**Notes on Number Theory and Discrete Mathematics**
**Print ISSN 1310–5132, Online ISSN 2367–8275**
**Vol. 23, 2017, No. 1, 54–62**

# Primitive Pythagorean triples and generalized Fibonacci sequences

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**Received:** 19 April 2016 **Accepted:** 10 November 2016

**Abstract:** It is proved that infinite sequences of generalized Fibonacci sequences obtained from generalizations of the Golden Ratio can generate all primitive Pythagorean triples. This is a consequence of the integer structure since the major component of a primitive Pythagorean triple always has the form $(4r_1 + 1)$ where $r_1$ belongs to the class in the modular ring $Z_4$.
**Keywords:** Unit digits (right-end-digits), Modular rings, Golden ratio, Fibonacci and Lucas numbers, Pythagorean triples.
**AMS Classification:** 11B39, 11B50.

## 1 Introduction

In the notation of the Fibonacci sequences of the Golden Ratio family, the generalized Fibonacci numbers satisfy the second order recurrence relation

$$F_{n+1}(a) = F_n(a) + r_1 F_{n-1}(a) \tag{1.1}$$

with unity as the two initial terms, and in which $r_1$ is the row of the variable $a$ for $\phi(a)$ of the Golden Ratio family, with

$$\varphi_a = \frac{1+\sqrt{a}}{2} \tag{1.2}$$

and $a = 4r_1 + 1 \in \bar{1}_4 \subset Z_4$ (Table 1) [5, 7].

| Row $r_i \downarrow$ | Class $i \rightarrow$ | $\bar{0}_4$ | $\bar{1}_4$ | $\bar{2}_4$ | $\bar{3}_4$ | Comments |
|---|---|---|---|---|---|---|
| **0** | | 0 | 1 | 2 | 3 | • $N = 4r_i + i$ |
| **1** | | 4 | 5 | 6 | 7 | • even $\bar{0}_4, \bar{2}_4$ |
| **2** | | 8 | 9 | 10 | 11 | • $\left(N^n, N^{2n}\right) \in \bar{0}_4$ |
| **3** | | 12 | 13 | 14 | 15 | • odd $\bar{1}_4, \bar{3}_4$ ; $N^{2n} \in \bar{1}_4$ |

Table 1. Classes and rows for $Z_4$

The generalized Binet formula in this notation is [6, 12]

$$F_n(a) = \frac{\left(\frac{1+\sqrt{a}}{2}\right)^n - \left(\frac{1-\sqrt{a}}{2}\right)^n}{\sqrt{a}}. \tag{1.3}$$

which is well-known for the ordinary Fibonacci numbers as

$$\varphi_a = \frac{1+\sqrt{a}}{2} \tag{1.4}$$

and any Golden Ratio family member is given by

$$\frac{F_n(a)}{F_{n-1}(a)} \rightarrow \varphi_a.$$

Other approaches to what has been called "generalized Golden numbers" are in the literature [1, 9], but they are more like generalized Pell numbers, which are in effect covered in [10].

The purpose of this paper is to demonstrate that all Pythagorean triples can be formed from these Golden Ratio generalized Fibonacci numbers

## 2 Primitive Pythagorean Fibonacci triples and triads

For the ordinary Fibonacci numbers

$$F_{2n+1}(5) = F_{n+1}^2(5) + F_n^2(5). \tag{2.1}$$

More generally this becomes

$$F_{2n+1}(a) = F_{n+1}(a)^2 + r_1 F_n(a)^2. \tag{2.2}$$

For example, when $r_1 = 1$ and $n = 5$ (2.2) becomes (2.1):

$$\begin{aligned} F_{11}(5) &= 89 \\ &= 64 + 25 \qquad , \\ &= F_6(5)^2 + 1 \times F_5(5)^2. \end{aligned}$$

Table 2 displays some of the first few terms of some of these $\{F_n(a)\}$.

| $r_1$ | *n* | *1* | *2* | *3* | *4* | *5* | *6* | *7* | *8* | *9* | *10* | *11* | *12* |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $F_n(1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | $F_n(5)$ | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 |
| 2 | $F_n(9)$ | 1 | 1 | 3 | 5 | 11 | 21 | 43 | 85 | 171 | 341 | 683 | 1365 |
| 3 | $F_n(13)$ | 1 | 1 | 4 | 7 | 19 | 40 | 97 | 217 | 508 | 1159 | 2683 | 6160 |
| 4 | $F_n(17)$ | 1 | 1 | 5 | 9 | 29 | 65 | 181 | 441 | 1165 | 2929 | 7589 | 19305 |
| 5 | $F_n(21)$ | 1 | 1 | 6 | 11 | 41 | 96 | 301 | 781 | 2286 | 6191 | 17621 | 48576 |
| 6 | $F_n(25)$ | 1 | 1 | 7 | 13 | 55 | 133 | 463 | 1261 | 4039 | 11605 | 35839 | 105469 |
| 7 | $F_n(29)$ | 1 | 1 | 8 | 15 | 71 | 176 | 673 | 1905 | 6616 | 19951 | 66263 | 205920 |
| 8 | $F_n(33)$ | 1 | 1 | 9 | 17 | 89 | 225 | 937 | 2737 | 10233 | 32129 | 113993 | 371025 |

Table 2: $F_n(a)$, $n = 1, 2, \ldots, 12$

Equation (2.2) can be extended with (1.1) to

$$F_n(a)^2 + F_{n+1}(a)^2 = 2F_n(a)F_{n+1}(a) + r_1^2 F_{n-1}(a)^2. \tag{2.3}$$

For example, when $r_1 = 1$ and $n = 5$ (2.3) becomes

$$\begin{aligned} F_5(5)^2 + F_6(5)^2 &= 25 + 64 \\ &= 89 \\ &= 80 + 9 \\ &= 2F_5(5)F_6(5) + 1^2 F_4(5)^2. \end{aligned}$$

Equation (2.2) only gives numbers already in the sequence, but for $a > 5$

$$N_n = F_{n+1}(a)^2 + F_n(a)^2. \tag{2.4}$$

will yield new sequences for each value of $a$ [6]. When $F_n(a), F_{n+1}(a)$ have opposite parity the sequences have an odd, even, odd pattern. In this case $N$ can be the major component of a Pythagorean triad [4] in which the minor components will be $2F_{n+1}(a)F_n(a)$ and $\left(F_{n+1}(a)^2 - F_n(a)^2\right)$ (Table 3, in which the $\{N_i\}$ values are calculated from (2.4) and Table 2).

| ***a*** | $r_1$ | $\{N_i\}$ | **Primitive Pythagorean triples from odd $N_i$** |
|---|---|---|---|
| 13 | 3 | 17,65,410,1961, 11009,… | (17,15,8);(65,56,33);(1961,1520,1239); (11009,7809,7760) |
| 21 | 5 | 37,157,1802,10897, 99817,700562,… | (37,35,12);(157,132,85);(10897,77535,787); (99817,81385,57792) |
| 29 | 7 | 65,289,5266,36017, 483905,4081954,… | (65,63,16);(289,240,161);(36017,25935,24); (483905,143165,236896) |
| 45 | 11 | 145,673,24554,190489, 44631233,40476770,… | (145,143,24); (673,552,385); (190489,126480,42439) |

Table 3: Examples of $\{N_i\}$ sequences & associated primitive Pythagorean triples

## 3 Sequences with $r_1$ even

When $r_1$ is even all the $F_n(a)$ are odd so that the sum of squares will be even. However, division by 2 can give odd values and these can be used to form primitive Pythagorean triples (pPts). For example, when $r_1 = 2$, the sequence is

$$\{M_n\} \equiv \tfrac{1}{2}\{N_n\} \equiv \{5,17,73,281,...\} \subset \bar{1}_4 .$$

The elements of $\{M_n\}$ can all be the major component of a pPt:

$$c = x^2 + y^2.$$

The values of $x$ and $y$ can be estimated from

$$x, y = \frac{A \pm \sqrt{2M_n - A^2}}{2} \tag{3.1}$$

in which $x$ is odd and $y$ is even with $A = x + y$ [8]. Then $a = x^2 - y^2$, $b = 2xy$. But

$$2M_n = F_{n+1}(a)^2 + F_n(a)^2$$

and

$$A = F_{n+1}(a)$$

so that

$$x, y = \tfrac{1}{2}\left(F_{n+1}(a) \pm F_n(a)\right).$$

For instance, when $r_1 = 8$, $F_6(33) = 225, F_7(33) = 937, M_6 = 464297$, and

$$x = \tfrac{1}{2}(937 + 225) = 581,$$

and

$$y = \tfrac{1}{2}(937 - 225) = 356,$$

so that

$$c = 581^2 + 356^2 = 464297,$$

$$b = 581^2 - 356^2 = 210825,$$

$$a = 2(581 \times 356) = 413672,$$

and

$$c^2 = 215, 571, 704, 209,$$

$$b^2 = 44, 447, 180, 625$$

$$a^2 = 171, 124, 523, 584.$$

Thus it can be readily confirmed that the components may also be calculated directly from the generalized Fibonacci numbers by

$$c = \tfrac{1}{2}\left(F_{n+1}(a)^2 + F_n(a)^2\right) \tag{3.2}$$

$$b = F_{n+1}(a)F_n(a) \tag{3.3}$$

$$a = \tfrac{1}{2}\left(F_{n+1}(a)^2 - F_n(a)^2\right), \quad (3.4)$$

so that $c^2 = a^2 + b^2$ becomes

$$\left(\tfrac{1}{2}\left(F_{n+1}(a)^2 + F_n(a)^2\right)\right)^2 = \left(\tfrac{1}{2}\left(F_{n+1}(a)^2 - F_n(a)^2\right)\right)^2 + \left(F_{n+1}(a)F_n(a)\right)^2,$$

which accords with known results [2,10] for generalized $F_n(5)$ and which appear in another form in Section 4. Some examples appear in Table 4.

| $a$ | $r_1$ | $\{M_i\}$ | Primitive Pythagorean triples from $M_i$ |
|---|---|---|---|
| 9 | 2 | 5,17,73,281,1145,4537, 18233, ... | (5,4,3);(17,15,8);(73,554,48);(281,231,160); (1145,1064,423);(4537,3655,26880); (18233,14535,11008). |
| 33 | 8 | 41,185,4105,29273, 464297,4184569, 56102729, ... | (41,40,9);(185,176,57);(4105,3816,1513); (29273,21352,20025);(464297,413672,210825); (4184569,3306600,2564569); (56102729,48611560,28007721). |

Table 4: Examples of $\{M_i\}$ sequences & associated primitive Pythagorean triples

This establishes the result that all Pythagorean triples are generalized Fibonacci triples generated from the generalized Golden Ratio. We expand this theme in the next section.

# 4 The generalized Golden Ratio triples equal primitive Pythagorean triples

We have shown that the major component of all primitive Pythagorean triples (pPts) ($c^2 = a^2 + b^2$) must fall in Class $\bar{1}_4 \in Z_4$, a modular ring, (Table 1); that is

$$c = 4r_1 + 1 \quad (4.1)$$

since only odd numbers in this class are sums of squares: $c = x^2 + y^2$. The generalized Golden Ratio family members [6] are given by

$$\varphi = \tfrac{1}{2}\left(1 + \sqrt{a}\right) \quad (4.2)$$

where

$$a = 4r_1 + 1$$

and hence

$$a = (2\varphi - 1)^2. \quad (4.3)$$

This means that the major components of all pPts may be obtained from the generalized Golden Ratio $\varphi_a$. Thus, we have not only displayed the role of the generalized Golden Ratio and thus extended the results of Horadam, but also provided insight into the integer structure through the associated modular ring.

When $a$ is prime, all $a = x^2 + y^2$ [8] and therefore are the major components of pPts. In this case $x$ and $y$ have no common factors. Moreover composites that only have factors in Class $\bar{3}_4$ can never be equal to a sum of squares. All other composites in $\bar{1}_4$ equal either one $(x, y)$ couple with a common factor or have the same number of $(x, y)$ couples as factors (Table 5).

| ***a*** | **Factors** | **Class** | ***x,y*** | **$(2\varphi\text{-}1)^2$ pPts** |
|---|---|---|---|---|
| 9 | 3,3 | $\bar{3}_4\,\bar{3}_4$ | --- | --- |
| 17 | prime | $\bar{1}_4$ | 1,4 | 17,15,8 |
| 25 | 5,5 | $\bar{1}_4\,\bar{1}_4$ | 3,4 | 25,7,24** |
| 41 | prime | $\bar{1}_4$ | 5,4 | 41,40,9 |
| 65 | 5,13 | $\bar{1}_4\,\bar{1}_4$ | 1,8 | 65,63,16 |
| | | | 7,4 | 65,56,33 |
| 101 | prime | $\bar{1}_4$ | 1,10 | 101,99,20 |
| 237 | 3,79 | $\bar{3}_4\,\bar{3}_4$ | --- | --- |
| 333 | 3,3,37 | $\bar{3}_4\,\bar{3}_4\,\bar{1}_4$ | 3,18 | 333,315,108*** |
| 577 | prime | $\bar{1}_4$ | 1,24 | 577,575,48 |
| 733 | prime | $\bar{1}_4$ | 2,27 | 733,725,108 |
| 3725 | 5,5,149 | $\bar{1}_4\,\bar{1}_4\,\bar{1}_4$ | 35,50 | 3725,1275,3500 |
| 13297 | prime | $\bar{1}_4$ | 79,84 | 13297,815,13272 |
| 69589 | 13,53,101 | $\bar{1}_4\,\bar{1}_4\,\bar{1}_4$ | 183,190 | 69589,2611,69540 |
| | | | 217,150 | 69589,24589,65100 |
| | | | 105,242 | 69589,47539,50820 |
| 1401953 | 7,19,83,127 | $\bar{3}_4\,\bar{3}_4\,\bar{3}_4\,\bar{3}_4$ | --- | --- |

Table 5: Factors, couples and pPts
[**Squares have only one $(x, y)$; *** $x$ and $y$ have a common factor]

The $(x, y)$ couples may be calculated from

$$x, y = \frac{A \pm \sqrt{2a - A^2}}{2} \tag{4.4}$$

in which x is odd and y is even with $A = x + y$ [8]. For primes $A \sim \sqrt{2a}$ and this is the maximum value otherwise.

From Equation (4.3) we can construct Table 6 for $n = 14$:

| $r_1$ | **1** | **2** | **3** | **4** | **5** | **6** |
|---|---|---|---|---|---|---|
| $c$ | 5.0 | 9.0 | 13.0 | 17.0 | 20.7 | 22.1 |
| $a$ | 5 | 9 | 13 | 17 | 21 | 25 |

Table 6: $c \sim a = \left((2F_{n+1} - F_n)/F_n\right)^2$

so that we need $n > 14$; for instance, as in Table 7.

| $r_1$ | $a = c$ | $F_{20}$ | $F_{20}/F_{19}$ | $\frac{1+\sqrt{a}}{2}$ |
|---|---|---|---|---|
| 1 | 5 | 6765 | 1.6180339 | 1.6180339 |
| 2 | 19 | 349525 | 1.9999942 | 2.0000000 |
| 3 | 13 | 4875913 | 2.3027037 | 2.3037756 |
| 4 | 17 | 35877321 | 2.5612022 | 2.5615528 |
| 5 | 21 | 179854741 | 2.7902858 | 2.7912878 |

Table 7: Approximation of $F_{20}/F_{19}$ to $\frac{1+\sqrt{a}}{2}$

In summary, $a = c$ for all primes since they equal $x^2 + y^2$ and for all composites provided they have factors in class $\bar{1}_4$. In this case there will be the same number of the $x$, $y$ pairs as factors, so multiple pPts can be obtained. For instance, $a = c = 65$ and $65 = 1^2 + 8^2 = 4^2 + 7^2$, and there are 2 pPts. Note that the first and third rows of Table 3 yield (65, 56, 33) and (65, 63, 16); and (65, 56, 33) = (65, (2 × 28), (49 – 16)) and (65, (64 – 1), (2 × 8)). The Table 3 values can be obtained from $a = 65$ directly (Table 5) so there is duplication in the process.

## 5 Final comments

Pythagorean triples have been studied since antiquity but this paper shows that we only need $a$ to get all pPts. The results here generalize the alternative methods [2,10] which in effect only duplicate those from $a$ and hence $\bar{1}_4 \in Z_4$. More generally though, since $a \in \bar{1}_4$ it can equal $x^2 + y^2$ which is essential for $c$.

When $a$ is prime there is always an $(x, y)$ pair, but if $a$ is composite there must be factors in class $\bar{1}_4$, otherwise there is no $(x, y)$ pair [8]. For example,

$$65 = 5 \times 13,\ 5, 13 \in \bar{1}_4,$$
$$65 = 1^2 + 8^2 = 4^2 + 7^2,$$

and there is the same number of pairs of squares as there are factors. Thus $c$ must equal

$$a = x_1^2 + y_1^2 = 4r_1 + 1 \in \bar{1}_4 \subset Z_4.$$

It follows that all pPts have components obtained from generalized Fibonacci number triples. Since

$$\varphi = \tfrac{1}{2}\left(1 + \sqrt{a}\right),$$
$$a = (2\varphi - 1)^2.$$

Thus, for sufficiently large $n$, and provided $a = x^2 + y^2$,

$$c = \left(2\varphi - 1^2\right) = \left(\left(2F_{n+1}(a) - F_n(a)\right) / F_n(a)\right)^2$$

There is still more to be studied. The interested reader might like to extend some of these results to Golden Ratio Lucas numbers, $\{L_n(a)\}$, (Table 7), where it can be seen that they are related to the Golden Ratio Fibonacci numbers by

$$L_n(a) = F_{n+1}(a) + r_1 F_{n-1}(a) \tag{5.1}$$

which is analogous to the well-known [16]

$$L_n(5) = F_{n+1}(5) + F_{n-1}(5), \tag{5.2}$$

since $r_1 = 1$.

| $r_1$ | $n$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | $L_n(1)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | $L_n(5)$ | 1 | 3 | 4 | 7 | 11 | 18 | 29 | 47 | 76 | 123 | 199 |
| 2 | $L_n(9)$ | 1 | 5 | 7 | 17 | 31 | 65 | 127 | 257 | 511 | 1025 | 2047 |
| 3 | $L_n(13)$ | 1 | 7 | 10 | 31 | 61 | 154 | 337 | 799 | 1810 | 4207 | 9637 |
| 4 | $L_n(17)$ | 1 | 9 | 13 | 49 | 101 | 297 | 701 | 1889 | 4693 | 12249 | 31021 |
| 5 | $L_n(21)$ | 1 | 11 | 16 | 71 | 151 | 506 | 1261 | 3791 | 10096 | 29051 | 79531 |
| 6 | $L_n(25)$ | 1 | 13 | 19 | 97 | 211 | 793 | 2059 | 6817 | 19171 | 60073 | 175099 |
| 7 | $L_n(29)$ | 1 | 15 | 22 | 127 | 281 | 1170 | 3137 | 11327 | 33286 | 112575 | 345577 |

Table 7: $L_n(a)$, $n = 1, 2, \ldots, 11$

Other similar generalizations include

$$F_n(a) = \frac{1}{a}\left(L_{n+1}(a) + r_1 L_{n-1}(a)\right) \tag{5.3}$$

which is analogous to

$$F_n(5) = \frac{1}{5}\left(L_{n+1}(5) + L_{n-1}(5)\right). \tag{5.4}$$

Similarly

$$F_{m+n}(a) = \frac{1}{2}\left(F_m(a)L_n(a) + F_n(a)L_m(a)\right) \tag{5.5}$$

so that the generalizations are quite extensive. Further studies could also investigate these sequences in finite fields [14], the graphs of their connections [11], and the intersections between sequences with the same value for $r_1$ and among sequences with different values for $r_1$ [3, 13, 15].

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