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Inequalities for φ and ψ functions (III)

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Abstract: A new arithmetic function is defined and some of its properties are studied.
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1 Introduction

In a series of papers (see, eg., [1, 2, 3, 4, 5], the author studied some inequalities related to the well-known φ, σ and ψ arithmetic functions, that are defined for the natural number

$$n = \prod_{i=1}^{k} p_i^{\alpha_i},$$

where $k, \alpha_1, ..., \alpha_k, k \geq 1$ are natural numbers and $p_1, ..., p_k$ are different primes, by:

$$\varphi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i - 1),\ \varphi(1) = 1,$$

$$\sigma(n) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i + 1} - 1}{p_i - 1},\ \sigma(1) = 1,$$

$$\psi(n) = \prod_{i=1}^{k} p_i^{\alpha_i - 1}(p_i + 1),\ \psi(1) = 1$$

(see, e.g. [6, 7]).

Here we use also the well known function

$$\omega(n) = k,$$

for the above natural numbers n and k.

For example. in [4] it is proved that if m is an odd number and a is a natural number, then (a) if $n = m$, or $n = 2^a m$, where $a \geq 4$, then,

$$\varphi(n) > 2^{\omega(n)-1}.\sqrt{n};$$

(b) if $n = 2m$, then,

$$\varphi(n) > 2^{\omega(n)-2-\frac{1}{2}}.\sqrt{n};$$

(c) if $n = 2^a m$, where $2 \leq a \leq 3$, then:

$$\varphi(n) > 2^{\omega(n)-2}.\sqrt{n}.$$

Here, we formulate and prove three new inequalities, related to φ and ψ functions.

2 Main results

Theorem 1. Let $n \geq 3$ be an odd number. Then

$$\varphi(n) > \frac{n}{2^{\frac{\omega(n)}{2}}}. \tag{1}$$

Proof: Let n be a prime number. Then $\omega(n) = 1$ and obviously

$$\varphi(n) = n - 1 > \frac{n}{\sqrt{2}} = \frac{n}{2^{\frac{\omega(n)}{2}}}. \tag{2}$$

Let us assume that (1) is valid for some odd number n and let for the prime number $p \geq 3$: $p \notin \underline{set}(n)$. Then by induction and from (2)

$$\varphi(np) = \varphi(n)(p-1) > \frac{n}{2^{\frac{\omega(n)}{2}}}(p-1) > \frac{n}{2^{\frac{\omega(n)}{2}}}.\frac{p}{\sqrt{2}} = \frac{np}{2^{\frac{\omega(n)+1}{2}}} = \frac{np}{2^{\frac{\omega(np)}{2}}}.$$

Let for the prime number $p \geq 3$: $p \in \underline{set}(n)$. Then $\omega(np) = \omega(n)$ and by induction

$$\varphi(np) = \varphi(n)p > \frac{n}{2^{\frac{\omega(n)}{2}}}.p = \frac{np}{2^{\frac{\omega(np)}{2}}}$$

that proves the Theorem 1. □

Corollary 1. If $n \geq 2$ is an even number, then

$$\varphi(n) > \frac{n}{2^{\frac{\omega(n)+1}{2}}}.$$

Theorem 2. For each natural number $n \geq 2$:

$$\varphi(n)\psi(n) \leq n^2 - \omega(n). \tag{3}$$

Proof: Let n be a prime number. Then $\omega(n) = 1$ and

$$n^2 - 1 = (n-1)(n+1) = \varphi(n)\psi(n).$$

Let us assume that (3) is valid for some natural number n and let for the prime number p: $p \notin \underline{set}(n)$. Then $\omega(np) = \omega(n) + 1$ and by induction we obtain:

$$(np)^2 - \omega(np) - \varphi(np)\psi(np) = n^2p^2 - \omega(n) - 1 - \varphi(n)\psi(n)(p^2 - 1)$$

$$\geq n^2p^2 - \omega(n) - 1 - (n^2 - \omega(n))(p^2 - 1)$$

$$= -\omega(n) - 1 + \omega(n)p^2 + n^2 - \omega(n) > 0.$$

Let for the prime number p: $p \in \underline{set}(n)$. Then $\omega(np) = \omega(n)$ and by induction

$$(np)^2 - \omega(np) - \varphi(np)\psi(np) = n^2p^2 - \omega(n) - \varphi(n)\psi(n)p^2$$

$$\geq n^2p^2 - \omega(n) - (n^2 - \omega(n))p^2 = \omega(n)(p^2 - 1)) > 0.$$

So, the Theorem 2 is proven. □

Let for the natural number n with the canonical form from the Introduction:

$$\underline{mult}(n) = p_1p_2 \dots p_k.$$

In the general case, the following inequality is stronger than the previous one.

Theorem 3. For each natural number $n \geq 2$:

$$\varphi(n)\psi(n) \leq n^2 - (\omega(n) - 1)\underline{mult}(n). \tag{4}$$

Proof: Let n be a prime number. Then $\omega(n) = 1$ and

$$n^2 - (\omega(n) - 1)\underline{mult}(n) = n^2 - 0.n > (n-1)(n+1) = \varphi(n)\psi(n).$$

Let $n = pq$ for two different prime numbers. Then

$$(pq)^2 - (\omega(pq) - 1)\underline{mult}(pq) - \varphi(pq)\psi(pq)$$

$$= p^2q^2 - pq - (p-1)(p+1)(q-1)(q+1)$$

$$= p^2q^2 - pq - p^2q^2 + p^2 + q^2 - 1$$

$$> p^2 + q^2 - pq - 1 > 0.$$

Let $n = p^2$. Then

$$(p^2)^2 - (\omega(p^2) - 1)\underline{mult}(p^2) - \varphi(p^2)\psi(p^2)$$

$$p^4 - p^2(p^2-1) > 0.$$

Therefore the Theorem is valid for the natural numbers n, for which $\omega(n) = 2$.

Let us assume that (4) is valid for some natural number n so that $\omega(n) \geq 2$ and let for the prime number p: $p \notin \underline{set}(n)$. Then $\omega(np) = \omega(n)+1$, $\underline{mult}(np) = \underline{mult}(n)p$ and by induction we obtain:

$$\begin{aligned}
&(np)^2 - (\omega(np)-1)\underline{mult}(np) - \varphi(np)\psi(np)\\
&= n^2p^2 - \omega(n)\underline{mult}(n)p - \varphi(n)\psi(n)(p^2-1)\\
&\geq n^2p^2 - \omega(n)\underline{mult}(n)p - (n^2 - (\omega(n)-1)\underline{mult}(n))(p^2-1)\\
&= n^2p^2 - \omega(n)\underline{mult}(n)p - n^2p^2 + n^2 + (\omega(n)-1)\underline{mult}(n)(p^2-1)\\
&= n^2 - \omega(n)\underline{mult}(n)p + \omega(n)\underline{mult}(n)(p^2-1) - \underline{mult}(n)(p^2-1)\\
&= n^2 + \omega(n)\underline{mult}(n)(p^2-p-1) - \underline{mult}(n)(p^2-1)\\
&\geq \underline{mult}(n)^2 + \underline{mult}(n)(\omega(n)p^2 - \omega(n)p - \omega(n) - p^2 + 1)\\
&= \underline{mult}(n)(\omega(n)p^2 - \omega(n)p - \omega(n) - p^2 + 1 + \underline{mult}(n))
\end{aligned}$$

(from $\omega(n) \geq 2$ it follows that $n \geq 6$ and hence $\underline{mult}(n) \geq 6$)

$$\geq \underline{mult}(n)(2p^2 - 2p - 2 - p^2 + 1 + 6)$$

$$= \underline{mult}(n)(p^2 - 2p + 5) > 0.$$

Let for the prime number p: $p \in \underline{set}(n)$. Then $\omega(np) = \omega(n)$, $\underline{mult}(np) = \underline{mult}(n)$ and by induction

$$\begin{aligned}
&(np)^2 - (\omega(np)-1)\underline{mult}(np) - \varphi(np)\psi(np)\\
&= n^2p^2 - (\omega(n)-1)\underline{mult}(n) - \varphi(n)\psi(n)p^2\\
&\geq n^2p^2 - (\omega(n)-1)\underline{mult}(n) - (n^2 - (\omega(n)-1)\underline{mult}(n))p^2\\
&= (\omega(n)-1)\underline{mult}(n)p^2 - (\omega(n)-1)\underline{mult}(n)\\
&= (\omega(n)-1)\underline{mult}(n)(p^2-1) > 0.
\end{aligned}$$

So, the Theorem 3 is proven. □

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