Notes on Number Theory and Discrete Mathematics
Print ISSN 1310–5132, Online ISSN 2367–8275
Vol. 23, 2017, No. 1, 91–98

On the bounds for the norms of circulant matrices with the Jacobsthal and Jacobsthal–Lucas numbers

Ş. Uygun and S. Yaşamalı

Department of Mathematics, Science and Art Faculty
Gaziantep University, Campus, 27310, Gaziantep, Turkey
E-mail: suygun@gantep.edu.tr

Received: 4 February 2016 **Accepted:** 10 November 2016

Abstract: In this study, we have found upper and lower bounds for the spectral norms of circulant matrices in the forms $A = C_r(j_0, j_1, ..., j_{n-1})$ and $B = C_r(c_0, c_1, ..., c_{n-1})$.

Keywords: Jacobsthal number, Jacobsthal–Lucas number, Circulant matrix, Norm.
AMS Classification: 15A36, 15A45, 15A60.

1 Introduction and Preliminaries

The Jacobsthal $\{j_n\}_{n\in\mathbb{N}}$, and the Jacobsthal–Lucas $\{c_n\}_{n\in\mathbb{N}}$ sequences are defined recurrently by

$$j_n = j_{n-1} + 2j_{n-2}, \quad j_0 = 0, \ j_1 = 1, \ n \geq 2, \tag{1}$$

$$c_n = c_{n-1} + 2c_{n-2}, \quad c_0 = 2, \ c_1 = 1, \ n \geq 2, \tag{2}$$

respectively.The first some Jacobsthal numbers are $0, 1, 1, 3, 5, 11$. The first some Jacobsthal–Lucas numbers are $2, 1, 5, 7, 17, 31$.

There have been several papers on the norms of special matrices [7–10]. Solak [8] has defined $A = [a_{ij}]$ and $B = [b_{ij}]$ as $n \times n$ circulant matrices, where $a_{ij} = F_{(\mathrm{mod}(j-i,n))}$ and $b_{ij} = L_{(\mathrm{mod}(j-i,n))}$, then he has given some bounds for the A and B matrices concerned with the spectral and Euclidean norms. Shen and Cen [10] have given upper and lower bounds for the spectral norms of r- circulant matrices $A = C_r(F_0^{(k,-1)}, F_1^{(k,-1)}, ..., F_{n-1}^{(k,-1)})$ and $B = C_r(L_0^{(k,-1)}, L_1^{(k,-1)}, ..., L_{n-1}^{(k,-1)})$. In addition, they also have obtained some bounds for the spectral norms of Hadamard and Kronecker products of these matrices.

In this paper we give lower and upper bounds for the spectral norms of the circulant matrices $A = C(j_0, j_1, ..., j_{n-1})$ and $B = C(c_0, c_1, ..., c_{n-1})$.

Recurrences (1) and (2) involve the characteristic equation

$$x^2 - x - 2 = 0$$

with roots

$$\alpha = 2, \quad \beta = -1.$$

Their Binet's formulas are defined by

$$j_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{3}$$

and

$$c_n = \alpha^n + \beta^n. \tag{4}$$

A matrix $C = [c_{ij}] \in M_{m,n}(C)$ is called a circulant matrix if it is of the form

$$c_{ij} = \begin{cases} c_{j-i}, & j \geq i \\ c_{n+j-i}, & j < i \end{cases}$$

For any $A = [a_{ij}] \in M_{m,n}(C)$. The Frobenius (or Euclidean) norm of matrix A is

$$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{\frac{1}{2}} \tag{5}$$

and the spectral norm of matrix A is

$$\|A\|_2 = \sqrt{\max_{1 \leq i \leq n} \lambda_i(A^H A)} \tag{6}$$

where $\lambda_i(A^H A)$ is eigenvalue of $A^H A$.

Lemma 1. *For any $A, B \in M_{m,n}(C)$, the Hadamard product of A, B is entrywise product and defined by [5,6]*

$$AoB = (a_{ij} b_{ij})$$

and have the following properties

$$\|AoB\|_2 \leq \|A\|_2 \|B\|_2$$

$$\|AoB\|_2 \leq r_1(A)\, c_1(B) \tag{7}$$

Lemma 2. *Let $A \in M_{m,n}(C)$, $B \in M_{p,q}(C)$ be given, then the Kronecker product of A, B is defined by*

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & \ddots & \vdots \\ a_{m1}B & \cdots & a_{mn}B \end{bmatrix}$$

and have the following property [11]

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2 .$$

Lemma 3. *Let $A \in M_{m,n}(C)$ be given, then the following inequality is fulfilled [4]*

$$\frac{1}{\sqrt{n}} \|A\|_F \leq \|A\|_2 \leq \|A\|_F \,. \tag{8}$$

2 The sum formulas of the square of Jacobsthal and Jacobsthal–Lucas numbers

Proposition 4. *The summation of the squares of Jacobsthal sequence is written by using Jacobsthal numbers as the following:*

$$\sum_{k=0}^{n-1} j_k^2 = \frac{1}{9}\left[j_{2n} + 2(-1)^n j_n + n\right]. \tag{9}$$

Proof. By using Binet forms we have

$$\begin{aligned}
\sum_{k=0}^{n-1} j_k^2 &= \left(\frac{2^k-(-1)^k}{3}\right)^2 = \sum_{k=0}^{n-1} \frac{2^{2k}+1-2(-2)^k}{9} \\
&= \frac{1}{9}\left[\frac{2^{2n}-1}{3} + n + 2\left[\frac{(-2)^n-1}{3}\right]\right] \\
&= \frac{1}{9}\left[\frac{2^{2n}-(-1)^{2n}}{3} + 2(-1)^n\left(\frac{2^n-(-1)^n}{3}\right) + n\right] \\
&= \frac{1}{9}\left[j_{2n} + 2(-1)^n j_n + n\right].
\end{aligned}$$

□

Proposition 5. *The summation of the squares of Jacobsthal numbers is written by using Jacobsthal–Lucas numbers as the following:*

$$\sum_{k=0}^{n-1} j_k^2 = \frac{1}{9}\left[\frac{c_n^2-4}{3} + n\right]. \tag{10}$$

Proof. From Binet forms we obtain

$$\begin{aligned}
\sum_{k=0}^{n-1} j_k^2 &= \left(\frac{2^k-(-1)^k}{3}\right)^2 = \sum_{k=0}^{n-1} \frac{2^{2k}+1-2(-2)^k}{9} \\
&= \frac{1}{9}\left[\frac{2^{2n}-1}{3} + n + 2\left[\frac{(-2)^n-1}{3}\right]\right] \\
&= \frac{1}{9}\left[\frac{c_n^2-4}{3} + n\right].
\end{aligned}$$

□

Proposition 6. *The summation of the squares of Jacobsthal–Lucas numbers is written by using Jacobsthal numbers as the following:*

$$\sum_{k=0}^{n-1} c_k^2 = 3j_n^2 + n = j_{2n} - 2(-1)^n j_n + n. \tag{11}$$

Proof. By using Binet forms we have

$$\begin{aligned}
\sum_{k=0}^{n-1} c_k^2 &= \sum_{k=0}^{n-1}\left(2^k+(-1)^k\right)^2=\sum_{k=0}^{n-1} 2^{2k}+1+2(-2)^k \\
&= \frac{2^{2n}-1}{3}+n+2\frac{(-2)^n-1}{-3} \\
&= 3j_n^2+n.
\end{aligned}$$

and for the other equality we have

$$\begin{aligned}
\sum_{k=0}^{n-1} c_k^2 &= \sum_{k=0}^{n-1}\left(2^k+(-1)^k\right)^2=\sum_{k=0}^{n-1} 2^{2k}+1+2(-2)^k \\
&= \frac{2^{2n}-1}{3}+n+2\frac{-(-2)^n+1}{3} \\
&= \frac{2^{2n}-(-1)^{2n}}{3}+n-2(-1)^n\left[\frac{2^n-(-1)^n}{3}\right] \\
&= j_{2n}-2(-1)^n j_n+n.
\end{aligned}$$

□

3 Lower and upper bounds of circulant matrices involving Jacobsthal numbers and Jacobsthal–Lucas numbers, depending on Jacobsthal and Jacobsthal–Lucas numbers

Theorem 7. *Let $A=C(j_0,j_1,...,j_{n-1})$ be circulant matrix, then we obtain*

$$\begin{aligned}
\sqrt{j_{2n}+2(-1)^n j_n+n} &\le \|A\|_2 \\
&\le \frac{1}{9}\sqrt{(j_{2n}+2(-1)^n j_n+n)(j_{2n}+2(-1)^n j_n+n+9)}
\end{aligned} \tag{12}$$

Proof. The matrix A is of the form

$$A=\begin{bmatrix} j_0 & j_1 & j_2 & \cdots & j_{n-1} \\ j_{n-1} & j_0 & j_1 & \cdots & j_{n-2} \\ j_{n-2} & j_{n-1} & j_0 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_1 & j_2 & j_3 & \cdots & j_0 \end{bmatrix} \tag{13}$$

From (5), (8), (9) we get

$$\|A\|_F^2=n\sum_{k=0}^{n-1} j_k^2=\frac{n}{9}\left[j_{2n}+2(-1)^n j_n+n\right]$$

$$\frac{1}{\sqrt{n}}\|A\|_F=\frac{1}{3}\sqrt{j_{2n}+2(-1)^n j_n+n}$$

$$\frac{1}{\sqrt{n}}\left\|A\right\|_F \leq \left\|A\right\|_2$$

$$\frac{1}{3}\sqrt{j_{2n}+2\left(-1\right)^n j_n+n} \leq \left\|A\right\|_2$$

On the other hand, let $A = BoC$ where

$$\begin{aligned} B=(b_{ij})&=\begin{cases} b_{ij}=j_{(\mathrm{mod}_{(j-i,n)})+1}, & i\geq j \\ b_{ij}=1, & i<j \end{cases} \\ C=(c_{ij})&=\begin{cases} c_{ij}=j_{(\mathrm{mod}_{(j-i,n)})+1}, & i<j \\ c_{ij}=1, & i\geq j \end{cases}. \end{aligned} \tag{14}$$

as

$$B=\begin{bmatrix} j_0 & 1 & 1 & \cdots & 1 \\ j_{n-1} & j_0 & 1 & \cdots & 1 \\ j_{n-2} & j_{n-1} & j_0 & \cdots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ j_1 & j_2 & j_3 & \cdots & j_0 \end{bmatrix}, C=\begin{bmatrix} 1 & j_1 & j_2 & \cdots & j_{n-1} \\ 1 & 1 & j_1 & \cdots & j_{n-2} \\ 1 & 1 & 1 & \cdots & j_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}$$

then

$$\begin{aligned} r_1\left(B\right) &= \max_{1\leq i\leq n}\sqrt{\sum_{j=1}^{n}\left|b_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|b_{nj}\right|^2}=\sqrt{\sum_{k=0}^{n-1}j_k^2}=\frac{1}{3}\sqrt{j_{2n}+2\left(-1\right)^n j_n+n} \\ c_1\left(C\right) &= \max_{1\leq j\leq n}\sqrt{\sum_{j=1}^{n}\left|c_{ij}\right|^2}=\sqrt{\sum_{j=1}^{n}\left|c_{jn}\right|^2}=\sqrt{1+\sum_{k=0}^{n-1}j_k^2}=\frac{1}{3}\sqrt{j_{2n}+2\left(-1\right)^n j_n+n+9} \end{aligned}$$

$$r_1\left(B\right)c_1\left(C\right)=\sqrt{\left(j_{2n}+2\left(-1\right)^n j_n+n\right)\left(j_{2n}+2\left(-1\right)^n j_n+n+1\right)}$$

$$\left\|A\right\|_2=\left\|BoC\right\|_2\leq r_1\left(B\right)c_1\left(C\right)\leq\frac{1}{9}\sqrt{\left(j_{2n}+2\left(-1\right)^n j_n+n\right)\left(j_{2n}+2\left(-1\right)^n j_n+n+9\right)}$$

Therefore, we complete the proof. □

Theorem 8. *Let $A = C(j_0, j_1, ..., j_{n-1})$ be circulant matrix, then we obtain*

$$\frac{1}{3}\sqrt{\frac{c_n^2-4}{3}+n}\leq\left\|A\right\|_2\leq\frac{1}{9}\sqrt{\left(\frac{c_n^2-4}{3}+n\right)\left(\frac{c_n^2-4}{3}+n+9\right)} \tag{15}$$

Proof. Let the matrix A is of the form above. From (5), (8), (10) we get

$$\left\|A\right\|_F^2=n\sum_{k=0}^{n-1}j_k^2=\frac{n}{9}\left[\frac{c_n^2-4}{3}+n\right]$$

$$\frac{1}{3}\sqrt{\frac{c_n^2-4}{3}+n}=\frac{1}{\sqrt{n}}\left\|A\right\|_F\leq\left\|A\right\|_2$$

On the other hand, let $A = BoC$ where B, C are defined in (14)

$$\begin{aligned} r_1(B) &= \max_{1\le i\le n}\sqrt{\sum_{j=1}^{n}|b_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} j_k^2} = \frac{1}{3}\sqrt{\frac{c_n^2-4}{3}+n} \\ c_1(C) &= \max_{1\le j\le n}\sqrt{\sum_{j=1}^{n}|c_{ij}|^2} = \sqrt{1+\sum_{k=0}^{n-1} j_k^2} = \frac{1}{3}\sqrt{\frac{c_n^2-4}{3}+n+9} \end{aligned}$$

$$\|A\|_2 = \|BoC\|_2 \le r_1(B)\,c_1(C) \le \frac{1}{9}\sqrt{\left(\frac{c_n^2-4}{3}+n\right)\left(\frac{c_n^2-4}{3}+n+9\right)}$$

The proof is completed. □

Theorem 9. *Let the elements of the circulant matrix be Jacobsthal–Lucas numbers,* $A = C(c_0, c_1, ..., c_{n-1})$*, then we obtain*

$$\begin{aligned} &\sqrt{j_{2n} - 2(-1)^n j_n + n} \le \|A\|_2 \\ &\qquad \le \sqrt{(j_{2n} - 2(-1)^n j_n + n)(j_{2n} - 2(-1)^n j_n + n + 1)} \end{aligned} \tag{16}$$

Proof. The matrix A is of the form

$$A = \begin{bmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{bmatrix}$$

$$\begin{aligned} \|A\|_E^2 &= n\sum_{k=0}^{n-1} c_k^2 = n\left(j_{2n} - 2(-1)^n j_n + n\right) \\ \frac{1}{\sqrt{n}}\|A\|_E &= \sqrt{j_{2n} - 2(-1)^n j_n + n}, \\ \frac{1}{\sqrt{n}}\|A\|_E &\le \|A\|_2 \\ \sqrt{j_{2n} - 2(-1)^n j_n + n} &\le \|A\|_2 \end{aligned}$$

On the other hand, let $A = BoC$ where B, C are

$$\begin{aligned} B = (b_{ij}) &= \begin{cases} b_{ij} = c_{(\mathrm{mod}_{(j-i,n)})+1}, & i \ge j \\ b_{ij} = 1, & i < j \end{cases} \\ C = (c_{ij}) &= \begin{cases} c_{ij} = c_{(\mathrm{mod}_{(j-i,n)})+1}, & i < j \\ c_{ij} = 1, & i \ge j \end{cases} . \end{aligned} \tag{17}$$

$$
\begin{aligned}
r_1(B) &= \max_i \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |b_{nj}|^2} = \sqrt{\sum_{k=0}^{n-1} c_k^2} = \sqrt{j_{2n} - 2(-1)^n j_n + n} \\
c_1(C) &= \max_j \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{1 + \sum_{k=0}^{n-1} c_k^2} = \sqrt{1 + j_{2n} - 2(-1)^n j_n + n}.
\end{aligned}
$$

$$
\|A\|_2 \le r_1(B)\, c_1(C) = \sqrt{(j_{2n} - 2(-1)^n j_n + n)(j_{2n} - 2(-1)^n j_n + n + 1)} \qquad \square
$$

Theorem 10. *Let $A = C(c_0, c_1, ..., c_{n-1})$ be circulant matrix, then we obtain*

$$
\sqrt{3j_n^2 + n} \le \|A\|_2 \le \sqrt{(3j_n^2 + n)(3j_n^2 + n + 1)}
$$

Proof.

$$
\begin{aligned}
\|A\|_E^2 &= n \sum_{k=0}^{n-1} c_k^2 = n(3j_n^2 + n) \\
\sqrt{3j_n^2 + n} &= \frac{1}{\sqrt{n}} \|A\|_E \le \|A\|_2
\end{aligned}
$$

On the other hand, let $A = BoC$ where B, C are defined in (16)

$$
\begin{aligned}
r_1(B) &= \max_{1 \le i \le n} \sqrt{\sum_{j=1}^{n} |b_{ij}|^2} = \sqrt{\sum_{k=0}^{n-1} c_k^2} = \sqrt{3j_n^2 + n} \\
c_1(C) &= \max_{1 \le j \le n} \sqrt{\sum_{j=1}^{n} |c_{ij}|^2} = \sqrt{\sum_{j=1}^{n} |c_{nj}|^2} = \sqrt{1 + \sum_{k=0}^{n-1} c_k^2} = \sqrt{3j_n^2 + n + 1}.
\end{aligned}
$$

$$
\|A\|_2 \le r_1(B)\, c_1(B) = \sqrt{(3j_n^2 + n)(3j_n^2 + n + 1)}.
$$

So the proof is completed. $\square$

References

[1] Horadam, A. F. (1996) Jacobsthal Representation Numbers, *The Fibonacci Quarterly*, 34(1), 40–54.

[2] Koshy, T. (2001)*Fibanacci and Lucas Numbers with Applications*, John Wiley and Sons Inc., New York.

[3] Sloane, N. J. A. (2006) The on-line Encyclopedia of Integer Sequences.

[4] Zielke, G. (1988) Some remarks on matrix norms, condition numbers and error estimates for linear equations, *Linear Algebra and its Applications*, 110, 29–41.

[5] Mathias, R. (1990) The spectral norm of nonnegative matrix, *Linear Algebra and its Applications*, 131, 269–284.

[6] Reams, R. (1999) Hadamard inverses square roots and products of almost semidefinite matrices, *Linear Algebra and its Applications*, 288, 35–43.

[7] Solak, S. & Bozkurt, D. (2003) On the spectral norms of Cauchy–Toeplitz and Cauchy–Henkel matrices, *Appl. Math. Comput.*, 140, 231–238.

[8] Solak, S. (2005) On the norms of circulant matrices with the Fibonacci and Lucas numbers, *Applied Mathematics and Computation*, 160, 125–132.

[9] Shen, S. & Cen, J. (2010) On the bounds for the norms of r-circulant matrices with the Fibonacci and Lucas numbers, *Appl. Math. Comput.*, 216, 2891–2897.

[10] Shen, S. (2010) On the spectral norms of r-circulant matrices with the k-Fibonacci and k-Lucas numbers, *Int. J. Contemp. Math. Sciences*, 5(12) 569–578.

[11] Horn, R. A. & Johnson, C. R. (1991) *Topics in Matrix Analysis*, Cambridge University Press, Cambridge.

[12] Uygun, Ş. (2015) The (s,t)-Jacobsthal and (s,t)-Jacobsthal Lucas sequences, *Applied Mathematical Sciences*, 70(9), 3467–3476.

[13] Uslu, K., Taşkara, N. & Uygun, Ş. (2011) The relations among k Fibonacci, k-Lucas and generalized k-Fibonacci numbers and the spectral norms of the matrices of involving these numbers, *Ars Combinatoria*, 102, 183–192.