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# Further results on arctangent sums with applications to generalized Fibonacci numbers

**Robert Frontczak**

Landesbank Baden-Wuerttemberg*
Am Hauptbahnhof 2, 70173 Stuttgart, Germany
e-mail: robert.frontczak@lbbw.de

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**Abstract:** In this article, we extend a classical trigonometric addition formula for the arctangent function and derive new summation identities for Fibonacci and Lucas numbers. While most of the results seem to be new, we also recover some known expressions.
**Keywords:** Arctangent sum, Fibonacci number, Lucas number.
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## 1 Introduction

This paper deals with generalized Fibonacci numbers $G_n$ defined through the second order recurrence relation $G_{n+1} = G_n + G_{n-1}$, where the initial terms $G_0$ and $G_1$ need to be specified. The most popular members of this family are the Fibonacci numbers $F_n$ and Lucas numbers $L_n$ defined by initial conditions $F_0 = 0, F_1 = 1$ and $L_0 = 2, L_1 = 1$, respectively. The Binet forms are given by

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad L_n = \alpha^n + \beta^n, \qquad n \geq 0, \tag{1.1}$$

where $\alpha$ and $\beta$ are roots of the quadratic equation $x^2 - x - 1 = 0$, i.e.

$$\alpha = \frac{1+\sqrt{5}}{2}, \qquad \beta = \frac{1-\sqrt{5}}{2}. \tag{1.2}$$

* Disclaimer: Statements and conclusions made in this article are entirely those of the author. They do not necessarily reflect the views of LBBW.

Clearly, we have $\alpha > 1, -1 < \beta < 0$, $\alpha + \beta = 1$, $\alpha\beta = -1$, $\alpha^2 = \alpha + 1$ and $\beta^2 = \beta + 1$ (see [7]). A celebrated classical result in the study of infinite series involving Fibonacci numbers is the so-called Lehmer identity (see [7]):

$$\sum_{n=1}^{\infty} \tan^{-1}\left(\frac{1}{F_{2n+1}}\right) = \frac{\pi}{4}. \tag{1.3}$$

Motivated by this striking result, many extensions and generalizations were obtained in [2], [5], [6], [8] and [9]. All the studies use the following trigonometric formula as a basis

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right), \qquad xy > -1 \tag{1.4}$$

where $\tan^{-1}(x)$ stands for the principal value of the arctangent function at $x$. If $xy < -1$ the formula reads as

$$\tan^{-1}(x) - \tan^{-1}(y) = \tan^{-1}\left(\frac{x-y}{1+xy}\right) + \pi sgn(x), \qquad xy < -1, \tag{1.5}$$

with $sgn(x)$ being the signum function of the argument $x$. In this paper we extend the above formula and give several applications.

# 2 Main results

Our main result is the following identity:

**Theorem 2.1.** *Let $f(x)$ and $g(x)$ be two real functions. For $k \geq 1$, consider the new function $h(x)$ defined as*

$$h(x) = \frac{f(g(x+k)) - f(g(x-k))}{1 + f(g(x+k))f(g(x-k))}. \tag{2.1}$$

*Then*

$$\sum_{i=1}^{n} \tan^{-1} h(i) = \sum_{m=-k+1}^{k} \tan^{-1} f(g(n+m)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) \qquad (mod\ \pi), \tag{2.2}$$

*and*

$$\sum_{i=1}^{\infty} \tan^{-1} h(i) = 2k \tan^{-1} f(g(\infty)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) \qquad (mod\ \pi). \tag{2.3}$$

*The above equations mean that for $f(g(i+k))f(g(i-k)) < -1$ the relations must be modified according to*

$$\sum_{i=1}^{n} \tan^{-1} h(i) = \sum_{m=-k+1}^{k} \tan^{-1} f(g(n+m)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) + \pi \sum sgn(f(g(i+k))), \tag{2.4}$$

*and*

$$\sum_{i=1}^{\infty} \tan^{-1} h(i) = 2k \tan^{-1} f(g(\infty)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)) + \pi \sum sgn(f(g(i+k))), \tag{2.5}$$

*where the sum is taken over all $i$ between 1 and $n$ (or infinity) for which $f(g(i+k))f(g(i-k)) < -1$.*

*Proof:* The proof is based on k-fold telescoping. Let $f(g(i+k))f(g(i-k)) > -1$. Then

$$\begin{aligned}
\sum_{i=1}^{n} \tan^{-1} h(i) &= \sum_{i=1}^{n} \Big( \tan^{-1} f(g(i+k)) - \tan^{-1} f(g(i-k)) \Big) \\
&= \sum_{i=1}^{n} \Big( \tan^{-1} f(g(i+k)) - \tan^{-1} f(g(i+k-1)) \\
&\quad + \tan^{-1} f(g(i+k-1)) - \tan^{-1} f(g(i+k-2)) + \cdots \\
&\quad + \tan^{-1} f(g(i-k+1)) - \tan^{-1} f(g(i-k)) \Big) \\
&= \sum_{i=1}^{n} \Big( \tan^{-1} f(g(i+k)) - \tan^{-1} f(g(i+k-1)) \Big) \\
&\quad + \sum_{i=1}^{n} \Big( \tan^{-1} f(g(i+k-1)) - \tan^{-1} f(g(i+k-2)) \Big) + \cdots \\
&\quad + \sum_{i=1}^{n} \Big( \tan^{-1} f(g(i-k+1)) - \tan^{-1} f(g(i-k)) \Big) \\
&= \tan^{-1} f(g(n+k)) - \tan^{-1} f(g(k)) \\
&\quad + \tan^{-1} f(g(n+k-1)) - \tan^{-1} f(g(k-1)) + \cdots \\
&\quad + \tan^{-1} f(g(n-k+1)) - \tan^{-1} f(g(1-k)) \\
&= \sum_{m=-k+1}^{k} \tan^{-1} f(g(n+m)) - \sum_{m=-k+1}^{k} \tan^{-1} f(g(m)).
\end{aligned}$$

This proves the first part of the Theorem. The second part follows straightforwardly by taking the limit $n \to \infty$. The case $f(g(i+k))f(g(i-k)) < -1$ is proved in exactly the same manner. $\square$

## 3 First applications

Before applying the Theorem to Fibonacci and Lucas numbers, we present some other interesting examples. For $k = 1$ and $f(g(i+1))f(g(i-1)) > -1$ the Theorem gives

$$\sum_{i=1}^{\infty} \tan^{-1} h(i) = 2\tan^{-1} f(g(\infty)) - \tan^{-1} f(g(0)) - \tan^{-1} f(g(1)). \tag{3.1}$$

A special case of this result may be found in [3].
Let $g(x) = ax, a > 0$, and $f(x) = x$. Then

$$h(x) = \frac{2a}{a^2x^2 - a^2 + 1}, \tag{3.2}$$

and

$$\sum_{i=1}^{\infty} \tan^{-1} \Big( \frac{2a}{a^2i^2 - a^2 + 1} \Big) = \pi - \tan^{-1}(a). \tag{3.3}$$

For $a = 1$ we get

$$\sum_{i=1}^{\infty} \tan^{-1} \Big( \frac{2}{i^2} \Big) = \frac{3\pi}{4}. \tag{3.4}$$

The last identity is known. It appears in Ramanujan's First Notebook [10] and was also proposed as a problem in [1]. The result is also derived in [3].
For $a = \sqrt{3}$ we get

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{2\sqrt{3}}{3i^2-2}\right) = \frac{2\pi}{3}. \tag{3.5}$$

For $a = \sqrt{3}/3$ we get

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{2\sqrt{3}}{i^2+2}\right) = \frac{5\pi}{6}. \tag{3.6}$$

Differentiating (3.3) with respect to $a$ gives

$$\sum_{i=1}^{\infty} \frac{a^2(i^2-1)-1}{(a^2(i^2-1)+1)^2+4a^2} = \frac{1}{2}\frac{1}{1+a^2}. \tag{3.7}$$

The special case $a = 1$ yields

$$\sum_{i=1}^{\infty} \frac{i^2-2}{i^4+4} = \frac{1}{4}. \tag{3.8}$$

Boros and Moll [3] show that

$$\sum_{i=1}^{\infty} \frac{i^2}{i^4+4} = \frac{\pi}{4}\coth(\pi). \tag{3.9}$$

Combining the last two results we obtain

$$\sum_{i=1}^{\infty} \frac{1}{i^4+4} = \frac{\pi}{8}\coth(\pi) - \frac{1}{8}. \tag{3.10}$$

Let $g(x) = a/x^2, a > 0$, and $f(x) = x$. Then

$$h(x) = \frac{-4ax}{(x^2-1)^2+a^2}, \tag{3.11}$$

and

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{4ai}{(i^2-1)^2+a^2}\right) = \frac{\pi}{2} + \tan^{-1}(a). \tag{3.12}$$

For $a = 1$ we have

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{4i}{(i^2-1)^2+1}\right) = \frac{3\pi}{4}. \tag{3.13}$$

For $a = \sqrt{3}$ we get

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{4\sqrt{3}i}{(i^2-1)^2+3}\right) = \frac{5\pi}{6}. \tag{3.14}$$

For $a = \sqrt{3}/3$ we get

$$\sum_{i=1}^{\infty} \tan^{-1}\left(\frac{4\sqrt{3}i}{3(i^2-1)^2+1}\right) = \frac{2\pi}{3}. \tag{3.15}$$

Differentiating (3.12) with respect to $a$ gives

$$\sum_{i=1}^{\infty} \frac{(i+1)(i^2(i+2)^2 - a^2)}{(i^2(i+2)^2 + a^2)^2 + 16a^2(i+1)^2} = \frac{5a^2 + 20}{4(1+a^2)(16+a^2)}. \tag{3.16}$$

The special case $a = 0$ yields

$$\sum_{i=1}^{\infty} \frac{i+1}{i^2(i+2)^2} = \frac{5}{16}. \tag{3.17}$$

In the next example, we choose $g(x) = ax, a > 0$ and $f(x) = be^{-x}, b > 0$. Then

$$h(x) = \frac{-2b\sinh(a)e^{-ax}}{1 + b^2e^{-2ax}}, \tag{3.18}$$

and

$$\sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{2b\sinh(a)e^{-ai}}{1+b^2e^{-2ai}}\Big) = \tan^{-1}(b) + \tan^{-1}(be^{-a}). \tag{3.19}$$

For $a = b = 1$ we obtain the striking identity

$$\sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{\sinh(1)}{\cosh(i)}\Big) = \frac{\pi}{4} + \tan^{-1}(\frac{1}{e}). \tag{3.20}$$

More generally, let $a$ be fixed and set $b = e^a$. This gives

$$\sum_{i=0}^{\infty} \tan^{-1}\Big(\frac{\sinh(a)}{\cosh(ai)}\Big) = \frac{\pi}{4} + \tan^{-1}(e^a). \tag{3.21}$$

Especially, for $a = 1$

$$\sum_{i=0}^{\infty} \tan^{-1}\Big(\frac{\sinh(1)}{\cosh(i)}\Big) = \frac{\pi}{4} + \tan^{-1}(e). \tag{3.22}$$

For $a = \ln\alpha$ we obtain using $\sinh(\ln\alpha) = 1/2$

$$\sum_{i=0}^{\infty} \tan^{-1}\Big(\frac{1}{\alpha^i + \alpha^{-i}}\Big) = \frac{\pi}{4} + \tan^{-1}(\alpha). \tag{3.23}$$

Differentiating (3.19) w.r.t. $b$ gives

$$\sum_{i=1}^{\infty} \frac{e^{ai} - b^2e^{-ai}}{(e^{ai} + b^2e^{-ai})^2 + 4b^2\sinh^2(a)} = \frac{1}{2\sinh(a)}\Big(\frac{1}{1+b^2} + \frac{e^{-a}}{1+b^2e^{-2a}}\Big), \tag{3.24}$$

from which for $b = 1$ we immediately get

$$\sum_{i=1}^{\infty} \frac{\sinh(ai)}{\cosh^2(ai) + \sinh^2(a)} = \frac{1}{2\sinh(a)}\Big(1 + \frac{1}{\cosh(a)}\Big) = \frac{1}{2\cosh(a)}\coth(\frac{a}{2}). \tag{3.25}$$

In a similar manner, we can differentiate (3.19) w.r.t. $a$. The result is

$$\sum_{i=1}^{\infty} \frac{i\sinh(a)(e^{ai} - b^2e^{-ai}) - \cosh(a)(e^{ai} + b^2e^{-ai})}{(e^{ai} + b^2e^{-ai})^2 + 4b^2\sinh^2(a)} = \frac{1}{2}\frac{e^{-a}}{1+b^2e^{-2a}}. \tag{3.26}$$

For $b = 1$ the expression reduces to

$$\sum_{i=1}^{\infty} \frac{i\sinh(a)\sinh(ai) - \cosh(a)\cosh(ai)}{\cosh^2(ai) + \sinh^2(a)} = \frac{1}{2\cosh(a)}. \tag{3.27}$$

**Remark 3.1.** *Due to the general nature of our main result many more examples could be stated. For instance, applying the Theorem to the Gamma function ($g(x) = \Gamma(a(x+1))$ and $f(x) = bx$, $a, b > 0$) it is easily shown that*

$$\sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{(i+1)! - (i-1)!}{1 + (i+1)!(i-1)!}\Big) = \frac{\pi}{2}. \tag{3.28}$$

*We have done much more research in this direction and obtained some striking relations for (arctangent) sums including special functions. However, since the focus of the paper is on generalized Fibonacci numbers we stop here listing isolated examples.*

# 4 Applications to generalized Fibonacci numbers

In this section we use the main Theorem to derive several identities for Fibonacci and Lucas numbers. We will make repeated use of the following known relations ([2] or [7]):

$$\begin{aligned} F_{u+v}F_{u-v} &= F_u^2 - (-1)^{u-v}F_v^2 \\ L_{u+v}L_{u-v} &= L_{2u} + (-1)^{u-v}L_{2v} \\ 5F_{u+v}F_{u-v} &= L_{2u} - (-1)^{u-v}L_{2v} \\ L_uF_v &= F_{v+u} + (-1)^u F_{v-u} \\ F_uL_v &= F_{v+u} - (-1)^u F_{v-u} \\ L_uL_v &= L_{u+v} + (-1)^u L_{v-u}. \end{aligned}$$

## 4.1 Case $k = 1$, $g(i) = G_{mi+n}, n \geq 0, m \geq 1$ and $f(i) = ai, a > 0$.

This choice produces

$$T(G,a,m,n) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{a(G_{m(i+1)+n} - G_{m(i-1)+n})}{1 + a^2 G_{m(i+1)+n}G_{m(i-1)+n}}\Big) = \pi - \tan^{-1}(aG_{m+n}) - \tan^{-1}(aG_n) \tag{4.1}$$

and after differentiation w.r.t. $a$

$$S(G,a,m,n) = \sum_{i=1}^{\infty} \frac{(a^2x_2 - 1)x_1}{(1 + a^2x_2)^2 + a^2x_1^2} = \frac{G_{m+n}}{1 + a^2G_{m+n}^2} + \frac{G_n}{1 + a^2G_n^2}, \tag{4.2}$$

where we have set

$$\begin{aligned} x_1 &= G_{m(i+1)+n} - G_{m(i-1)+n} \\ x_2 &= G_{m(i+1)+n}G_{m(i-1)+n}. \end{aligned}$$

Equations (4.1) and (4.2) contain many remarkable identities as special cases.
We start with the most simple two:

$$T(F,a,1,0) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{aF_i}{1 + a^2(F_i^2 + (-1)^i)}\Big) = \pi - \tan^{-1}(a), \tag{4.3}$$

and

$$S(F,a,1,0)=\sum_{i=1}^{\infty}\frac{(a^2(F_i^2+(-1)^i)-1)F_i}{(a^2(F_i^2+(-1)^i)+1)^2+a^2F_i^2}=\frac{1}{1+a^2}. \tag{4.4}$$

Especially,

$$T(F,1,1,0)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{F_i}{1+F_i^2+(-1)^i}\Big)=\frac{3\pi}{4}. \tag{4.5}$$

In the last equation we can split the summation into odd and even indexes and use Lehmer's formula to derive

$$\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{F_{2i}}{2+F_{2i}^2}\Big)=\frac{\pi}{4}. \tag{4.6}$$

In a similar fashion, starting with $S(F,1,1,0)$, splitting the summation into odd and even parts, after simple algebraic modifications we obtain the following result:

$$\sum_{i=1}^{\infty}\frac{1}{F_{2i-1}}=\frac{3}{2}\sum_{i=1}^{\infty}\frac{F_{2i-1}}{F_{2i-1}^2+1}+\frac{1}{2}\sum_{i=1}^{\infty}\frac{F_{2i}^3}{(2+F_{2i}^2)^2+F_{2i}^2}-\frac{1}{4}. \tag{4.7}$$

Equation (4.7) is interesting since no simple expression for the reciprocal sum on the LHS exists. The only known evaluation is to the best of our knowledge the classical result of Landau in terms of Theta functions (see [4]):

$$\begin{aligned}\sum_{i=0}^{\infty}\frac{1}{F_{2i+1}} &= \frac{\sqrt{5}}{4}\Theta_2^2\Big(\frac{3-\sqrt{5}}{2}\Big)\\ &= \frac{\sqrt{5}}{4}\Big[\Theta_3^2\Big(\frac{\sqrt{5}-1}{2}\Big)-\Theta_3^2\Big(\frac{3-\sqrt{5}}{2}\Big)\Big],\end{aligned}$$

where

$$\Theta_2(q)=2q^{1/4}\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n})^2,\quad |q|<1, \tag{4.8}$$

and

$$\Theta_3(q)=\prod_{n=1}^{\infty}(1-q^{2n})(1+q^{2n-1})^2,\quad |q|<1. \tag{4.9}$$

The corresponding identities for Lucas numbers are

$$T(L,a,1,0)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{aL_i}{1+a^2(L_i^2+(-1)^{i-1}5)}\Big)=\pi-\tan^{-1}(a)-\tan^{-1}(2a), \tag{4.10}$$

and

$$S(L,a,1,0)=\sum_{i=1}^{\infty}\frac{(a^2(L_i^2+(-1)^{i-1}5)-1)L_i}{(a^2(L_i^2+(-1)^{i-1}5)+1)^2+a^2L_i^2}=\frac{1}{1+a^2}+\frac{2}{1+4a^2}. \tag{4.11}$$

Especially,

$$T(L,1,1,0)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{L_i}{1+L_i^2+(-1)^{i-1}5}\Big)=\frac{3\pi}{4}-\tan^{-1}(2). \tag{4.12}$$

From $S(L, 1/\sqrt{5}, 1, 0)$ we are able to derive

$$\sum_{i=1}^{\infty} \frac{1}{L_{2i}} = \frac{3}{2} \sum_{i=1}^{\infty} \frac{L_{2i}}{L_{2i}^2 + 5} + \frac{1}{2} \sum_{i=1}^{\infty} \frac{L_{2i-1}^3}{(10 + L_{2i-1}^2)^2 + 5L_{2i-1}^2} - \frac{7}{36}. \tag{4.13}$$

This equation gives another expression for the even indexed reciprocal Lucas sum and may be compared to the following Theta function expression found in [4]:

$$\sum_{i=1}^{\infty} \frac{1}{L_{2i}} = \Big[\Theta_3^2\Big(\frac{3-\sqrt{5}}{2}\Big) - 1\Big]/4. \tag{4.14}$$

Further examples of equations (4.1) and (4.2) are presented next.
Let $m$ and $n$ be even numbers. Then

$$T(F, a, 2j, 2p) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{aF_{2j} L_{2ji+2p}}{1 + a^2(F_{2ji+2p}^2 - F_{2j}^2)}\Big) = \pi - \tan^{-1}(aF_{2j+2p}) - \tan^{-1}(aF_{2p}), \tag{4.15}$$

and

$$S(F, a, 2j, 2p) = F_{2j} \sum_{i=1}^{\infty} \frac{(a^2(F_{2ji+2p}^2 - F_{2j}^2) - 1)L_{2ji+2p}}{(a^2(F_{2ji+2p}^2 - F_{2j}^2) + 1)^2 + a^2 F_{2j}^2 L_{2ji+2p}^2} = \frac{F_{2j+2p}}{1 + a^2 F_{2j+2p}^2} + \frac{F_{2p}}{1 + a^2 F_{2p}^2}. \tag{4.16}$$

Special cases are

$$T(F, 1, 2, 0) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{L_{2i}}{F_{2i}^2}\Big) = \frac{3\pi}{4}, \tag{4.17}$$

$$T(F, 1, 2, 2) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{L_{2i+2}}{F_{2i+2}^2}\Big) = \frac{3\pi}{4} - \tan^{-1}(3), \tag{4.18}$$

$$T(F, 2, 2, 0) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{2L_{2i}}{4F_{2i}^2 - 3}\Big) = \pi - \tan^{-1}(2), \tag{4.19}$$

$$T(F, 1/F_{2j}, 2j, 0) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{F_{2j}^2 L_{2ji}}{F_{2ji}^2}\Big) = \frac{3\pi}{4}, \tag{4.20}$$

$$T(F, 1/F_{2j}, 2j, 2j) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{F_{2j}^2 L_{2j(i+1)}}{F_{2j(i+1)}^2}\Big) = \frac{3\pi}{4} - \tan^{-1}\Big(\frac{F_{4j}}{F_{2j}}\Big). \tag{4.21}$$

These results complete some identities found by Adegoke in [2]. Focusing on $S(F, a, 2j, 2p)$ we present the following few special cases:

$$S(F, 1, 2, 0) = \sum_{i=1}^{\infty} \frac{(F_{2i}^2 - 2)L_{2i}}{F_{2i}^4 + L_{2i}^2} = \frac{1}{2}, \tag{4.22}$$

$$S(F, 1, 2, 2) = \sum_{i=1}^{\infty} \frac{(F_{2i+2}^2 - 2)L_{2i+2}}{F_{2i+2}^4 + L_{2i+2}^2} = \frac{4}{5}, \tag{4.23}$$

$$\frac{1}{F_{2j}^3} S(F, 1/F_{2j}, 2j, 0) = \sum_{i=1}^{\infty} \frac{(F_{2ji}^2 - 2F_{2j}^2)L_{2ji}}{F_{2ji}^4 + F_{2j}^4 L_{2ji}^2} = \frac{1}{2F_{2j}^2}. \tag{4.24}$$

We also have

$$T(L,a,2j,2p)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{5aF_{2j}F_{2ji+2p}}{1+a^2(L_{4ji+4p}+L_{4j})}\Big)=\pi-\tan^{-1}(aL_{2j+2p})-\tan^{-1}(aL_{2p}), \tag{4.25}$$

and

$$S(L,a,2j,2p)= \\ 5F_{2j}\sum_{i=1}^{\infty}\frac{(a^2(L_{4ji+4p}+L_{4j})-1)F_{2ji+2p}}{(a^2(L_{4ji+4p}+L_{4j})+1)^2+a^2 25F_{2j}^2F_{2ji+2p}^2}=\frac{L_{2j+2p}}{1+a^2L_{2j+2p}^2}+\frac{L_{2p}}{1+a^2L_{2p}^2}. \tag{4.26}$$

Special cases are

$$T(L,1,2,0)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{5F_{2i}}{8+L_{4i}}\Big)=\pi-\tan^{-1}(2)-\tan^{-1}(3), \tag{4.27}$$

$$T(L,1/2,2,0)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{10F_{2i}}{11+L_{4i}}\Big)=\frac{3\pi}{4}-\tan^{-1}(\frac{3}{2}), \tag{4.28}$$

$$T(L,1/3,2,2)=\sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{15F_{2i+2}}{16+L_{4i+4}}\Big)=\frac{3\pi}{4}-\tan^{-1}(\frac{7}{3}), \tag{4.29}$$

as well as

$$\frac{1}{5}S(L,1,2,0)=\sum_{i=1}^{\infty}\frac{(L_{4i}+6)F_{2i}}{(L_{4i}+8)^2+25F_{2i}^2}=\frac{7}{50}, \tag{4.30}$$

$$\frac{1}{5}S(L,1,2,2)=\sum_{i=1}^{\infty}\frac{(L_{4i+4}+6)F_{2i+2}}{(L_{4i+4}+8)^2+25F_{2i+2}^2}=\frac{11}{125}, \tag{4.31}$$

and

$$\begin{aligned}\frac{1}{5F_{2j}L_{4j}}S(L,1/\sqrt{L_{4j}},2j,2p) &= \sum_{i=1}^{\infty}\frac{L_{4ji+4p}F_{2ji+2p}}{(L_{4ji+4p}+2L_{4j})^2+25L_{4j}F_{2j}^2F_{2ji+2p}^2}\\ &= \frac{1}{5F_{2j}}\Big(\frac{L_{2j+2p}}{L_{4j}+L_{2j+2p}^2}+\frac{L_{2p}}{L_{4j}+L_{2p}^2}\Big).\end{aligned} \tag{4.32}$$

## 4.2 Case $k=1$, $g(i)=G_{mi}/G_{mi+n}, m,n\geq 1$ and $f(i)=ai, a>0$.

First, we observe that for $G=F$ and $G=L$, $\lim_{i\to\infty} g(i)=\alpha^{-n}$. Therefore,

$$\begin{aligned}T(G,a,m,n) &= \sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{a(G_{m(i+1)}G_{m(i-1)+n}-G_{m(i-1)}G_{m(i+1)+n})}{G_{m(i+1)+n}G_{m(i-1)+n}+a^2G_{m(i+1)}G_{m(i-1)}}\Big)\\ &= 2\tan^{-1}(a\alpha^{-n})-\tan^{-1}\Big(a\frac{G_0}{G_n}\Big)-\tan^{-1}\Big(a\frac{G_m}{G_{m+n}}\Big)\end{aligned} \tag{4.33}$$

and after differentiation w.r.t. $a$

$$S(G,a,m,n)=\sum_{i=1}^{\infty}\frac{x_1(x_2-a^2x_3)}{(x_2+a^2x_3)^2+a^2x_1^2}=\frac{2\alpha^n}{\alpha^{2n}+a^2}-\frac{G_0G_n}{G_n^2+a^2G_0^2}-\frac{G_mG_{m+n}}{G_{m+n}^2+a^2G_m^2}, \tag{4.34}$$

where we have set

$$\begin{aligned} x_1 &= G_{m(i+1)}G_{m(i-1)+n} - G_{m(i-1)}G_{m(i+1)+n} \\ x_2 &= G_{m(i+1)+n}G_{m(i-1)+n} \\ x_3 &= G_{m(i+1)}G_{m(i-1)}. \end{aligned}$$

For $G = F$ the above sums can be written more compactly according to

$$\begin{aligned} T(F,a,m,n) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{(-1)^{m(i+1)}aF_nF_{2m}}{F_{m(i+1)+n}F_{m(i-1)+n} + a^2(F_{mi}^2 - (-1)^{m(i-1)}F_m^2)}\Big) \\ &= 2\tan^{-1}(a\alpha^{-n}) - \tan^{-1}\Big(a\frac{F_m}{F_{m+n}}\Big), \end{aligned} \tag{4.35}$$

as well as

$$\begin{aligned} S(F,a,m,n) &= F_nF_{2m}\sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}(F_{m(i+1)+n}F_{m(i-1)+n} - a^2(F_{mi}^2 - (-1)^{m(i-1)}F_m^2))}{(F_{m(i+1)+n}F_{m(i-1)+n} + a^2(F_{mi}^2 - (-1)^{m(i-1)}F_m^2))^2 + a^2F_n^2F_{2m}^2} \\ &= \frac{2\alpha^n}{\alpha^{2n} + a^2} - \frac{F_mF_{m+n}}{F_{m+n}^2 + a^2F_m^2}. \end{aligned} \tag{4.36}$$

Special cases are

$$T(F,1,1,1) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{(-1)^{i+1}}{F_iF_{i+2} + F_i^2 + (-1)^i}\Big) = 2\tan^{-1}\Big(\frac{1}{\alpha}\Big) - \frac{\pi}{4}, \tag{4.37}$$

$$T(F,\alpha,1,1) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{\alpha(-1)^{i+1}}{F_iF_{i+2} + \alpha^2(F_i^2 + (-1)^i)}\Big) = \frac{\pi}{2} - \tan^{-1}(\alpha), \tag{4.38}$$

and

$$\begin{aligned} T(F,a,2j,n) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{aF_nF_{4j}}{F_{2j(i+1)+n}F_{2j(i-1)+n} + a^2(F_{2ji}^2 - F_{2j}^2)}\Big) \\ &= 2\tan^{-1}(a\alpha^{-n}) - \tan^{-1}\Big(a\frac{F_{2j}}{F_{2j+n}}\Big). \end{aligned} \tag{4.39}$$

Two explicit evaluations of $T(F,a,2j,n)$ are

$$T(F,1,2,1) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{3}{F_{2i-1}F_{2i+3} + F_{2i}^2 - 1}\Big) = 2\tan^{-1}\Big(\frac{1}{\alpha}\Big) - \tan^{-1}\Big(\frac{1}{2}\Big), \tag{4.40}$$

and

$$T(F,1,2,2) = \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{3}{F_{2i}F_{2i+4} + F_{2i}^2 - 1}\Big) = 2\tan^{-1}\Big(\frac{1}{\alpha^2}\Big) - \tan^{-1}\Big(\frac{1}{3}\Big). \tag{4.41}$$

From $S(F,a,m,n)$ we deduce the following beautiful identity concerning Fibonacci reciprocals

$$\frac{1}{F_nF_{2m}}S(F,0,m,n) = \sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}}{F_{mi-m+n}F_{mi+m+n}} = \frac{1}{F_nF_{2m}}\Big(\frac{2}{\alpha^n} - \frac{F_m}{F_{m+n}}\Big). \tag{4.42}$$

Especially for $m = n$,

$$\frac{1}{F_m F_{2m}} S(F, 0, m, m) = \sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}}{F_{mi} F_{m(i+2)}} = \frac{2}{F_m F_{2m} \alpha^m} - \frac{1}{F_{2m}^2}. \tag{4.43}$$

Observe that depending on $m$ the above reciprocal sums exhibit an alternating or a non-alternating structure. Explicit examples are

$$S(F, 0, 1, 1) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_i F_{i+2}} = \frac{2}{\alpha} - 1, \tag{4.44}$$

$$S(F, 0, 1, 2) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{i+1} F_{i+3}} = \frac{2}{\alpha^2} - \frac{1}{2}, \tag{4.45}$$

$$\frac{1}{3} S(F, 0, 2, 1) = \sum_{i=1}^{\infty} \frac{1}{F_{2i-1} F_{2i+3}} = \frac{2}{3\alpha} - \frac{1}{6}, \tag{4.46}$$

$$\frac{1}{3} S(F, 0, 2, 2) = \sum_{i=1}^{\infty} \frac{1}{F_{2i} F_{2i+4}} = \frac{2}{3\alpha^2} - \frac{1}{9}, \tag{4.47}$$

$$\frac{1}{8} S(F, 0, 3, 1) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{3i-2} F_{3i+4}} = \frac{1}{4\alpha} - \frac{1}{12}, \tag{4.48}$$

$$\frac{1}{8} S(F, 0, 3, 2) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{3i-1} F_{3i+5}} = \frac{1}{4\alpha^2} - \frac{1}{20}. \tag{4.49}$$

$$\frac{1}{16} S(F, 0, 3, 3) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{3i} F_{3i+6}} = \frac{1}{8\alpha^3} - \frac{1}{64}, \tag{4.50}$$

$$\frac{1}{63} S(F, 0, 4, 4) = \sum_{i=1}^{\infty} \frac{1}{F_{4i} F_{4i+8}} = \frac{2}{63\alpha^4} - \frac{1}{441}. \tag{4.51}$$

The evaluation of $S(F, 0, 1, 1)$ is known. This is a special case of a classical result of Brousseau

$$\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_i F_{i+n}} = \frac{1}{F_n} \Big( \frac{n}{\alpha} - \sum_{j=1}^{n} \frac{F_{j-1}}{F_j} \Big). \tag{4.52}$$

See for instance [11]. It is also worth to mention that $S(F, 0, 1, n)$ is closely related to a recently established identity in [6]:

$$S^*(n) := \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{F_{i+n} F_{i+n+1}} = \frac{1}{F_{n+1} \alpha^{n+1}}. \tag{4.53}$$

Using $\alpha^n = \alpha F_n + F_{n-1}$ we have

$$\frac{1}{F_n} S(F, 0, 1, n) = S^*(n-1) - S^*(n). \tag{4.54}$$

The other summations seem to be new.
Two other results that follow from (4.36) are:

$$S(F,a,1,1)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}(F_iF_{i+2}-a^2((-1)^i+F_i^2))}{(F_iF_{i+2}+a^2((-1)^i+F_i^2))^2+a^2}=\frac{2\alpha}{\alpha^2+a^2}-\frac{1}{1+a^2}, \tag{4.55}$$

and

$$\begin{aligned}\frac{1}{F_nF_{4j}}S(F,a,2j,n) &= \sum_{i=1}^{\infty}\frac{F_{2ji-2j+n}F_{2ji+2j+n}-a^2(F_{2ji}^2-F_{2j}^2)}{(F_{2ji-2j+n}F_{2ji+2j+n}+a^2(F_{2ji}^2-F_{2j}^2))^2+a^2F_n^2F_{4j}^2}\\ &= \frac{1}{F_nF_{4j}}\Big(\frac{2\alpha^n}{\alpha^{2n}+a^2}-\frac{F_{2j}F_{2j+n}}{F_{2j+n}^2+a^2F_{2j}^2}\Big). \end{aligned}\tag{4.56}$$

Especially,

$$S(F,1,1,1)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}F_iF_{i+1}+1}{(F_iF_{i+2}+(-1)^i+F_i^2))^2+1}=\frac{2\alpha}{\alpha+2}-\frac{1}{2}, \tag{4.57}$$

and

$$\frac{1}{3}S(F,1,2,2)=\sum_{i=1}^{\infty}\frac{F_{2i}F_{2i+4}-F_{2i}^2+1}{(F_{2i}F_{2i+4}+F_{2i}^2-1)^2+9}=\frac{11}{90}. \tag{4.58}$$

Now, we consider the case $G=L$. We present the following central results:

$$\begin{aligned}T(L,a,m,n) &= \sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{5(-1)^{m(i+1)+1}aF_nF_{2m}}{L_{m(i+1)+n}L_{m(i-1)+n}+a^2(L_{2mi}+(-1)^{m(i-1)}L_{2m})}\Big)\\ &= 2\tan^{-1}(a\alpha^{-n})-\tan^{-1}\Big(\frac{2a}{L_n}\Big)-\tan^{-1}\Big(a\frac{L_m}{L_{m+n}}\Big), \end{aligned}\tag{4.59}$$

as well as

$$\begin{aligned}&S(L,a,m,n)\\ &=5F_nF_{2m}\sum_{i=1}^{\infty}\frac{(-1)^{m(i+1)+1}(L_{m(i+1)+n}L_{m(i-1)+n}-a^2(L_{2mi}+(-1)^{m(i-1)}L_{2m}))}{(L_{m(i+1)+n}L_{m(i-1)+n}+a^2(L_{2mi}+(-1)^{m(i-1)}L_{2m}))^2+25a^2F_n^2F_{2m}^2}\\ &=\frac{2\alpha^n}{\alpha^{2n}+a^2}-\frac{2L_n}{L_n^2+4a^2}-\frac{L_mL_{m+n}}{L_{m+n}^2+a^2L_m^2}. \end{aligned}\tag{4.60}$$

Special cases of these results are stated next. We start with

$$\begin{aligned}T(L,a,1,1) &= \sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{5a(-1)^i}{L_iL_{i+2}+a^2((-1)^{i-1}5+L_i^2)}\Big)\\ &= 2\tan^{-1}\Big(\frac{a}{\alpha}\Big)-\tan^{-1}(2a)-\tan^{-1}\Big(\frac{a}{3}\Big). \end{aligned}\tag{4.61}$$

Hence,

$$\begin{aligned}T(L,3,1,1) &= \sum_{i=1}^{\infty}\tan^{-1}\Big(\frac{15(-1)^i}{L_iL_{i+2}+9L_i^2+45(-1)^{i-1}}\Big)\\ &= 2\tan^{-1}\Big(\frac{3}{\alpha}\Big)-\tan^{-1}(6)-\frac{\pi}{4}, \end{aligned}\tag{4.62}$$

or

$$\begin{aligned} T(L,\alpha,1,1) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{5\alpha(-1)^i}{L_iL_{i+2}+\alpha^2(L_i^2+5(-1)^{i-1})}\Big) \\ &= \frac{\pi}{2} - \tan^{-1}(2\alpha) - \tan^{-1}\Big(\frac{\alpha}{3}\Big). \end{aligned} \tag{4.63}$$

If $m$ is even ($m = 2j$), then

$$\begin{aligned} (-1)\cdot T(L,a,2j,n) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{5aF_nF_{4j}}{L_{2j(i+1)+n}L_{2j(i-1)+n}+a^2(L_{4ji}+L_{4j})}\Big) \\ &= \tan^{-1}\Big(\frac{2a}{L_n}\Big) + \tan^{-1}\Big(a\frac{L_{2j}}{L_{2j+n}}\Big) - 2\tan^{-1}(a\alpha^{-n}). \end{aligned} \tag{4.64}$$

This yields for instance

$$\begin{aligned} (-1)\cdot T(L,1,2,2) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{15}{L_{2i}L_{2i+4}+L_{4i}+7}\Big) \\ &= \tan^{-1}\Big(\frac{2}{3}\Big) + \tan^{-1}\Big(\frac{3}{7}\Big) - 2\tan^{-1}\Big(\frac{1}{\alpha^2}\Big). \end{aligned} \tag{4.65}$$

From $S(L,a,m,n)$ we deduce the following striking identity for Lucas reciprocals

$$\frac{-1}{5F_nF_{2m}}S(L,0,m,n) = \sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}}{L_{mi-m+n}L_{mi+m+n}} = \frac{1}{5F_nF_{2m}}\Big(\frac{2}{L_n} + \frac{L_m}{L_{m+n}} - \frac{2}{\alpha^n}\Big). \tag{4.66}$$

Especially for $m = 1$,

$$\frac{-1}{5F_n}S(F,0,1,n) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{L_{i+n-1}L_{i+n+1}} = \frac{L_{n+3}}{5F_nL_nL_{n+1}} - \frac{2}{5F_n\alpha^n}, \tag{4.67}$$

and for $m = n$

$$\frac{-1}{5F_mF_{2m}}S(L,0,m,m) = \sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}}{L_{mi}L_{m(i+2)}} = \frac{1}{5F_mF_{2m}}\Big(\frac{2}{L_m} + \frac{L_m}{L_{2m}} - \frac{2}{\alpha^m}\Big). \tag{4.68}$$

These sums also exhibit an alternating or a non-alternating structure. The first explicit examples are

$$\frac{-1}{5}S(L,0,1,1) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{L_iL_{i+2}} = \frac{7}{15} - \frac{2}{5\alpha}, \tag{4.69}$$

$$\frac{-1}{5}S(L,0,1,2) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{L_{i+1}L_{i+3}} = \frac{11}{60} - \frac{2}{5\alpha^2}, \tag{4.70}$$

$$\frac{-1}{15}S(L,0,2,1) = \sum_{i=1}^{\infty} \frac{1}{L_{2i-1}L_{2i+3}} = \frac{11}{60} - \frac{2}{15\alpha}, \tag{4.71}$$

$$\frac{-1}{15}S(L,0,2,2) = \sum_{i=1}^{\infty} \frac{1}{L_{2i}L_{2i+4}} = \frac{23}{315} - \frac{2}{15\alpha^2}, \tag{4.72}$$

$$\frac{-1}{40}S(L,0,3,1) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{L_{3i-2}L_{3i+4}} = \frac{9}{140} - \frac{1}{20\alpha}, \tag{4.73}$$

$$\frac{-1}{80}S(L,0,3,3) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{L_{3i}L_{3i+6}} = \frac{13}{1440} - \frac{1}{40\alpha^3}. \tag{4.74}$$

Two other identities that follow from the central result for $S(L,a,m,n)$ are

$$S(L,a,1,1) = \sum_{i=1}^{\infty} \frac{(-1)^i 5(L_i^2 + L_iL_{i+1} - a^2((-1)^{i-1}5 + L_i^2))}{(L_i^2 + L_iL_{i+1} + a^2((-1)^{i-1}5 + L_i^2))^2 + 25a^2} = \frac{2\alpha}{\alpha^2 + a^2} - \frac{2}{1+4a^2} - \frac{3}{9+a^2} \tag{4.75}$$

and

$$\begin{aligned} \frac{-1}{5F_nF_{4j}}S(L,a,2j,n) &= \sum_{i=1}^{\infty} \frac{L_{2j(i+1)+n}L_{2j(i-1)+n} - a^2(L_{4ji} + L_{4j})}{(L_{2j(i+1)+n}L_{2j(i-1)+n} + a^2(L_{4ji} + L_{4j}))^2 + 25a^2F_n^2F_{4j}^2} \\ &= \frac{1}{5F_nF_{4j}}\Big(\frac{2L_n}{L_n^2 + 4a^2} + \frac{L_{2j}L_{2j+n}}{L_{2j+n}^2 + a^2L_{2j}^2} - \frac{2\alpha^n}{\alpha^{2n} + a^2}\Big). \end{aligned} \tag{4.76}$$

Especially,

$$\frac{1}{5}S(L,1,1,1) = \sum_{i=1}^{\infty} \frac{(-1)^i L_iL_{i+1} + 5}{(2L_i^2 + L_iL_{i+1} + 5(-1)^{i-1})^2 + 25} = \frac{2}{5}\frac{\alpha}{\alpha+2} - \frac{7}{50}, \tag{4.77}$$

$$\frac{-1}{15}S(L,1,2,1) = \sum_{i=1}^{\infty} \frac{L_{2i-1}L_{2i+3} - L_{4i} - 7}{(L_{2i-1}L_{2i+3} + L_{4i} + 7)^2 + 225} = \frac{22}{375} - \frac{2}{15}\frac{\alpha}{\alpha+2}, \tag{4.78}$$

and

$$\frac{-1}{15}S(L,1,2,2) = \sum_{i=1}^{\infty} \frac{L_{2i}L_{2i+4} - L_{4i} - 7}{(L_{2i}L_{2i+4} + L_{4i} + 7)^2 + 225} = \frac{7}{669}. \tag{4.79}$$

Our final application deals with a quotient of a Fibonacci and a Lucas number.

## 4.3 Case $k = 1$, $g(i) = F_{mi}/L_{mi}, m \geq 1$ and $f(i) = ai, a > 0$.

Using $\lim_{i\to\infty} g(i) = 1/(\alpha - \beta) = 1/\sqrt{5}$ the Theorem gives the following identities:

$$\begin{aligned} T(a,m) &= \sum_{i=1}^{\infty} \tan^{-1}\Big(\frac{2(-1)^{m(i+1)}F_{2m}a}{L_{m(i+1)}L_{m(i-1)} + a^2F_{m(i+1)}F_{m(i-1)}}\Big) \\ &= 2\tan^{-1}\Big(\frac{a}{\sqrt{5}}\Big) - \tan^{-1}\Big(a\frac{F_m}{L_m}\Big) \end{aligned} \tag{4.80}$$

and after differentiation w.r.t. $a$

$$\begin{aligned} S(a,m) &= 2F_{2m}\sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}(L_{m(i+1)}L_{m(i-1)} - a^2F_{m(i+1)}F_{m(i-1)})}{(L_{m(i+1)}L_{m(i-1)} + a^2F_{m(i+1)}F_{m(i-1)})^2 + 4a^2F_{2m}^2} \\ &= \frac{2\sqrt{5}}{5+a^2} - \frac{F_{2m}}{L_m^2 + a^2F_m^2}. \end{aligned} \tag{4.81}$$

Especially for $a = 0$

$$\frac{1}{2F_{2m}}S(0,m) = \sum_{i=1}^{\infty} \frac{(-1)^{m(i+1)}}{L_{m(i+1)}L_{m(i-1)}} = \frac{\sqrt{5}}{5F_{2m}} - \frac{1}{2L_m^2}. \tag{4.82}$$

Our final explicit examples take the form

$$T(\sqrt{5},1)=\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{2\sqrt{5}(-1)^{i+1}}{L_{i+1}L_{i-1}+5F_{i+1}F_{i-1}}\right)=\frac{\pi}{2}-\tan^{-1}(\sqrt{5}), \tag{4.83}$$

$$T(3,2)=\sum_{i=1}^{\infty}\tan^{-1}\left(\frac{18}{L_{2i+2}L_{2i-2}+9F_{2i+2}F_{2i-2}}\right)=2\tan^{-1}\left(\frac{3}{\sqrt{5}}\right)-\frac{\pi}{4}, \tag{4.84}$$

$$\frac{1}{6}S(0,2)=\sum_{i=1}^{\infty}\frac{1}{L_{2i-2}L_{2i+2}}=\frac{\sqrt{5}}{15}-\frac{1}{18}, \tag{4.85}$$

$$\frac{1}{16}S(0,3)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{L_{3i-3}L_{3i+3}}=\frac{\sqrt{5}}{40}-\frac{1}{32}, \tag{4.86}$$

and

$$\frac{1}{2}S(1,1)=\sum_{i=1}^{\infty}\frac{(-1)^{i+1}(L_{i+1}L_{i-1}-F_{i+1}F_{i-1})}{(L_{i+1}L_{i-1}+F_{i+1}F_{i-1})^2+4}=\frac{\sqrt{5}}{6}-\frac{1}{4}. \tag{4.87}$$

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