity. One reason for this is that 2 is the only positive integer n for which there exist n-powerfully t-perfect numbers that are not associated to integers [2]. For example, in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $3+9i$ is 2-powerfully perfect, and $30+30i$ is 2-powerfully 3-perfect. We will also utilize the helpful fact that, for any $d \in K$ and $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\backslash\{0\}$, we have $N(z), \delta_2(z) \in \mathbb{N}$. In this section, we will focus on the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, which are the only rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$ in which 2 is not inert.

Theorem 2.1. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in \{-1, -2\}$. Then 2 ramifies in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, so we may write $2 \sim \xi^2$ for some prime ξ satisfying $\xi \sim \overline{\xi}$ and $N(\xi) = 2$. Suppose z is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ and $\xi|z$. Then we may write $z = \xi^\gamma x$, where $\gamma \in \mathbb{N}$, $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $\xi \nmid x$, and $2^{\gamma+1} - 1$ is a Mersenne prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Furthermore, there exists an odd positive integer m such that $\delta_2(x) = 2^{\gamma+1}m$ and $N(x) = (2^{\gamma+1} - 1)m$.*

Proof. We know the first part of the theorem, which is stated simply to introduce notation. All that we need to prove is the final sentence of the theorem, as well as the fact that $2^{\gamma+1} - 1$ is a Mersenne prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. As z is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, we have

$$\delta_2(z) = 2N(z) = 2N(\xi^\gamma)N(x) = 2^{\gamma+1}N(x).$$

However, we also have

$$\delta_2(z) = \delta_2(\xi^\gamma)\delta_2(x) = \left(\sum_{j=0}^{\gamma} N(\xi^j)\right)\delta_2(x) = \left(\sum_{j=0}^{\gamma} 2^j\right)\delta_2(x) = (2^{\gamma+1}-1)\delta_2(x).$$

Therefore, $2^{\gamma+1}N(x) = (2^{\gamma+1}-1)\delta_2(x)$. As $2^{\gamma+1}-1$ is odd, we find that $2^{\gamma+1}|\delta_2(x)$ in $\mathbb{Z}$. We may then write $\delta_2(x) = 2^{\gamma+1}m$ for some positive integer m. Substituting this new expression for $\delta_2(x)$ into the equation $2^{\gamma+1}N(x) = (2^{\gamma+1}-1)\delta_2(x)$, we find $N(x) = (2^{\gamma+1}-1)m$. This tells us that m is odd because $\xi \nmid x$ (implying that $2 \nmid N(x)$ in $\mathbb{Z}$). Suppose that $2^{\gamma+1}-1$ is not a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ so that we may write $2^{\gamma+1}-1 = y_1y_2$, where $y_1, y_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ satisfy $1 < N(y_1) \leq N(y_2) < N(2^{\gamma+1}-1) = (2^{\gamma+1}-1)^2$. Then, because $N(y_1)N(y_2) = N(2^{\gamma+1}-1) = (2^{\gamma+1}-1)^2$, we see that $N(y_1) \leq 2^{\gamma+1}-1$. Now, let π_0 be a prime that divides y_1. Then $\pi_0|N(x)$, which implies that either $\pi_0|x$ or $\overline{\pi_0}|x$. If $\pi_0|x$, write $\pi = \pi_0$. Otherwise, write $\pi = \overline{\pi_0}$. Then $N(\pi) \leq N(y_1) \leq 2^{\gamma+1}-1$, and $\dfrac{x}{\pi}$ is a nonunit proper divisor of x. This implies that

$$\delta_2(x) \geq 1 + N\left(\frac{x}{\pi}\right) + N(x) = 1 + \frac{N(x)}{N(\pi)} + N(x) = 1 + \frac{(2^{\gamma+1}-1)m}{N(\pi)} + (2^{\gamma+1}-1)m$$

$$\geq 1 + \frac{(2^{\gamma+1}-1)m}{2^{\gamma+1}-1} + (2^{\gamma+1}-1)m = 1 + 2^{\gamma+1}m.$$

However, this contradicts the fact that $\delta_2(x) = 2^{\gamma+1}m$, so we conclude that $2^{\gamma+1}-1$ is a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. Furthermore, because $2^{\gamma+1}-1$ is an integer, we conclude that $2^{\gamma+1}-1$ is an inert integer prime that is also a Mersenne prime. $\square$

Theorem 2.2. *Let z, m, γ, and x be as in Theorem 2.1. Write $q = 2^{\gamma+1}-1$ and $m = q^k v$, where $k \in \mathbb{N}_0$, $v \in \mathbb{N}$, and $q \nmid v$ in $\mathbb{Z}$. Then k is odd, $v \geq q+2$, and*

$$m \geq q^{k+1} + (q+3)\sum_{j=0}^{\frac{k-1}{2}} q^{2j} \geq q^2 + q + 3.$$

Proof. First, note that q is inert and $v_q(N(x)) = k+1$. Therefore, Fact 1.3 implies that k is odd and $\rho_q(x) = \dfrac{k+1}{2}$. Next, assume that $v = 1$. Then $m = q^k$, so $x \sim q^{\frac{k+1}{2}}$. This implies that $\delta_2(x) = \displaystyle\sum_{j=0}^{\frac{k+1}{2}} q^{2j} \equiv 1 \pmod{q}$. However, this contradicts Theorem 2.1, which tells us, under the assumption $m = q^k$, that $\delta_2(x) = 2^{\gamma+1}m = (q+1)m = (q+1)q^k \equiv 0 \pmod{q}$. Therefore, $v > 1$. Now, write $y = \dfrac{x}{q^{(k+1)/2}}$. Then, using Theorem 2.1,

$$N(y) = \frac{N(x)}{N(q^{\frac{k+1}{2}})} = \frac{qm}{q^{k+1}} = \frac{q^{k+1}v}{q^{k+1}} = v.$$

Because $\rho_q(x) = \dfrac{k+1}{2}$, we see that y and q^{k+1} are relatively prime. Therefore,

$$\delta_2(x) = \delta_2(y)\delta_2(q^{\frac{k+1}{2}}) = \delta_2(y)\sum_{j=0}^{\frac{k+1}{2}} q^{2j} \geq (v+1)\sum_{j=0}^{\frac{k+1}{2}} q^{2j}.$$

Theorem 2.1 states that $\delta_2(x) = 2^{\gamma+1}m = (q+1)m$, so we have

$$(q+1)m \geq (v+1)\sum_{j=0}^{\frac{k+1}{2}} q^{2j} = q^{k+1}v + q^{k+1} + (v+1)\sum_{j=0}^{\frac{k-1}{2}} q^{2j} = qm + q^{k+1} + (v+1)\sum_{j=0}^{\frac{k-1}{2}} q^{2j}.$$

We can simplify this last inequality to get

$$m \geq q^{k+1} + (v+1)\sum_{j=0}^{\frac{k-1}{2}} q^{2j}. \qquad (1)$$

Therefore, $v = \dfrac{m}{q^k} \geq q + (v+1)\displaystyle\sum_{j=0}^{\frac{k-1}{2}} q^{2j-k} > q$. As v and q are both odd and $v > q$, we conclude that $v \geq q+2$. Substituting this into (1), we have

$$m \geq q^{k+1} + (q+3)\sum_{j=0}^{\frac{k-1}{2}} q^{2j} \geq q^2 + q + 3,$$

which completes the proof. $\square$

It is interesting to note that, in the case $z = 3+9i$ in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, the inequalities in Theorem 2.2 are, in fact, equalities. That is, $q = 3$, $v = q+2 = 5$, and $m = q^2+q+3 = 15$. It seems likely, in light of the inequalities in Theorem 2.2, that the value of k in Theorem 2.2 should have to be 1.

We now prove results similar to Theorems 2.1 and 2.2 in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$.

Theorem 2.3. *Let us work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ so that 2 splits as $2 = \varepsilon\overline{\varepsilon}$, where $\varepsilon = \frac{1+\sqrt{-7}}{2}$. Suppose z is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ and $2|N(z)$ in $\mathbb{Z}$. Then either $z = \varepsilon^{\gamma}x$ or $z = \overline{\varepsilon}^{\gamma}x$, where $\gamma \in \mathbb{N}$, $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, $2 \nmid N(x)$ in $\mathbb{Z}$, and $2^{\gamma+1}-1$ is a Mersenne prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. Furthermore, there exists an odd positive integer m such that $\delta_2(x) = 2^{\gamma+1}m$ and $N(x) = (2^{\gamma+1}-1)m$.*

Proof. We know that we may write $z = \varepsilon^{\gamma_1}\overline{\varepsilon}^{\gamma_2}x$, where $\gamma_1, \gamma_2 \in \mathbb{N}_0$, $x \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, and $2 \nmid N(x)$ in $\mathbb{Z}$. Furthermore, we know from the fact that $2|N(z)$ in $\mathbb{Z}$ that γ_1 and γ_2 are not both zero. We must prove that either $\gamma_1 = 0$ or $\gamma_2 = 0$. Then, after setting $\gamma = \gamma_1 + \gamma_2$, we need to prove the final sentence of the theorem and the fact that $2^{\gamma+1}-1$ is a Mersenne prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$.

As z is 2-powerfully perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, we have

$$\delta_2(z) = 2N(z) = 2N(\varepsilon^{\gamma_1})N(\overline{\varepsilon}^{\gamma_2})N(x) = 2^{\gamma_1+\gamma_2+1}N(x).$$

However, we also have

$$\delta_2(z) = \delta_2(\varepsilon^{\gamma_1})\delta_2(\overline{\varepsilon}^{\gamma_2})\delta_2(x) = \left(\sum_{j=0}^{\gamma_1} N(\varepsilon^j)\right)\left(\sum_{j=0}^{\gamma_2} N(\overline{\varepsilon}^j)\right)\delta_2(x)$$

$$= \left(\sum_{j=0}^{\gamma_1} 2^j\right)\left(\sum_{j=0}^{\gamma_2} 2^j\right)\delta_2(x) = (2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)\delta_2(x).$$

Therefore, $2^{\gamma_1+\gamma_2+1}N(x) = (2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)\delta_2(x)$. As $(2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)$ is odd, we find that $2^{\gamma_1+\gamma_2+1}|\delta_2(x)$ in $\mathbb{Z}$. We may then write $\delta_2(x) = 2^{\gamma_1+\gamma_2+1}m$ for some positive integer m. Substituting this new expression for $\delta_2(x)$ into the equation $2^{\gamma_1+\gamma_2+1}N(x) = (2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)\delta_2(x)$, we find $N(x) = (2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)m$. This tells us that m is odd because $2 \nmid N(x)$ in $\mathbb{Z}$. Now, $2^{\gamma_1+\gamma_2+1}m = \delta_2(x) \geq 1+N(x) = 1+(2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1)m$, so $2^{\gamma_1+\gamma_2+1} > (2^{\gamma_1+1}-1)(2^{\gamma_2+1}-1) = 2\cdot 2^{\gamma_1+\gamma_2+1} - 2^{\gamma_1+1} - 2^{\gamma_2+1} + 1$. Simplifying this inequality, we have $2^{\gamma_1+1} + 2^{\gamma_2+1} > 2^{\gamma_1+\gamma_2+1} + 1$, which is impossible unless $\gamma_1 = 0$ or $\gamma_2 = 0$. Therefore, either $z = \varepsilon^{\gamma_1}x$ or $z = \overline{\varepsilon}^{\gamma_2}x$. Either way, if we write $\gamma = \gamma_1 + \gamma_2$, then we have $\delta_2(x) = 2^{\gamma+1}m$ and $N(x) = (2^{\gamma+1}-1)m$. Suppose that $2^{\gamma+1}-1$ is not a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ so that we may write $2^{\gamma+1}-1 = y_1y_2$, where $y_1, y_2 \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ satisfy $1 < N(y_1) \leq N(y_2) < N(2^{\gamma+1}-1) = (2^{\gamma+1}-1)^2$. Then, because $N(y_1)N(y_2) = N(2^{\gamma+1}-1) = (2^{\gamma+1}-1)^2$, we see that $N(y_1) \leq 2^{\gamma+1}-1$. Now, let π_0 be a prime that divides y_1. Then $\pi_0|N(x)$, which implies that either $\pi_0|x$ or $\overline{\pi_0}|x$. If $\pi_0|x$, write $\pi = \pi_0$. Otherwise, write $\pi = \overline{\pi_0}$. Then $N(\pi) \leq N(y_1) \leq 2^{\gamma+1}-1$, and $\frac{x}{\pi}$ is a nonunit proper divisor of x. This implies that

$$\delta_2(x) \geq 1 + N\left(\frac{x}{\pi}\right) + N(x) = 1 + \frac{N(x)}{N(\pi)} + N(x) = 1 + \frac{(2^{\gamma+1}-1)m}{N(\pi)} + (2^{\gamma+1}-1)m$$

$$\geq 1 + \frac{(2^{\gamma+1}-1)m}{2^{\gamma+1}-1} + (2^{\gamma+1}-1)m = 1 + 2^{\gamma+1}m.$$

However, this contradicts the fact that $\delta_2(x) = 2^{\gamma+1}m$, so we see that $2^{\gamma+1}-1$ is a prime in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. Furthermore, because $2^{\gamma+1}-1$ is an integer, we find that $2^{\gamma+1}-1$ is an inert integer prime that is also a Mersenne prime. □

Theorem 2.4. *Let z, m, γ, and x be as in Theorem 2.3. Write $q = 2^{\gamma+1}-1$ and $m = q^k v$, where $k \in \mathbb{N}_0$, $v \in \mathbb{N}$, and $q \nmid v$ in $\mathbb{Z}$. Then k is odd, $v \geq q+2$, $\gamma \equiv 1 \pmod 3$, $q \equiv 3 \pmod 7$, and*

$$m \geq q^{k+1} + (q+3)\sum_{j=0}^{\frac{k-1}{2}} q^{2j} \geq q^2 + q + 3.$$

Proof. Fact 1.4 tells us that an integer prime is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ if and only if that integer prime is congruent to 3, 5, or 6 modulo 7. Also, it is easy to see that powers of 2 cannot be congruent to 6 or 7 modulo 7. Therefore, as q is a Mersenne prime that is inert in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, we must have $q \equiv 3 \pmod 7$. This implies that $2^{\gamma+1} \equiv 4 \pmod 7$, so $\gamma \equiv 1 \pmod 3$. The proof of the rest of the theorem is identical to the proof of Theorem 2.2, except all references to Theorem 2.1 should be replaced with references to Theorem 2.3. □

Within the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, Theorems 2.1 through 2.4 examine some properties of 2-powerfully perfect numbers with even norms. These numbers are somewhat analogous to perfect numbers in $\mathbb{Z}$. The analogues of odd perfect numbers are then 2-powerfully perfect numbers with odd norms. We now briefly explore some of the properties that such numbers would need to exhibit.

Theorem 2.5. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$. Suppose $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\setminus\{0\}$ is such that $I_2(z) = 2$ and $N(z)$ is odd (suppose such a z exists). Then we may write $z \sim \pi^k x^2$, where $\pi, x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}\setminus\{0\}$, π is prime, and $k \in \mathbb{N}$. Furthermore, $k \equiv N(\pi) \equiv 1 \pmod 4$.*

Proof. First, let π_0 be a prime whose norm is odd, and let α be a positive integer. As

$$\delta_2(\pi_0^\alpha) = \sum_{j=0}^{\alpha} N(\pi_0^j) = \sum_{j=0}^{\alpha} N(\pi_0)^j$$

and $N(\pi_0)$ is odd, we see that α and $\delta_2(\pi_0^\alpha)$ have opposite parities.

Now, from $I_2(z) = 2$, we have $\delta_2(z) = 2N(z)$. Because $N(z)$ is odd, we find that $\delta_2(z) \equiv 2 \pmod 4$. Write $z = \prod_{j=1}^{r} \pi_j^{\alpha_j}$, where, for all distinct $j, l \in \{1, 2, \ldots, r\}$, π_j is prime, α_j is a positive integer, and $\pi_j \not\sim \pi_l$. Then $\delta_2(z) = \prod_{j=1}^{r} \delta_2(\pi_j^{\alpha_j})$. Because $\delta_2(z) \equiv 2 \pmod 4$, we find that there must be exactly one value of $j \in \{1, 2, \ldots, r\}$ such that $\delta_2(\pi_j^{\alpha_j})$ is even. This means that there is exactly one value of $j \in \{1, 2, \ldots, r\}$ such that α_j is odd. Therefore, $z \sim \pi^k x^2$, where $\pi, x \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, π is prime, and k is an odd positive integer. Furthermore, $\delta_2(\pi^k) \equiv 2 \pmod 4$.

If $N(\pi) = q^2$, where q is an inert integer prime, then

$$\delta_2(\pi^k) = \sum_{l=0}^{k} N(\pi^l) = \sum_{l=0}^{k} q^{2l} \equiv \sum_{l=0}^{k} 1 \equiv k + 1 \pmod 4.$$

Therefore, in this case, we have $k \equiv 1 \pmod 4$. Also, because $N(\pi) = q^2$ and q is odd, we know that $N(\pi) \equiv 1 \pmod 4$.

On the other hand, if $N(\pi) = p$ is an integer prime, then

$$\delta_2(\pi^k) = \sum_{l=0}^{k} N(\pi^l) = \sum_{l=0}^{k} p^l \equiv 2 \pmod 4,$$

which implies that $p \equiv k \equiv 1 \pmod 4$. □

Theorem 2.6. *Let us work in a ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in \{-1, -2\}$. Let $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})} \backslash \{0\}$ be such that $I_2(z) = 2$ and $N(z)$ is odd (suppose such a z exists). Then z has at least five nonassociated prime divisors.*

Proof. Suppose z has four or fewer nonassociated prime divisors. Then we may write $z \sim \pi_1^{\alpha_1}\pi_2^{\alpha_2}\pi_3^{\alpha_3}\pi_4^{\alpha_4}$, where, for all distinct $j, l \in \{1, 2, 3, 4\}$, π_j is prime, α_j is a nonnegative integer, and $\pi_j \not\sim \pi_l$.

First, let us deal with the case $d = -1$. In the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, the five primes (up to units) that have the smallest odd norms are $2 + i$, $1 + 2i$, 3, $3 + 2i$, and $2 + 3i$, which have norms 5, 5, 9, 13, and 13, respectively. Therefore,

$$I_2(z) = I_2(\pi_1^{\alpha_1}\pi_2^{\alpha_2}\pi_3^{\alpha_3}\pi_4^{\alpha_4})$$

$$= \left(\sum_{j=0}^{\alpha_1} \frac{1}{N(\pi_1)^j}\right)\left(\sum_{j=0}^{\alpha_2} \frac{1}{N(\pi_2)^j}\right)\left(\sum_{j=0}^{\alpha_3} \frac{1}{N(\pi_3)^j}\right)\left(\sum_{j=0}^{\alpha_4} \frac{1}{N(\pi_4)^j}\right)$$

$$< \left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_1)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_2)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_3)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_4)^j}\right)$$

$$\leq \left(\sum_{j=0}^{\infty}\frac{1}{5^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{5^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{9^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{13^j}\right) = \frac{5}{4}\cdot\frac{5}{4}\cdot\frac{9}{8}\cdot\frac{13}{12} < 2,$$

which is a contradiction.

Second, let us deal with the case $d = -2$. In the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, the integer prime 3 splits as $3 = (1+\sqrt{-2})(1-\sqrt{-2})$. Suppose $1+\sqrt{-2}|z$ and $1-\sqrt{-2}|z$. Then, because $N(1+\sqrt{-2}) = N(1-\sqrt{-2}) = 3 \not\equiv 1 \pmod 4$, Theorem 2.5 implies that $1+\sqrt{-2}$ and $1-\sqrt{-2}$ must both appear with even exponents in the prime factorization of z. In particular, $(1+\sqrt{-2})^2(1-\sqrt{-2})^2|z$. Therefore, by part (d) of Theorem 2.2,

$$I_2(z) \geq I_2((1+\sqrt{-2})^2)I_2((1-\sqrt{-2})^2) = \left(1+\frac{1}{3}+\frac{1}{9}\right)^2 > 2,$$

which is a contradiction. This implies that $1+\sqrt{-2}$ and $1-\sqrt{-2}$ cannot both divide z. Now, the six primes (up to units) that have the smallest odd norms are $1+\sqrt{-2}$, $1-\sqrt{-2}$, $3+\sqrt{-2}$, $3-\sqrt{-2}$, $3+2\sqrt{-2}$, and $3-2\sqrt{-2}$, which have norms 3, 3, 11, 11, 17, and 17, respectively. Because $1+\sqrt{-2}$ and $1-\sqrt{-2}$ cannot both divide z, we have

$$I_2(z) = I_2(\pi_1^{\alpha_1}\pi_2^{\alpha_2}\pi_3^{\alpha_3}\pi_4^{\alpha_4})$$

$$= \left(\sum_{j=0}^{\alpha_1}\frac{1}{N(\pi_1)^j}\right)\left(\sum_{j=0}^{\alpha_2}\frac{1}{N(\pi_2)^j}\right)\left(\sum_{j=0}^{\alpha_3}\frac{1}{N(\pi_3)^j}\right)\left(\sum_{j=0}^{\alpha_4}\frac{1}{N(\pi_4)^j}\right)$$

$$< \left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_1)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_2)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_3)^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_4)^j}\right)$$

$$\leq \left(\sum_{j=0}^{\infty}\frac{1}{3^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{11^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{11^j}\right)\left(\sum_{j=0}^{\infty}\frac{1}{17^j}\right) = \frac{3}{2}\cdot\frac{11}{10}\cdot\frac{11}{10}\cdot\frac{17}{16} < 2,$$

which is a contradiction. □

Theorem 2.7. *Let us work in the ring $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$. Let $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{-7})}\backslash\{0\}$ be such that $I_2(z) = 2$ and $N(z)$ is odd (suppose such a z exists). Then z has at least eleven nonassociated prime divisors.*

Proof. Suppose z has ten or fewer nonassociated prime divisors. Then we may write $z \sim \prod_{m=1}^{10}\pi_m^{\alpha_m}$, where, for all distinct $m, l \in \{1, 2, \ldots, 10\}$, π_m is prime, α_m is a nonnegative integer, and $\pi_m \not\sim \pi_l$. In $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, the eleven primes (up to units) that have the smallest odd norms are $\sqrt{-7}$, 3, $2+\sqrt{-7}$, $2-\sqrt{-7}$, $4+\sqrt{-7}$, $4-\sqrt{-7}$, 5, $1+2\sqrt{-7}$, $1-2\sqrt{-7}$, $3+2\sqrt{-7}$, and $3-2\sqrt{-7}$, which have norms 7, 9, 11, 11, 23, 23, 25, 29, 29, 37, and 37, respectively. Therefore,

$$I_2(z) = \prod_{m=1}^{10} I_2(\pi_m^{\alpha_m}) = \prod_{m=1}^{10}\left(\sum_{j=0}^{\alpha_m}\frac{1}{N(\pi_m)^j}\right) < \prod_{m=1}^{10}\left(\sum_{j=0}^{\infty}\frac{1}{N(\pi_m)^j}\right)$$

$$\leq \left(\sum_{j=0}^{\infty} \frac{1}{7^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{9^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{11^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{11^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{23^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{23^j}\right)$$
$$\cdot \left(\sum_{j=0}^{\infty} \frac{1}{25^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{29^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{29^j}\right) \left(\sum_{j=0}^{\infty} \frac{1}{37^j}\right)$$
$$= \frac{7}{6} \cdot \frac{9}{8} \cdot \frac{11}{10} \cdot \frac{11}{10} \cdot \frac{23}{22} \cdot \frac{23}{22} \cdot \frac{25}{24} \cdot \frac{29}{28} \cdot \frac{29}{28} \cdot \frac{37}{36} < 2,$$
which is a contradiction. □

We conclude this section with a remark about 2-powerfully perfect numbers in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ that have odd norms. In each of these three rings, there is a prime, say ξ, with norm 2. If $d \in \{-1, -2, -7\}$, $z \in \mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, $I_2(z) = 2$, and $N(z)$ is odd, then ξz is 2-powerfully 3-perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$. This is simply because, under these assumptions, we find that $I_2(\xi z) = I_2(\xi) I_2(z) = \frac{1+2}{2} I_2(z) = \frac{3}{2} \cdot 2 = 3$.

3 Further ideas and a conjecture

We admit that we directed almost all of our attention toward 2-powerfully perfect numbers, rather than the more general 2-powerfully t-perfect numbers. Hence, the subject of 2-powerfully t-perfect numbers awaits exploration. We also concentrated so heavily on the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$, $\mathcal{O}_{\mathbb{Q}(\sqrt{-2})}$, and $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$ when dealing with 2-powerfully perfect numbers that we left open all questions about the rings $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$ with $d \in K$ in which 2 is inert. We mentioned that $3+9i$ and $9+3i$ are 2-powerfully perfect and that $30+30i$ is 2-powerfully 3-perfect in $\mathcal{O}_{\mathbb{Q}(\sqrt{-1})}$. Are there other 2-powerfully t-perfect numbers in this ring? What about in other rings?

Referring to the concluding paragraph of Section 2, we might ask if there are other relationships between different types of n-powerfully t-perfect numbers. More specifically, in a given ring $\mathcal{O}_{\mathbb{Q}(\sqrt{d})}$, are there certain criteria which would guarantee that some specific multiple of an n_1-powerfully t_1-perfect number is n_2-powerfully t_2-perfect (for some $n_1, n_2, t_1, t_2 \in \mathbb{N}$ with $t_1, t_2 \geq 2$)?

Conjecture 3.1. *The value of k in Theorem 2.2 must be 1. Similarly, if there is a 2-powerfully perfect number in $\mathcal{O}_{\mathbb{Q}(\sqrt{-7})}$, then the value of k in Theorem 2.4 must be 1.*

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