Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 1, 99–100

A note on the density of quotients of primes in arithmetic progressions

Brian D. Sittinger

Department of Mathematics, California State University Channel Islands
1 University Drive, Camarillo, CA 93010, United States
e-mail: brian.sittinger@csuci.edu

Received: 4 July 2016 **Accepted:** 31 March 2017

Abstract: We give an alternate proof to the density of quotients of primes in an arithmetic progression which has been established by Micholson [2] and Starni [4].
Keywords: Arithmetic progression, Prime number.
AMS Classification: 11A25, 11A41.

1 Introduction

It is a standard fact from Real Analysis that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$. However, it is not as well-known that the set of quotients of *prime* numbers (from $\mathbb{Z}$) is dense in $\mathbb{R}$. One of the earliest appearances of this fact is in Sierpiński's textbook on number theory [3]. Inspired by [1], Starni in [4] gave a proof for a generalization of this to primes belonging in an arithmetic progression. More recently in [2], Micholson corrected Starni's proof. We give another proof of this result, albeit in a stronger form, below.

2 Main result

To prove this result, we use a prime number theorem version of Dirichlet's theorem concerning primes in an arithmetic progression.

Theorem 1. *(Dirichlet's Prime Number Theorem) Suppose a and m are positive integers such that $\gcd(a, m) = 1$. If $\pi(x; a, m)$ denote the number of primes less than or equal to x that are* *congruent to* a *modulo* m*. Then,*

$$\pi(x; a, m) \sim \frac{x}{\phi(m) \ln x}.$$

In other words, $\lim_{x\to\infty} \frac{\pi(x;a,m)}{\frac{x}{\phi(m)\ln x}} = 1$.

As a reminder, ϕ denotes Euler's phi function. Moreover, this theorem readily implies that there are infinitely many primes in the arithmetic progression $a \mod m$. Now, we state the main theorem of this note.

Theorem 2. *(Density of quotients of primes from arithmetic progressions)*
Fix $a, b, m, n \in \mathbb{N}$ *such that* $\gcd(a, m) = \gcd(b, n) = 1$*. Then,*

$$\left\{\frac{p}{q} \mid p, q \textit{ prime in } \mathbb{Z}, p \equiv a \textit{ mod } m, q \equiv b \textit{ mod } n\right\}$$

is dense in $\mathbb{R}$*.*

Proof: Without loss of generality, assume that $0 < c < d$. We need to show that any interval (c, d) contains a quotient of primes as prescribed.

To this end, we first observe that

$$\begin{aligned}
\lim_{x\to\infty}[\pi(dx; a, m) - \pi(cx; a, m)] &= \lim_{x\to\infty} \pi(dx; a, m)\left[1 - \frac{\pi(cx; a, m)}{\pi(dx; a, m)}\right] \\
&= \lim_{x\to\infty} \pi(dx; a, m)\left[1 - \frac{cx \ln(dx)}{dx \ln(cx)}\right] \\
&= \left(1 - \frac{c}{d}\right) \lim_{x\to\infty} \pi(dx; a, m) \\
&= \infty.
\end{aligned}$$

Therefore, for any sufficiently large x, there exists a prime $p \equiv a$ mod m such that $cx < p < dx$. Next, since there are infinitely many primes of the form b mod n, set $x = q$ where q is a sufficiently large prime that is congruent to b mod n. Hence, $cq < p < dq$ or equivalently $c < \frac{p}{q} < d$, as required. □

As a special case, note that setting $a = b$ and $m = n$ yields Starni and Micholson's result.

References

[1] Hobby, D. & Silberger D. M. (1993) Quotients of primes, *Amer. Math. Monthly*, 100, 50–52.

[2] Micholson A. (2012) Quotients of primes in arithmetic progressions, *Notes on Number Theory and Discrete Mathematics*, 18(2), 56–57.

[3] Sierpiński, W. (1988) *Elementary Theory of Numbers, 2nd Edition*. North-Holland, Amsterdam.

[4] Starni P. (1995) Answers to two questions concerning quotients of primes, *Amer. Math. Monthly*, 102, 347–349.