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Some combinatorial formulas for the partial r-Bell polynomials

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Abstract: The partial r-Bell polynomials generalize the classical partial Bell polynomials (coinciding with them when $r = 0$) by assigning a possibly different set of weights to the blocks containing the r smallest elements of a partition no two of which are allowed to belong to the same block. In this paper, we study the partial r-Bell polynomials from a combinatorial standpoint and derive several new formulas. We prove some general identities valid for arbitrary values of the parameters as well as establish formulas for some specific evaluations. Several of our results extend known formulas for the partial Bell polynomials and reduce to them when $r = 0$. Our arguments are largely combinatorial, and therefore provide, alternatively, bijective proofs of these formulas, many of which were shown by algebraic methods.

Keywords: Partial Bell polynomial, r-Stirling number, Combinatorial identity.
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1 Introduction

Given $n \geq k \geq 0$, $r \geq 0$ and sequences of indeterminates $(x_i)_{i\geq 1}$ and $(y_i)_{i\geq 1}$, the partial r-Bell polynomial $B_{n,k}^{(r)}(x_i; y_i)$ (see [10, Theorem 5]) is defined explicitly by

$$B_{n,k}^{(r)}(x_i; y_i) = \sum_{\Lambda(n,k,r)} \left[\frac{n!}{k_1!k_2!\cdots}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\right]\left[\frac{r!}{r_0!r_1!\cdots}\left(\frac{b_1}{0!}\right)^{r_0}\left(\frac{b_2}{1!}\right)^{r_1}\cdots\right], \quad (1)$$

where $\Lambda(n, k, r)$ denotes the set of all non-negative integer sequences $(k_i)_{i\geq 1}$ and $(r_i)_{i\geq 0}$ such that $\sum_{i\geq 1} k_i = k$, $\sum_{i\geq 0} r_i = r$ and $\sum_{i\geq 1} i(k_i + r_i) = n$. Note that $B_{n,k}^{(0)}(x_i; y_i) = B_{n,k}(x_i)$, the classical partial Bell polynomial [3]. The $B_{n,k}^{(r)}(x_i;y_i)$ have exponential generating function (egf) given by

$$\sum_{n\geq k} B_{n,k}^{(r)}(x_i;y_i)\frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{j\geq 1} x_j \frac{t^j}{j!}\right)^k \left(\sum_{j\geq 0} y_{j+1}\frac{t^j}{j!}\right)^r, \tag{2}$$

see [10, Corollary 4], which reduces to the well-known egf formula for $B_{n,k}(x_i)$ when $r = 0$. In cases when $x_i = y_i$ for all i, we denote $B_{n,k}^{(r)}(x_i;y_i)$ simply by $B_{n,k}^{(r)}(x_i)$. For appropriate choices of the indeterminates, the partial r-Bell polynomials reduce to some special combinatorial sequences:

$B_{n,k}^{(r)}(0!, 1!, \ldots) = c_r(n,k)$, unsigned r-Stirling number of the first kind [5],
$B_{n,k}^{(r)}(1, 1, \ldots) = S_r(n,k)$, r-Stirling number of the second kind [5],
$B_{n,k}^{(r)}(1, m, m^2, \ldots; 1, 1, \ldots) = W_{m,r}(n,k)$, r-Whitney number of the second kind [6],
$B_{n,k}^{(r)}(1!, 2!, \ldots) = L_r(n,k)$, r-Lah number [11].

In this paper, using a combinatorial approach, we derive various identities satisfied by the $B_{n,k}^{(r)}(x_i;y_i)$. In several cases, these extend previous formulas for the partial Bell polynomials which follow from taking $r = 0$. Since our proofs are largely bijective in nature, one thus obtains combinatorial explanations of these identities for $B_{n,k}(x_i)$, many of which were shown previously by algebraic methods. Moreover, through a combinatorial approach, one is likely to be led to further (and perhaps in some cases new) identities for the partial Bell and r-Bell polynomials, upon introducing and modifying certain parameters within the proofs.

The organization of this paper is as follows. In the next section, we describe the relevant combinatorial structure that will be made subsequent use of. In the third section, we establish some general identities satisfied by $B_{n,k}^{(r)}(x_i;y_i)$ for arbitrary values of the parameters which follow from our combinatorial description. In the fourth section, we give formulas for several specific evaluations of $B_{n,k}^{(r)}(x_i;y_i)$ further establishing its connection to several well-known sequences. We provide in the final section combinatorial proofs of some identities for $B_{n,k}(x_i;y_i)$ that were shown in [10] using generating function techniques.

2 Preliminaries

In this section, we describe the combinatorial interpretation for $B_{n,k}^{(r)}(x_i;y_i)$ that will be used. It is equivalent to and expands upon an interpretation mentioned in [10, Section 2]. Recall the notation $[m] = \{1, 2, \ldots, m\}$ if $m \geq 1$, with $[0] = \varnothing$. If m and n are positive integers, then $[m,n] = \{m, m+1, \ldots, n\}$ for $m \leq n$, with $[m,n] = \varnothing$ for $m > n$. By a *partition* of a set, we mean a collection of pairwise disjoint subsets, called *blocks*, whose union is the set. An *r-partition* of $[m]$ is one in which the elements $1, 2, \ldots, r$ belong to distinct blocks.

Definition 1. *Given $n \geq k \geq 0$ and $r \geq 0$, let $\mathcal{B}_{n,k}^{(r)}$ denote the set of all r-partitions of $[n+r]$ into $k+r$ blocks.*

Recall that $|\mathcal{B}_{n,k}^{(r)}| = S_r(n,k)$, the r-Stirling number of the second kind [5]. Given $\lambda \in \mathcal{B}_{n,k}^{(r)}$, blocks of λ containing an element of $[r]$ will be described as *special*, with all other blocks referred to as *non-special*. Thus, members of $\mathcal{B}_{n,k}^{(r)}$ contain r special and k non-special blocks. The members of $[r]$ themselves within λ will be referred to as *special* (elements), and members of $[r+1, r+n]$ as non-special. The set $[r+1, r+n]$ will frequently be denoted by I. An element that is the smallest in its block within a member of $\mathcal{B}_{n,k}^{(r)}$ (including special elements) will often be described as *minimal*, with all other elements *non-minimal*.

We define the following statistics on $\mathcal{B}_{n,k}^{(r)}$.

Definition 2. *Given $i \geq 1$ and $\lambda \in \mathcal{B}_{n,k}^{(r)}$, let $\nu_i(\lambda)$ be the number of non-special blocks of λ of size i and $\omega_i(\lambda)$ be the number of special blocks of size i.*

From (1), it follows that

$$B_{n,k}^{(r)}(x_i; y_i) = \sum_{\lambda \in \mathcal{B}_{n,k}^{(r)}} x_1^{\nu_1(\lambda)} x_2^{\nu_2(\lambda)} \cdots y_1^{\omega_1(\lambda)} y_2^{\omega_2(\lambda)} \cdots . \tag{3}$$

We will make use of this interpretation to prove several identities for $B_{n,k}^{(r)}(x_i; y_i)$ in the subsequent sections.

At times, in the proofs that follow, we will use similar terminology for permutations. By an *r-permutation* of $[m]$, we mean one in which the elements $1, 2, \ldots, r$ belong to distinct cycles. *Special* cycles are those that contain a member of $[r]$, with all others referred to as *non-special*. Recall that the cardinality of the set of r-permutations of $[n+r]$ having $k+r$ cycles is given by the signless r-Stirling number, which we will denote here by $c_r(n,k)$. Let $s_r(n,k) = (-1)^{n-k} c_r(n,k)$ denote the r-Stirling number of the first kind [5]. Note that when $r = 0$, the $s_r(n,k)$ and $S_r(n,k)$ reduce, respectively, to the classical Stirling numbers of the first and second kind [12, A008275 and A008277].

3 General formulas

In this section, we prove some formulas for $B_{n,k}^{(r)}(x_i; y_i)$ which hold for all values of its parameters.

Theorem 3. *If $n \geq k \geq 0$ and $r \geq 0$, then*

$$\begin{aligned}
&\sum_{\alpha=1}^{n-k} \binom{n}{\alpha} \left(k + 1 - \frac{n+1}{\alpha+1}\right) x_{\alpha+1} B_{n-\alpha,k}^{(r)}(x_i; y_i) \\
&\qquad = (n-k) x_1 B_{n,k}^{(r)}(x_i; y_i) - r \sum_{\alpha=0}^{n-k-1} \sum_{\beta=1}^{n-\alpha} \binom{n}{\alpha} \binom{n-\alpha}{\beta} x_\beta y_{\alpha+2} B_{n-\alpha-\beta,k}^{(r-1)}(x_i; y_i).
\end{aligned} \tag{4}$$

Proof. Both sides of (4) are trivial if $n = k$, so let us assume throughout that $n > k$. We prove the following equivalent form of (4):

$$\begin{aligned}\sum_{\alpha=1}^{n-k}\binom{n}{\alpha}\left(k+1-\frac{n+1}{\alpha+1}\right)x_{\alpha+1}B_{n-\alpha,k}^{(r)}(x_i;y_i)\\ +r\sum_{\alpha=0}^{n-k-1}\sum_{\beta=2}^{n-\alpha}\binom{n}{\alpha}\binom{n-\alpha}{\beta}x_\beta y_{\alpha+2}B_{n-\alpha-\beta,k}^{(r-1)}(x_i;y_i)\\ =(n-k)x_1B_{n,k}^{(r)}(x_i;y_i)-rx_1\sum_{\alpha=0}^{n-k-1}(n-\alpha)\binom{n}{\alpha}y_{\alpha+2}B_{n-\alpha-1}^{(r-1)}(x_i;y_i).\end{aligned}\tag{5}$$

Let $\widehat{\mathcal{B}}$ denote the subset of $\mathcal{B}_{n+1,k+1}^{(r)}$ whose members contain at least one non-special singleton block and in which the element $n+r+1$ belongs to a non-special non-singleton block whose smallest element is smaller than at least one of the elements contained within non-special singleton blocks. Let m denote the smallest element of the block containing $n+r+1$ within a member of $\widehat{\mathcal{B}}$. Let $\mathcal{B}^*$ denote the set of "circled" extended set partitions whose members are obtained by taking partitions in $\widehat{\mathcal{B}}$ and circling exactly one of the non-special singleton blocks whose element is greater than m. Define the weight of a member of $\mathcal{B}^*$ as that of the underlying member of $\widehat{\mathcal{B}}$. We argue now that both sides of (5) give the sum of the weights of all members of $\mathcal{B}^*$.

To show that the right-hand side of (5) achieves this, we use a subtraction argument as follows. Let $\widetilde{\mathcal{B}}$ denote the set of configurations defined the same way as $\mathcal{B}^*$ except that the element $n+r+1$ is also allowed to occupy a special block, with weights defined as before. Then the sum of the weights of all members of $\widetilde{\mathcal{B}}$ is given by $(n-k)x_1B_{n,k}^{(r)}(x_i;y_i)$. To see this, we start with any member of $\mathcal{B}_{n,k}^{(r)}$, remove a non-minimal element s from one of the blocks, replace s by $n+r+1$ within its block, and then add back the singleton block $\{s\}$, which we circle. Note that there are $n-k$ choices for s and that the block $\{s\}$ contributes weight x_1. Since all members of $\widetilde{\mathcal{B}}$ arise from those in $\mathcal{B}_{n,k}^{(r)}$ as described, the weight of the members of $\widetilde{\mathcal{B}}$ is as given.

We now determine the weight of all members of $\widetilde{\mathcal{B}}$ in which $n+r+1$ belongs to a special block and subtract that from the total weight of $\widetilde{\mathcal{B}}$ to obtain the weight of $\mathcal{B}^*$. To do so, consider the cardinality, $\alpha+2$, of the block containing the element $n+r+1$. This block contributes weight $y_{\alpha+2}$ and there are $\binom{n}{\alpha}r$ choices for the elements in this block since there are exactly α members of I to be selected together with some member of $[r]$. Then there are $n-\alpha$ choices for the element comprising the circled non-special singleton block, which contributes a weight of x_1. Finally, the remaining $n+r+1-(\alpha+3)=n+r-\alpha-2$ members of $[n+r+1]$ are to form a partition having $k+r-1$ blocks, $r-1$ of which are special. Thus, there are $B_{n-\alpha-1,k}^{(r-1)}(x_i;y_i)$ possibilities concerning these elements. Summing over $0\le\alpha\le n-k-1$ implies that the weight of all members of $\widetilde{\mathcal{B}}$ in which $n+r+1$ belongs to a special block is given by the second term on the right-hand side of (5), as desired.

To show that the left-hand side of (5) also gives the weight of all members of $\mathcal{B}^*$, we again use a subtraction argument. First observe that the sum

$$\sum_{\alpha=1}^{n-k}(k+1)\binom{n}{\alpha}x_{\alpha+1}B_{n-\alpha,k}^{(r)}(x_i;y_i)$$

gives the total weight of all (extended) members of $\mathcal{B}^{(r)}_{n+1,k+1}$ in which the element $n+r+1$ belongs to a non-special, non-singleton block and one of the $k+1$ non-special blocks is circled, by considering the cardinality of the block containing $n+r+1$. On the other hand, the sum

$$\sum_{\alpha=1}^{n-k}\binom{n+1}{\alpha+1}x_{\alpha+1}B^{(r)}_{n-\alpha,k}(x_i;y_i)$$

gives the weight of all members of $\mathcal{B}^{(r)}_{n+1,k+1}$ containing at least one non-special, non-singleton block where one such block is circled, by considering the cardinality $\alpha+1$ of the circled block.

Let $X,Y\subseteq\mathcal{B}^{(r)}_{n+1,k+1}$, where X consists of those partitions in which the element $n+r+1$ belongs to a non-special non-singleton and containing at least one non-special singleton and Y consists of partitions in which $n+r+1$ comprises a singleton and containing at least one non-special non-singleton block. Let X^* and Y^* be obtained from X and Y by circling one of the non-special singletons or non-special non-singletons, respectively. By subtraction, the left side of (5) gives the difference in the weights of the sets X^* and Y^*. Note that the double-sum on the left side of (5) is there to cancel out the sum of the weights of all members of $\mathcal{B}^{(r)}_{n+1,k+1}$ accounted for in the sum in the preceding paragraph in which $n+r+1$ belongs to a special block (note $n+r+1$ cannot belong to a special block within a member of Y^*). Given $\lambda\in Y^*$, consider removing the singleton $\{n+r+1\}$, adding the element $n+r+1$ to the circled block B (and uncircling B), removing the smallest element s from B, and adding back the singleton $\{s\}$, which we circle. This operation defines a weight-preserving bijection between Y^* and $Z\subseteq X^*$ consisting of those members in which the element occupying the circled singleton is smaller than the smallest element of the block containing $n+r+1$. By the definitions, the difference in the weights between X^* and Z, and hence between X^* and Y^*, equals the weight of the set $\mathcal{B}^*$. Thus, the left-hand side of (5) gives the weight of $\mathcal{B}^*$, as desired, which completes the proof. □

Taking $r=0$ in (4), we get (via a combinatorial proof) the following formula for $B_{n,k}(x_i)$ which was shown by algebraic [8] and later by probabilistic [9] arguments.

Corollary 4 (Cvijović [8])**.** *If $n\geq k\geq 0$, then*

$$\sum_{\alpha=1}^{n-k}\binom{n}{\alpha}\left(k+1-\frac{n+1}{\alpha+1}\right)x_{\alpha+1}B_{n-\alpha,k}(x_i)=(n-k)x_1B_{n,k}(x_i). \tag{6}$$

In the next two results, we give recurrence formulas for $B^{(r)}_{n,k}(x_i;y_i)$ in terms of $B^{(j)}_{n,k}(x_i;y_i)$ for $j<r$.

Theorem 5. *If $n\geq k\geq 0$ and $r\geq 1$, then*

$$B^{(r)}_{n+1,k}(x_i;y_i)=\sum_{j=0}^{n-k+1}\binom{n}{j}x_{j+1}B^{(r)}_{n-j,k-1}(x_i;y_i)+r\sum_{j=0}^{n-k}\binom{n}{j}y_{j+2}B^{(r-1)}_{n-j,k}(x_i;y_i) \tag{7}$$

and

$$B^{(r)}_{n,k}(x_i;y_i)=y_1^rB_{n,k}(x_i)+\sum_{j=0}^{r-1}\sum_{\ell=1}^{n-k}\binom{n}{\ell}y_1^jy_{\ell+1}B^{(r-j-1)}_{n-\ell,k}(x_i;y_i). \tag{8}$$

Proof. To show (7), we consider the number $j+1$ of non-special elements within the block containing the element $r+1$ (counting $r+1$) within a member of $\mathcal{B}_{n+1,k}^{(r)}$. If $r+1$ belongs to a special block, then there are $r\binom{n}{j}$ ways to choose the special and the j other non-special elements contained within it, with this block contributing y_{j+2} towards the weight. The remaining $r-1$ special and $n-j$ non-special elements may then be arranged in $B_{n-j,k}^{(r-1)}(x_i;y_i)$ ways. Considering all possible j gives the second sum on the right-hand side of (7). On the other hand, if $r+1$ belongs to a non-special block, then there are $\binom{n}{j}$ ways to choose the other elements of this block, which contributes weight x_{j+1}, and $B_{n-j,k-1}^{(r)}(x_i;y_i)$ possibilities for the remaining elements of $[n+r]$.

To show (8), consider whether or not all of the special blocks within a member of $\mathcal{B}_{n,k}^{(r)}$ are singletons. If this is the case, then there are $y_1^r B_{n,k}(x_i)$ possibilities. If not, then consider the smallest j, $0 \le j \le r-1$, such that the element $r-j$ does not comprise its own block. Suppose further that there are exactly ℓ non-special elements in the block containing $r-j$, where $1 \le \ell \le n-k$. Then there are $\binom{n}{\ell}$ ways to select these elements, with the resulting block contributing $y_{\ell+1}$ towards the weight. The singleton blocks containing the elements in $[r-j+1, r]$ contribute weight y_1^j, and there are $B_{n-\ell,k}^{(r-j-1)}(x_i;y_i)$ possibilities concerning the elements contained within the non-special and the first $r-j-1$ special blocks. Considering all possible j and ℓ implies that the sum on the right-hand side of (8) gives the total weight of all members of $\mathcal{B}_{n,k}^{(r)}$ whose special blocks are not all singletons, which completes the proof. $\square$

Theorem 6. *If $n \ge k \ge 0$ and $r, s, \ell \ge 0$, then*

$$B_{n,k}^{(r+s)}(x_i;y_i) = \sum_{j=k}^{n} \binom{n}{j} B_{j,k}^{(r)}(x_i;y_i) B_{n-j,0}^{(s)}(x_i;y_i) \tag{9}$$

and

$$\binom{k+\ell}{k} B_{n,k+\ell}^{(r+s)}(x_i;y_i) = \sum_{j=k}^{n-\ell} \binom{n}{j} B_{j,k}^{(r)}(x_i;y_i) B_{n-j,\ell}^{(s)}(x_i;y_i). \tag{10}$$

Proof. For (9), consider the number j of non-special elements belonging to non-special blocks or the first r special blocks within a member of $\mathcal{B}_{n,k}^{(r+s)}$. Note that there are $\binom{n}{j}$ choices for these elements and $B_{j,k}^{(r)}(x_i;y_i)$ possibilities concerning their arrangement. The remaining elements of $[n+r]$ may then be arranged according to a member of $\mathcal{B}_{n-j,0}^{(s)}$. Considering all j for $k \le j \le n$ gives (9).

To show (10), we circle exactly k of the $k+\ell$ non-special blocks within a (weighted) member of $\mathcal{B}_{n,k+\ell}^{(r+s)}$. Alternatively, such configurations may be obtained by considering the number j of non-special elements belonging to either a circled block or to one of the first r special blocks where $k \le j \le n-\ell$. There are $\binom{n}{j} B_{j,k}^{(r)}(x_i;y_i)$ possibilities concerning the choice and arrangement of these elements. The remaining elements of $[n+r]$ then must go in either one of the ℓ uncircled non-special blocks or in one of the final s special blocks, which gives $B_{n-j,\ell}^{(s)}(x_i;y_i)$ possibilities. Summing over all possible j gives (10). $\square$

The $s=1$ case of formula (9) is equivalent to the second identity in [10, Proposition 8], which is slightly misstated. Taking $r=s=0$ in formula (10) gives [8, Equation 1.4]. We remark that the proof presented here in these cases differs from the earlier one.

Remark: Note that the $B^{(s)}_{n-j,0}(x_i; y_i)$ factor in (9) above does not depend on x_i (since no non-special blocks occur) and is given explicitly by

$$B^{(s)}_{n-j,0}(x_i; y_i) = \sum_{n_1+\cdots+n_s=n-j} \binom{n-j}{n_1, n_2, \ldots, n_s} y_{n_1+1} y_{n_2+1} \cdots y_{n_s+1}.$$

Taking $r = 0$ in (9) then gives an explicit formula for the r-Bell polynomials $B^{(r)}_{n,k}(x_i; y_i)$ in terms of the ordinary Bell polynomials $B_{j,k}(x_i)$ where $j \leq n$.

We have the following formula for $B^{(r)}_{n,k}(x_i; y_i)$ in terms of $B_{n,k}(x_i)$ which seems to be new.

Theorem 7. *If $n \geq k \geq 0$, $r \geq 1$, and $y_1 = 1$, then*

$$B^{(r)}_{n,k}(x_i; y_i) = B_{n,k}(x_i) + \frac{1}{\binom{n+r}{r}} \sum_{j=0}^{n-1} \binom{n}{j} B_{j,k}(x_i) B_{n+r-j,r}(i y_i). \tag{11}$$

Proof. To show this, we make use of the potential functions $P^{(\lambda)}_n = P^{(\lambda)}_n(a_1, a_2, \ldots)$ (see, e.g., [7, p. 141]), defined for each complex number λ by

$$1 + \sum_{n \geq 1} P^{(\lambda)}_n \frac{t^n}{n!} = \left(1 + \sum_{n \geq 1} a_n \frac{t^n}{n!}\right)^{\lambda}. \tag{12}$$

Note that since

$$\sum_{n \geq k} B_{n,k}(x_i) \frac{t^n}{n!} = \frac{1}{k!} \left(\sum_{j \geq 1} x_j \frac{t^j}{j!}\right)^k,$$

we have by (2),

$$\sum_{n \geq k} B^{(r)}_{n,k}(x_i; y_i) \frac{t^n}{n!} = \sum_{j \geq k} B_{j,k}(x_i) \frac{t^j}{j!} \left(a_0 + \sum_{n \geq 1} a_n \frac{t^n}{n!}\right)^r,$$

where $a_n = y_{n+1}$ for all n. If $a_0 = y_1 = 1$, then (12) implies

$$\sum_{n \geq k} B^{(r)}_{n,k}(x_i; y_i) \frac{t^n}{n!} = \sum_{j \geq k} B_{j,k}(x_i) \frac{t^j}{j!} \left(1 + \sum_{i \geq 1} P^{(r)}_i(y_2, y_3, \ldots) \frac{t^i}{i!}\right). \tag{13}$$

Comparing coefficients of t^n on both sides of (13) gives

$$B^{(r)}_{n,k}(x_i; y_i) = B_{n,k}(x_i) + \sum_{j=0}^{n-1} \binom{n}{j} B_{j,k}(x_i) P^{(r)}_{n-j}(y_2, y_3, \ldots). \tag{14}$$

Formula (11) follows from (14) and the fact that

$$P^{(r)}_n(a_1, a_2, \ldots) = \frac{1}{\binom{n+r}{r}} B_{n+r,r}(1, 2a_1, 3a_2, \ldots), \qquad n \geq 0,$$

for any sequence $(a_i)_{i \geq 0}$ and positive integer r, which can be shown by verifying that both sides possess the same exponential generating function. $\square$

Remark: Comparing generating functions gives

$$P_n^{(\lambda)}(a_1, a_2, \ldots) = \sum_{k=1}^{n} \binom{\lambda}{k} k! B_{n,k}(a_1, a_2, \ldots), \qquad n \geq 1.$$

Thus, the preceding proof yields the further identity

$$B_{n,k}^{(r)}(x_i; y_i) = B_{n,k}(x_i) + \sum_{j=0}^{n-1} \sum_{\ell=1}^{r} \binom{n}{j} \binom{r}{\ell} \ell! B_{j,k}(x_i) B_{n-j,\ell}(y_2, y_3, \ldots), \qquad n, r \geq 1, \quad (15)$$

where $y_1 = 1$.

4 Some particular evaluations

In this section, we consider several particular values assumed by partial r-Bell polynomials. Let B_n denote the n-th Bell number [12, A000110]. Taking $x_i = iB_{i-1}$ and $y_i = B_{i-1}$ for $i \geq 1$, and then $x_i = y_i = iB_{i-1}$, gives the following formulas.

Theorem 8. *If $n \geq k \geq 0$ and $r \geq 0$, then*

$$B_{n,k}^{(r)}(B_0, 2B_1, 3B_2, \ldots; B_0, B_1, B_2, \ldots) = \binom{n}{k} \sum_{j=0}^{n-k} S(n-k, j)(k+r)^j \qquad (16)$$

and

$$B_{n,k}^{(r)}(B_0, 2B_1, 3B_2, \ldots) = \binom{n}{k} \sum_{i=0}^{r} \sum_{j=0}^{n-k} \binom{r}{i} \binom{n-k}{r-i} (r-i)! S_{r-i}(n-k-r+i, j)(k+r)^j. \quad (17)$$

Proof. We first show (16). To do so, let C denote the set whose members are obtained from those in $\mathcal{B}_{n,k}^{(r)}$ by (i) dividing up the elements within each non-special block according to a partition containing at least one singleton block where one of the singletons is circled, and (ii) dividing up the non-special elements within each special block according to an arbitrary partition. By the definition of the partial r-Bell polynomials, we have

$$|C| = B_{n,k}^{(r)}(B_0, 2B_1, 3B_2, \ldots; B_0, B_1, B_2, \ldots).$$

We now determine $|C|$ in a different way. First select k elements of I that are to constitute the set of circled singletons belonging to the non-special blocks of a member of C, which can be done in $\binom{n}{k}$ ways. Add each of these selected elements as circled singletons as well as the members of $[r]$ to distinct urns. Once this is done, arrange the remaining $n-k$ elements of I according to an arbitrary partition containing say j blocks. Then add these blocks to the urns described above with no restriction, which can be done in $(k+r)^j$ ways. The contents of the urns then become the blocks of a member of C. Considering all possible j implies that the cardinality of C is also given by the right-hand side of (16).

To show (17), let D denote the set whose members are obtained from those in $\mathcal{B}_{n,k}^{(r)}$ in the same way as those in C are except that part (i) is applied to *all* blocks (not just the non-special ones). Then $|D|$ is given by the left-hand side of (17). Alternatively, the cardinality of D may be found by first considering the elements comprising the set S of circled singletons within non-special blocks, which may be chosen in $\binom{n}{k}$ ways. Once this is done, consider the number i of special blocks in which the circled singleton within corresponds to a special element. There are $\binom{r}{i}$ choices for the set T whose elements comprise these blocks and $\binom{n-k}{r-i}(r-i)!$ ways in which to choose and arrange $r-i$ elements of $I-S$ as circled singletons within the remaining special blocks. Let us denote the elements of the latter (ordered) set by $a_1, a_2, \ldots, a_{r-i}$, and the elements of $[r]-T$ by $b_1 < b_2 < \cdots < b_{r-i}$. Arrange the remaining $n-k-(r-i)$ non-special elements, together with $b_1, b_2, \ldots, b_{r-i}$, according to a member $\lambda \in \mathcal{B}^{(r-i)}_{n-k-r+i,j}$ for some j.

Now place the elements of $S \cup T$ in urns as circled singletons, one per urn. We also insert the elements a_ℓ into $r-i$ additional urns (as circled singletons). To the urn containing the element a_ℓ, we add the block of λ containing the element b_ℓ for each ℓ. At this point, there are $k+r$ urns that are distinguished from one another by their contents, with each containing minimally a circled singleton block. We then add the j non-special blocks of λ to these urns, which can be done in $(k+r)^j$ ways. The (partitioned) contents of an urn then become the elements comprising a block within a member of D. Allowing S, T, λ and j to vary, it is seen that all members of D arise uniquely in the manner described above. Considering all possible i and j then implies that $|D|$ is given by the right-hand side of (17), which completes the proof. $\square$

Letting $r=0$ in (16) and (17) yields [1, Corollary 7]:

$$B_{n,k}(B_0, 2B_1, 3B_2, \ldots) = \binom{n}{k} \sum_{j=0}^{n-k} S(n-k,j)k^j, \qquad n \geq k,$$

which was shown by generating functions. Next, taking $x_i = i - \delta_{i,1}$ and $y_i = i$ for $i \geq 1$, and then $x_i = y_i = i - \delta_{i,1}$, gives the following formulas.

Theorem 9. *If $r, k \geq 0$ and $n \geq 2k$, then*

$$B^{(r)}_{n,k}(0,2,3,\ldots;1,2,3,\ldots) = \sum_{j=0}^{r} \binom{r}{j} \binom{n}{k+r-j} (k+r-j)! S_r(n-k-r+j,k) \tag{18}$$

and

$$B^{(r)}_{n,k}(0,2,3,\ldots) = \sum_{j=0}^{r} \binom{r}{j} \binom{k+j}{j} \binom{n}{k+r-j} j!(k+r-j)! S_{r-j}(n-k-r+j,k+j). \tag{19}$$

Proof. Let $\mathcal{U}$ be the set of configurations derived from the subset of $\mathcal{B}^{(r)}_{n,k}$ whose members contain no non-special singleton blocks by circling one of the elements within each block. Then the left side of (18) gives $|\mathcal{U}|$ and we show that the right side gives this as well. To do so, let $\mathcal{U}_j$ be the subset of $\mathcal{U}$ in which exactly j elements of $[r]$ are circled. To determine $|\mathcal{U}_j|$, first select a subset S of I of size $k+r-j$. Then arrange the elements of $(I-S) \cup [r]$ according to an r-partition λ of size $n-k+j$ having $k+r$ blocks, which can be done in $S_r(n-k-r+j,k)$ ways. Select j of the special blocks of λ and circle the special element contained within each chosen block. Then add the elements of S to the non-special blocks of λ and the remaining special blocks so that one element, which we circle, goes in each block, which can be done in $(k+r-j)!$ ways. The resulting partition of $k+r$ blocks belongs to $\mathcal{U}_j$, and each member of $\mathcal{U}_j$ is seen to arise uniquely in this manner, upon allowing S and λ to vary. It follows that $|\mathcal{U}_j|$ is given by the summand on the right side of (18) for $0 \le j \le r$, which implies (18).

To show (19), let $\mathcal{V}$ be the set of configurations that are obtained in the same way as those in $\mathcal{U}$ above except now no member of $\mathcal{B}_{n,k}^{(r)}$ containing a singleton may be used. By the definitions, the left side of (19) gives $|\mathcal{V}|$. Alternatively, we derive a formula for $|\mathcal{V}_j|$ for $0 \le j \le r$, where $\mathcal{V}_j \subseteq \mathcal{V}$ is defined the same way as $\mathcal{U}_j$ above. To do so, first select a subset P of $[r]$ of size j and a subset Q of I of size $k+r-j$. Then arrange the elements of $([r]-P)\cup(I-Q)$ according to an $(r-j)$-partition ρ of size $n-(k+r-j)+(r-j)=n-k$ having $k+r$ blocks. Select j of the $k+r-(r-j)=k+j$ non-special blocks of ρ and add the elements of P which we circle to these chosen blocks, one per block, which can achieved in $\binom{k+j}{j}j!$ ways. Finally, distribute in any one of $(k+r-j)!$ ways the elements of Q within the special blocks of ρ and the remaining non-special blocks so that each block receives one element which we circle. The partition obtained by considering the now augmented blocks of ρ is seen to belong to $\mathcal{V}_j$. Thus, the cardinality of $\mathcal{V}_j$ is given by the summand on the right side of (19). Summing over j completes the proof. □

When $r=0$, formulas (18) and (19) both give [13, Example 4.2]:

$$B_{n,k}(0,2,3,\ldots)=\binom{n}{k}k!S(n-k,k), \qquad n \ge 2k.$$

Our next result generalizes [13, Example 4.3], reducing to it when $r=0$.

Theorem 10. *If $r,k \ge 0$ and $n \ge 2k$, then*

$$B_{n,k}^{(r)}(0,2\cdot 0!,-3\cdot 1!,4\cdot 2!,-5\cdot 3!,\ldots;-0!,1!,-2!,3!,\ldots)=(-1)^r\binom{n}{k}k!s_r(n-k,k). \quad (20)$$

Proof. Let $\mathcal{W}$ denote the set of configurations derived from the subset of $\mathcal{B}_{n,k}^{(r)}$ containing no non-special singletons as follows: (i) within each non-special block, one element is circled (and set aside), while the remaining elements are listed in any order such that the smallest (uncircled) element is first, and (ii) within each special block, elements are listed in any order so that the smallest element is first. Since the number of elements belonging to blocks of odd cardinality within a member of $\mathcal{W}$ has the same parity as $n+r$, it follows from the definition of r-partial Bell polynomials that

$$B_{n,k}^{(r)}(0,2\cdot 0!,-3\cdot 1!,4\cdot 2!,-5\cdot 3!,\ldots;-0!,1!,-2!,3!,\ldots)=(-1)^{n+r}|\mathcal{W}|.$$

To complete the proof, we must show that $|\mathcal{W}|=\binom{n}{k}k!c_r(n-k,k)$. To do so, first select a subset S of I of size k and then arrange the members of $(I-S)\cup[r]$ according to an r-permutation λ of size $n-k+r$ having $k+r$ cycles. Then distribute the elements of S, one per cycle, among the non-special cycles of λ, circling these elements. The resulting configuration is seen to belong to $\mathcal{W}$ and all members of $\mathcal{W}$ are seen to so arise. Thus, the cardinality of $\mathcal{W}$ is as claimed. □

Our last two results in this section correspond when $r=0$ to [4, Identities 1 and 4], which were shown algebraically. The $r=0$ case of (21) below also occurs (in a different form) as [2, Theorem 4.1].

Theorem 11. *If $n, k, \ell \geq 1$ and $r \geq 0$, then*

$$B_{n,k}^{(r)}(1!, \ldots, \ell!, 0, 0, \ldots; 1!, 2!, \ldots) = \frac{n!}{k!} \sum_{j=0}^{m} (-1)^j \binom{k}{j} \binom{n+2r-\ell j-1}{k+2r-1}, \tag{21}$$

where $m = \min\{k, \lfloor (n-k)/\ell \rfloor\}$.

Proof. By an *r-Lah distribution* of $[m]$, we mean a partition of $[m]$ into contents-ordered blocks in which the elements of $[r]$ belong to distinct blocks (cf. [11]). Let $\mathcal{K}$ denote the set of r-Lah distributions of $[n+r]$ having $k+r$ blocks in which no non-special block has size greater than ℓ. To show (21), we argue that the right side gives the cardinality of $\mathcal{K}$. First note that there are $n!$ possible orderings for the elements of I within a left-to-right scan of the contents of the blocks of $\lambda \in \mathcal{K}$ (where for now non-special blocks may be listed in any order). To completely determine λ, we also need to know the number of elements of I to the left and to the right of each special element within its block as well as the number of elements in each non-special block. This information corresponds to a (weak) composition of n having $k+2r$ parts in which the first $2r$ parts are non-negative and the final k parts are positive but at most ℓ. We use the principle of inclusion-exclusion to count all such compositions. Let $(x_1, x_2, \ldots, x_{k+2r})$ be a composition of n in which $x_i \geq 0$ for $1 \leq i \leq 2r$ and $x_i \geq 1$ for $2r+1 \leq i \leq 2r+k$. To count the compositions in question, suppose that it is the case that for j indices $i \in [2r+1, 2r+k]$ (and possibly others), we have $x_i > \ell$. Then there are $\binom{k}{j}$ choices for these indices, and once they are selected, the corresponding composition is equivalent to one of the form $y_1 + y_2 + \cdots + y_{k+2r} = n - \ell j$, where y_i satisfies the same requirements as the x_i above, of which there are $\binom{n+2r-\ell j-1}{k+2r-1}$. By the principle of inclusion-exclusion, the sum $\sum_{j=0}^{m} (-1)^j \binom{k}{j} \binom{n+2r-\ell j-1}{k+2r-1}$ then counts the compositions in question. Multiplying this sum by $n!$, and dividing by $k!$ (since the order of non-special blocks is not important), thus gives the number of ways of arranging the elements of $[n+r]$ according to a member of $\mathcal{K}$, which completes the proof. $\square$

Theorem 12. *If $n \geq k \geq 0$ and $r, s \geq 0$, then*

$$B_{n,k}^{(r)}(1!t_1(s), 2!t_2(s), \ldots; 1!, 2!, \ldots) = \frac{n!}{k!} \binom{n+(s+1)k+2r-1}{n-k}, \tag{22}$$

where $t_j(s) = \binom{s+j}{s+1}$ for $j \geq 1$.

Proof. Suppose $k > 0$, for how to handle the easier $k = 0$ case will be apparent from what follows. Let $\mathcal{L} = \mathcal{L}_{n,k}^{(r)}$ denote the set of configurations obtained from members of $\mathcal{B}_{n,k}^{(r)}$ by ordering elements within all blocks and adding indistinguishable objects that we will refer to as *dots* to the non-special blocks subject to the following rules: (i) dots can occur between any two elements and after the last element of a block, but not prior to the first, and (ii) the number of dots added to each block is $s+1$, with no restriction as to the number of dots (possibly none) following a particular element. Note that if a non-special block within a member of $\mathcal{L}$ contains j elements, then there are $t_j(s)$ possible ways to add dots to the block subject to (i) and (ii). Thus, the left-hand side of (22) is seen to give the cardinality of $\mathcal{L}$.

Alternatively, members of $\mathcal{L}$ may be formed as follows. Start with a linear sequence of $n + (s+1)k + 2r - 1$ dots and consider replacing exactly $n-k$ of the dots by open circles. Then add an open circle at the beginning and let S denote the resulting sequence of dots and circles. Note that there are $\binom{n+(s+1)k+2r-1}{n-k}$ possibilities for S. Given S, we first partition it into k parts B_i as follows. Let B_1 be the portion of S to the left of the $(s+2)$-nd dot (but not including it). For $2 \leq i \leq k-1$, let B_i be the portion of S starting with dot $(i-1)s+2(i-1)$ from the left and ending one position to the left of dot $is+2i$. Finally, if $r>0$, then let B_k be defined like the previous B_i for $i \geq 2$, while if $r=0$, let $B_k = S - \cup_{i=1}^{k-1} B_i$. Next replace the initial dot of each B_i for $2 \leq i \leq k$ with an open circle. Set aside for now each B_i and consider the remainder R of the sequence S if non-empty (R is non-empty iff $r>0$). Note that R starts with a dot, by the definition of B_k. If j represents the number of open circles in S initially, excluding the one at the beginning, to the left of the $k(s+2)$-nd dot, then R contains $n-k-j$ circles and

$$(n+(s+1)k+2r-1)-(n-k)-((s+2)k-1)=2r$$

dots. Since R starts with a dot, one can express R if non-empty as a sequence

$$R = dc^{i_1}dc^{i_2}\cdots dc^{i_{2r}},$$

where c and d stand for circle and dot, respectively, and the i_ℓ are non-negative with $\sum_{\ell=1}^{2r} i_\ell = n-j-k$.

We next form r blocks by putting into the a-th block for $1 \leq a \leq r$, i_{2a-1} open circles, followed by the element a, followed by i_{2a} open circles. Label the block containing the element a as C_a for $a \in [r]$. We next write the blocks $C_1, C_2, \ldots, C_r$, followed by $B_1, B_2, \ldots, B_k$. Note that starting with the C_i and B_i, reversing the procedure described above to obtain them recovers S. Thus, there are $\binom{n+(s+1)k+2r-1}{n-k}$ possibilities for these blocks. Observe that there are a total of $(n-k-j)+(k+j)=n$ open circles within all of the blocks combined. Now write the elements of I in the open circles within the blocks according to any permutation. Let $\mathcal{L}^*$ denote the set of configurations obtained from those in $\mathcal{L}$ by ordering the non-special blocks. Note that writing the elements of I in the open circles as described yields a member of $\mathcal{L}^*$ and that all members of $\mathcal{L}^*$ arise uniquely in this fashion, upon allowing S and the order in which elements are written to vary. It follows that $k!|\mathcal{L}| = |\mathcal{L}^*| = n!\binom{n+(s+1)k+2r-1}{n-k}$, which implies (22). $\square$

5 Combinatorial proofs of some prior formulas

In this section, we provide combinatorial proofs of some results from [10]. The following identity expresses $B_{n,k}^{(r)}(x_i; y_i)$ in terms of the partial Bell polynomials and occurs as [10, Proposition 9].

Theorem 13. *If $n \geq k \geq 0$ and $r \geq 0$, then*

$$\binom{n+r}{r} B_{n,k}^{(r)}(x_i; y_i) = \sum_{j=k}^{n} \binom{n+r}{j} B_{j,k}(x_i) B_{n+r-j,r}(iy_i). \tag{23}$$

Proof. Let E denote the set of partitions of $[n+r]$ having $k+r$ blocks in which r elements, each belonging to separate blocks, are circled. Assign the weight x_i and y_i, respectively, to blocks of size i that do not or do contain a circled element, with the weight of a member of E given by the product of the weights of its blocks. Then the sum of the weights of members of E is given by $\binom{n+r}{r}B_{n,k}^{(r)}(x_i;y_i)$. To see this, first choose a subset S of $[n+r]$ of size r and then arrange the elements of $[n+r]$ according to any member of $\mathcal{B}_{n,k}^{(r)}$ (treating the members of S as the special elements, which we circle). To show that the right-hand side of (23) also gives the weight of E, consider the number j of elements belonging to blocks which do *not* contain a circled element; note that $k \leq j \leq n$ since circled elements are to belong to distinct blocks. There are $\binom{n+r}{j}$ choices for these elements and $B_{j,k}(x_i)$ possible ways in which to arrange them in k blocks. The remaining $n+r-j$ members of $[n+r]$ then go in r blocks in each of which one of the elements is to be circled. Assigning each such block of size i a weight of iy_i, it is seen that there are $B_{n+r-j,r}(iy_i)$ possibilities for the blocks containing circled elements. Summing over all possible j implies (23). □

The following r-Bell polynomial formulas occur as [10, Proposition 8].

Theorem 14. *If* $n \geq k \geq 1$ *and* $r \geq 1$, *then*

$$kB_{n,k}^{(r)}(x_i;y_i) = \sum_{j=1}^{n}\binom{n}{j}x_jB_{n-j,k-1}^{(r)}(x_i;y_i), \tag{24}$$

$$B_{n,k}^{(r)}(x_i;y_i) = \sum_{j=1}^{n}\binom{n}{j-1}y_jB_{n+1-j,k}^{(r-1)}(x_i;y_i), \tag{25}$$

and

$$\begin{aligned}(n+r)B_{n,k}^{(r)}(x_i;y_i) = &\sum_{j=1}^{n}j\binom{n}{j}x_jB_{n-j,k-1}^{(r)}(x_i;y_i)\\ &+ r\sum_{j=1}^{n}j\binom{n}{j-1}y_jB_{n+1-j,k}^{(r-1)}(x_i;y_i).\end{aligned} \tag{26}$$

Proof. To show (24), consider circling a non-special block within members of $\mathcal{B}_{n,k}^{(r)}$. Then the right-hand side gives the sum of the weights of such configurations by considering the number j of elements belonging to the circled block. Note that (25) is equivalent to the $s=1$ case of (9) above and can be obtained by considering the number of elements of $[n]$ belonging to the block containing the element 1. Finally, to show (26), we circle an element s of $[n+r]$ within a member of $\mathcal{B}_{n,k}^{(r)}$. The first sum on the right side of (26) accounts for the case in which s belongs to a non-special block. If j denotes the cardinality of this block, then there are $j\binom{n}{j}$ ways to select it and circle one of its elements, with it contributing x_j towards the weight, while the remaining elements of $[n+r]$ comprise a member of $\mathcal{B}_{n-j,k-1}^{(r)}$. On the other hand, if s belongs to a special block, then there are r choices for the special element belonging to this block, and once this is determined, $\binom{n}{j-1}$ choices for some $j \in [n]$ for the members of I belonging to this block. Then there are j choices for the circled element s, which may or may not be special, and the remaining elements of $[n+r]$ comprise a member of $\mathcal{B}_{n+1-j,k}^{(r-1)}$. Considering all possible j gives the second sum on the right side of (26) and completes the proof. □

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