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A Fibonacci integral lattice approach to Pythagoras' Theorem

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Abstract: Square integral lattices with basis vector pairs $\{(a, b), (-b, a)\}$, where a and b are successive Fibonacci numbers, are employed to develop intermediate convergence forms of Pythagoras' Theorem for triangles with integral sides.
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1 Introduction

The purpose of this paper is to employ square integral lattices with basis vector pairs $\{(a, b), (-b, a)\}$, where a and b are successive Fibonacci numbers, to develop intermediate convergence forms of Pythagoras' Theorem for triangles with integral sides [1].

In a limiting sense these intermediate forms may be made to approach the exact form of Pythagoras' Theorem as closely as we please. Links between the Fibonacci numbers and Pythagorean triples have previously been found in an algebraic and combinatorial ways [2, 3].

Certain square grid sub-lattice configurations are defined in place of what is usually considered to be an integer coordinate grid. A 'unit square' with a side common to the hypotenuse of any allowable slope has its 'area' defined in terms of the number of certain sub-lattice coordinates contained within its boundaries (*cf.* [4]). Necessary and sufficient conditions for such a definition to permit convergence to the absolute conditions imposed by Pythagoras' Theorem are identified and justified

2 Integral lattices

In this paper, the term 'integral lattice' denotes a subspace of $\mathbb{Z}^2$ under vector addition and integer scalar multiplication, where such a lattice has basis vectors $\{(x_0, 1), (m, 0)\}$, where $0 < x_0 < m$ such that $gcd(x_0, m) = 1$. Evidently coordinates of any such lattice occur with period m in every row and column of $\mathbb{Z}^2$.

The matrix identity $AI = A$ where

$$A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}, \ \gcd(a,b) = 1, . \tag{2.1}$$

may be considered to be a mapping by the matrix operator A of coordinate pairs $\{(1, 0), (0, 1)\}$ to the basis vector pair $\{(a, b), (-b, a)\}$ of a square integral S of row and column period (modulus $(a^2 + b^2 = \det A)$) [5]. We define coordinates $[X, Y]$ with respect to the basis vectors of such a square grid by

$$[X, Y] = A^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \tag{2.2}$$

which yields $[1, 0] = (a, b)$ and $[0, 1] = (-b, a)$. A unit length of this grid is thus $\sqrt{a^2 + b^2}$. Furthermore, one unit in the direction of integer coordinates (x, y) is equivalent to finding

$$\left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}\right).$$

The lattice approach is shown to be adaptable to this idea rather than determining an approximation to $\sqrt{x^2 + y^2}$ directly.

Let $\{F_n\}$ denote the Fibonacci sequence defined by the linear homogenous second order recurrence relation and initial terms:

$$F_n = F_{n-1} + F_{n-2}, \ n \ge 2, \ F_0 = F_1 = 1. \tag{2.3}$$

A relevant result is [6]:

$$(F_n F_{n+3})^2 + \left(2F_{n+2}(F_{n+2} - F_n)\right)^2 = \left(F_n^2 + 2F_{n+2} - F_n\right)^2. \tag{2.4}$$

Let S_α and S_β denote the square lattices where $(a, b) = (F_n, F_{n-1})$ and $(a, b) = (F_{n-1}, F_n)$, respectively. Evidently as n becomes relatively large, the gradients of the X and Y axes of S_α and S_β with respect to the x-axis approach the Golden ratio α and its negative reciprocal β. The elements of these alpha-lattices belong to $\mathbb{Z}[\alpha]$, the field formed by the adjunction to $\mathbb{Z}$ of α, (the golden ratio), the dominant root of the characteristic equation of the recurrence relation in (2.3).

From the properties of S_α type lattices, it follows that a unit length of any S grid is given by

$$\det A = \sqrt{F_{n-1}^2 + F_n^2} = \sqrt{F_{2n-1}}\,, \tag{2.5}$$

the Fibonacci part being a result due to Lucas [7]. Note that these S lattices are undefined when a is 0 or 1 in either of the Fibonacci initial conditions since both cases correspond to a self-mapping over $\mathbb{Z}^2$. In this paper we arbitrarily focus on some properties of S_α type lattices, since the properties of the other lattice are analogous.

3 Conclusion

This paper establishes a fundamental link between generalized Fibonacci numbers and Pythagoras' Theorem (*cf.* [8]). It also highlights the role that integral lattices can play in non-elementary analysis. In this respect we demonstrate the use of integral lattice arrangements as alpha-integral arrangements (elements of the field $\mathbb{Z}[\alpha]$), where too often integral arrangements are viewed as absolute integral arrangements, thereby creating the perception that such lattices only have a role in terms of elementary number theory

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