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New theorems on explicit evaluation of a parameter of Ramanujan's $\chi(q)$ function

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Abstract: We prove many new theorems for the explicit values of the parameter $I_{k,n}$, for positive real number n and k, involving Ramanujan's $\chi(q)$ function by establishing its connection with some other parameters of Ramanujan's theta-functions. As applications of the parameter $I_{k,n}$ we offer formulas for the explicit values of Ramanujan's cubic continued fraction and $\chi(e^{-\pi n})$.
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1 Introduction

For any complex number a, define

$$(a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n), \quad |q|<1. \tag{1.1}$$

Ramanujan's general theta-function $f(a,b)$ is defined by

$$f(a,b)=\sum_{k=-\infty}^{\infty} a^{k(k+1)/2}b^{k(k-1)/2}, \tag{1.2}$$

where $|ab| < 1$. If we set $a = qe^{2iz}$, $b = qe^{-2iz}$, and $q = e^{\pi i\tau}$, where z is complex and $\mathrm{Im}(\tau) > 0$, then $f(a,b) = \vartheta_3(z,\tau)$, where $\vartheta_3(z,\tau)$ denotes one of the classical theta-functions in its standard notation [15, p. 464].

Three special cases of $f(a,b)$ are the theta-functions ϕ, ψ and f [5, p.36, Entry 22(i)-(iii)] defined by

$$\phi(q) := f(q,q) = 1 + 2\sum_{k=1}^{\infty} q^{k^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty(-q^2;q^2)_\infty}, \tag{1.3}$$

$$\psi(q) := f(q,q^3)\sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{1.4}$$

$$f(-q) := f(-q,-q^2)\sum_{k=0}^{\infty}(-1)^k q^{k(3k-1)/2} + \sum_{k=1}^{\infty}(-1)^k q^{k(3k+1)/2} = (q;q)_\infty. \tag{1.5}$$

If $q = e^{2\pi iz}$ with $\mathrm{Im}(z) > 0$, then $f(-q) = q^{-1/24}\eta(z)$, where $\eta(z)$ denotes the classical Dedekind eta-function.

Ramanujan further defined the $\chi(q)$ function [5, p.36, Entry 22(iv)] as

$$\chi(q) := (-q;q^2)_\infty. \tag{1.6}$$

The functions ϕ, ψ, f, and χ are connected by

$$f(q) = \frac{f^3(-q^2)}{f(-q)f(-q^4)}, \qquad \phi(q) = \frac{f^5(-q^2)}{f^2(-q)f^2(-q^4)}, \qquad \phi(-q) = \frac{f^2(-q)}{f(-q^2)}, \tag{1.7}$$

$$\psi(q) = \frac{f^2(-q^2)}{f(-q)}, \qquad \psi(-q) = \frac{f(-q)f(-q^4)}{f(-q^2)}, \qquad \chi(q) = \frac{f(q)}{f(-q^2)} = \frac{f^2(-q^2)}{f(-q)f(-q^4)}. \tag{1.8}$$

and follows easily from [5, p. 39-40, Entries 24 & 25] or see [3]. In his first notebook [11, Vol. I, p. 248] Ramanujan recorded many explicit values of $\phi(\pm q), \psi(\pm q), f(\pm q)$ and $\chi(\pm q)$. All these values were proved by Berndt [7, p. 325] and Berndt and Chan [8] by using values of Ramanujan's class invariants G_n and g_n, which are defined, respectively, by

$$G_n = 2^{-1/4}q^{-1/24}\chi(q) \quad \text{and} \quad g_n = 2^{-1/4}q^{-1/24}\chi(-q), \qquad q := e^{-\pi\sqrt{n}} \tag{1.9}$$

An account of this can be found in [7]. Yi [17] evaluated many new values of $f(\pm q)$ by considering the parameters $r_{k,n}$ and $r'_{k,n}$, where k and n are positive real numbers, defined, respectively, as

$$r_{k,n} := \frac{f(-q)}{k^{1/4}q^{(k-1)/24}f(-q^k)}, \qquad q = e^{-2\pi\sqrt{n/k}}, \tag{1.10}$$

$$r'_{k,n} := \frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.11}$$

Yi [16] also introduced the parameters $h_{k,n}$ and $h'_{k,n}$ to find explicit values of theta-function $\phi(\pm q)$ which are defined respectively, as

$$h_{k,n} := \frac{\phi(q)}{k^{1/4}\phi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.12}$$

$$h'_{k,n} := \frac{\phi(-q)}{k^{1/4}\phi(-q^k)}, \qquad q = e^{-2\pi\sqrt{n/k}}. \tag{1.13}$$

Baruah and Saikia [4] found explicit values of the theta-function $\psi(\pm q)$ by finding explicit values of the parameters $g_{k,n}$ and $g'_{k,n}$ which are defined, respectively, by

$$g_{k,n} := \frac{\psi(-q)}{k^{1/4}q^{(k-1)/8}\psi(-q^k)} \qquad q = e^{-\pi\sqrt{n/k}}, \tag{1.14}$$

$$g'_{k,n} := \frac{\psi(q)}{k^{1/4}q^{(k-1)/8}\psi(q^k)}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.15}$$

Saikia [12] introduced another parameter $A_{k,n}$ involving theta-functions $\phi(q)$ and $\psi(q)$ to find explicit values of $\psi(q)$ and is defined by

$$A_{k,n} = \frac{\phi(-q)}{2\,k^{1/4}q^{k/4}\psi(q^{2k})}, \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.16}$$

In sequel, Saikia [13] defined the parameter $I_{k,n}$ involving Ramanujan's $\chi(q)$ function as

$$I_{k,n} := \frac{\chi(q)}{q^{(-k+1)/24}\chi(q^k)}; \qquad q = e^{-\pi\sqrt{n/k}}. \tag{1.17}$$

Saikia [13] proved many properties of $I_{k,n}$ and evaluated its explicit values to prove some new and known values of Ramanujan's class invariant G_n.

This paper is mainly concerned with the parameter $I_{k,n}$. We prove some new general theorems for the explicit values of the parameter $I_{k,n}$ by establishing its connection with the parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, $g_{k,n}$ defined above and the Ramanujan's remarkable product of theta-functions $a_{k,n}$ defined by

$$a_{k,n} = k\ e^{-(k-1)/4\sqrt{n/k}}\frac{\psi^2(e^{-\pi\sqrt{kn}})\phi^2(-e^{-2\pi\sqrt{kn}})}{\psi^2(e^{-\pi\sqrt{n/k}})\phi^2(-e^{-2\pi\sqrt{n/k}})}. \tag{1.18}$$

The parameter $a_{k,n}$ along with 18 particular values is recorded by Ramanujan on page 338 and 339 of his first notebook [11] (also see [7, p. 337, (8.1)]). All 18 values are established by Berndt, Chan, and Zhang [10]. Saikia [14] proved many properties of $a_{k,n}$.

The layout of the paper is as follows: Section 2 is devoted to record some preliminary results. In Section 3, we prove general theorems for explicit values of $I_{k,n}$ by establishing its connections with the parameters $r_{k,n}$, $r'_{k,n}$, $h_{k,n}$, $g_{k,n}$ and $a_{k,n}$ and give examples. In section 4, we use the parameter $I_{k,n}$ to prove a general theorems for the explicit evaluation of Ramanujan's cubic continued fraction $G(q)$, defined by

$$G(q) := \frac{q^{1/3}}{1}_{+}\frac{q+q^2}{1}_{+}\frac{q^2+q^4}{1}_{+}\frac{q^3+q^6}{1}_{+\dots}, \qquad |q|<1. \tag{1.19}$$

2 Preliminaries

Using transformation formulas [5, p. 43, Entry 27] for theta-functions it is established in [4, 13, 14, 16, 17] that, for $\Delta := r, r', h, g, I$, and a

$$\Delta_{m,n} = \Delta_{n,m}. \tag{2.1}$$

Lemma 2.1. *[14, p. 138 , Theorem 3.9]We have*

$$a_{k,n} = (r_{k,n} r'_{k,n})^{-2}.$$

Lemma 2.2. *[1, p. 10]; [2, p. 2157] If* $P = \dfrac{\psi(-q)}{q^{1/4}\psi(-q^3)}$ *and* $Q = \dfrac{\phi(q)}{\phi(q^3)}$, *then*

$$Q^4 + P^4 Q^4 = 9 + P^4. \tag{2.2}$$

Lemma 2.3. *[1, p. 10]; [2, p. 2156]). If* $P = \dfrac{\psi(-q)}{q^{1/2}\psi(-q^5)}$ *and* $Q = \dfrac{\phi(q)}{\phi(q^5)}$, *then*

$$Q^2 + P^2 Q^2 = 5 + P^2.$$

Lemma 2.4. *[1, p. 10]; [2, p. 2156]. If* $P = \dfrac{\psi(-q)}{q\psi(-q^9)}$ *and* $Q = \dfrac{\phi(q)}{\phi(q^9)}$, *then*

$$Q + PQ = 3 + P.$$

Lemma 2.5. *[6, p. 207, (53.3)] If* $R = \dfrac{f^2(q)}{q^{1/6} f^2(q^3)}$ *and* $Q = \dfrac{f^2(-q^2)}{q^{1/3} f^2(-q^6)}$, *then*

$$(RQ) - \frac{9}{RQ} = \left(\frac{Q}{R}\right)^3 - \left(\frac{R}{Q}\right)^3.$$

Lemma 2.6. *[6, p. 207, (53.3)] If* $R = \dfrac{f(q)}{q^{1/6} f(q^5)}$ *and* $Q = \dfrac{f(-q^2)}{q^{1/3} f(-q^{10})}$, *then*

$$(RQ) - \frac{5}{RQ} = \left(\frac{Q}{R}\right)^3 - \left(\frac{R}{Q}\right)^3.$$

Lemma 2.7. *[6, p. 211, (56.7)] If* $R = \dfrac{f(q)}{q^{1/3} f(q^9)}$ *and* $Q = \dfrac{f(-q^2)}{q^{2/3} f(-q^{18})}$, *then*

$$(RQ)^2 - Q^3 = 3QR - R^3.$$

Lemma 2.8. *[6, p. 210, (55.8)] If* $R = \dfrac{f^2(q)}{q^{1/2} f^2(q^7)}$ *and* $Q = \dfrac{f^2(-q^2)}{q f^2(-q^{14})}$, *then*

$$(RQ) - \frac{49}{RQ} = \left(\frac{Q}{R}\right)^3 - \left(\frac{R}{Q}\right)^3 + 8\left(\frac{Q}{R} - \frac{R}{Q}\right).$$

Lemma 2.9. *[6, p. 212, (57.6)] If* $R = \dfrac{f(q)}{q^{1/2} f(q^{13})}$ *and* $Q = \dfrac{f(-q^2)}{q f(-q^{26})}$, *then*

$$(RQ) - \frac{13}{RQ} = \left(\frac{Q}{R}\right)^3 - \left(\frac{R}{Q}\right)^3 + 4\left(\frac{Q}{R} - \frac{R}{Q}\right).$$

Lemma 2.10. *[6, p. 213, (58.7)] If* $R = \dfrac{f(q)}{q f(q^{25})}$ *and* $Q = \dfrac{f(-q^2)}{q^2 f(-q^{50})}$, *the*

$$(RQ) + \frac{25}{RQ} = \left(\frac{Q}{R}\right)^3 + \left(\frac{R}{Q}\right)^3 + 4\left(\frac{Q}{R}\right)^2 + 4\left(\frac{R}{Q}\right)^2.$$

Lemma 2.11. *[5, p. 345, Entry 1(i)] If $G(q)$ is as defined in* (1.19), *then*

$$1+\frac{1}{G^3(q)}=\frac{\psi^4(q)}{q\,\psi^4(q^3)} \tag{2.3}$$

Lemma 2.12. *[5, p. 347] If $G(q)$ is as defined in* (1.19), *then*

$$1-8G^3(q)=\frac{\phi^4(-q)}{\phi^4(-q^3)}.$$

3 General theorems for explicit values of $I_{k,n}$

Theorem 3.1. *We have*

$$I_{k,n}=\frac{G_{n/k}}{G_{kn}}=\frac{r'_{k,n}}{r_{k,n}}.$$

Proof. First equality follows directly from the definitions of $I_{k,n}$ and G_n. Using (1.8), we rewrite $I_{k,n}$ as

$$I_{k,n}=\frac{f(q)}{k^{1/4}q^{(k-1)/24}f(q^k)}\frac{k^{1/4}q^{(k-1)/12}f(-q^{2k})}{f(-q^2)};\qquad q=e^{-\pi\sqrt{n/k}}. \tag{3.1}$$

Employing the definition of $r_{k,n}$ and $r'_{k,n}$ in (3.1), we arrive at the second equality. □

Remark 3.2. (i) *From Theorem 3.1 it is obvious that if we know the explicit values of $r'_{k,n}$ and $r_{k,n}$ then $I_{k,n}$ can be evaluated for the corresponding values of k and n. Yi [17] evaluated several values of $r'_{k,n}$ and $r_{k,n}$ some of which are also listed in the [4, Theorems 3.1 and 3.2]. For example, employing the values $r_{3,35}=\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right)/2$ and $r'_{3,25}=(1+\sqrt{5})/2$ in Theorem 3.1, we evaluate*

$$I_{3,25}=\left(1+\sqrt{5}\right)/\left(1+\sqrt[3]{10}+\sqrt{5+2\sqrt[3]{10}+\sqrt[3]{10^2}}\right).$$

(ii) *From Theorem 3.1 we can also evaluate the values of Ramanujan's class invariant G_n if we know the value of $I_{k,n}$. For example, noting $r_{5,5}=(25+10\sqrt{5})^{1/6}=\sqrt{(5+\sqrt{5})/2}$ and $r'_{5,5}=\sqrt{(5-\sqrt{5})/2}$, we evaluate $I_{5,5}=(\sqrt{5}-1)/2$. Then, $G_{25}=1/I_{5,5}=(\sqrt{5}+1)/2$, where we used the well known $G_1=1$. For evaluation of G_n from $I_{k,n}$ when $k\neq n$ see [13].*

Theorem 3.3. *We have*

$$I^2_{k,n}=\frac{h_{k,n}}{r_{k,n}}.$$

Proof. Employing (1.7) and (1.8) in the definition of $I_{k,n}$, we obtain

$$\begin{aligned} I^2_{k,n}&=\frac{f^5(-q^2)}{q^{(-k+1)/12}f^2(-q)f^2(-q^4)}\frac{f^2(-q^k)f^2(-q^{4k})}{f^5(-q^{2k})},\frac{f(-q^{2k})}{f(-q^2)};\qquad q=e^{-\pi\sqrt{n/k}}\\ &=\frac{\phi(q)}{k^{1/4}\phi(q^k)}\frac{q^{(k-1)/12}k^{1/4}f(-q^{2k})}{f(-q^2)}. \end{aligned} \tag{3.2}$$

Employing the definitions of $r_{k,n}$ and $h_{k,n}$ in (3.2), we complete the proof. □

Remark 3.4. *From Theorem 3.3 it is easily seen that that if we know the explicit values of* $r_{k,n}$ *and* $h_{k,n}$ *then the explicit value of* $I_{k,n}$ *can be determined. Several values of* $h_{k,n}$ *are evaluated in [16]. For example, noting* $r_{3,5} = \left((\sqrt{5}+1)/2\right)^{5/6}$ *and* $h_{3,5} = \left((\sqrt{5}-1)/2\right)^{1/2}$*, we evaluate* $I_{3,5}^2 = \left((\sqrt{5}-1)/2\right)^{4/3}$.

Theorem 3.5. *We have*

$$I_{k,n} = \frac{r_{k,n}}{g_{k,n}}.$$

Proof. Employing (1.7) and (1.8) in the definition of $I_{k,n}$, we obtain

$$\begin{aligned} I_{k,n} &= \frac{1}{q^{(-k+1)/24}} \frac{f(-q^2)}{f(-q)f(-q^4)} \frac{f(-q^{2k})}{f(-q^k)f(-q^{4k})} \frac{f(-q^2)}{f(-q^{2k})}; \qquad q = e^{-\pi\sqrt{n/k}} \\ &= \frac{k^{1/4}q^{(k-1)/8}\psi(-q^k)}{\psi(-q)} \frac{f(-q^2)}{k^{1/4}q^{(k-1)/12}f(-q^{2k})}. \end{aligned} \tag{3.3}$$

Employing the definition of $r_{k,n}$ and $g_{k,n}$ in (3.3), we arrive at the desired result. □

Remark 3.6. *From Theorem 3.5 it is easily seen that that if we know the explicit values of* $r_{k,n}$ *and* $g_{k,n}$ *then the explicit value of* $I_{k,n}$ *can be determined. Several values of* $g_{k,n}$ *are evaluated in [4]. For example, noting* $r_{3,3} = 3^{1/12}\left(3+2\sqrt{3}\right)^{1/12}$ *and* $g_{3,3} = \left(3+2\sqrt{3}\right)^{1/4}$*, we evaluate* $I_{3,3} = 3^{-1/12}\left(2\sqrt{3}-3\right)^{1/6}$.

Theorem 3.7. *We have*

$$I_{k,n}^3 = \frac{h_{k,n}}{g_{k,n}}.$$

Proof. This follows easily from Theorem 3.3 and Theorem 3.5, respectively. □

Remark 3.8. *It is obvious from Theorem 3.7 that if we know the explicit values of* $h_{k,n}$ *and* $g_{k,n}$ *then the corresponding value of* $I_{k,n}$ *can be evaluated. Noting* $h_{2,2} = \sqrt{2\sqrt{2}-2}$ *and* $g_{2,2} = 2^{-1/16}(\sqrt{2}+1)^{1/4}$ *from [16] and [4], respectively, we evaluate* $I_{2,2} = 2^{3/16}\left(\sqrt{2}-1\right)^{1/4}$.

Theorem 3.9. *We have*

$$I_{3,n}^{12} = 3\left(\frac{1}{g_{3,n}^4} - h_{3,n}^4\right) + 1.$$

Proof. Dividing (2.2) by P^4, we obtain

$$\frac{Q^4}{P^4} + Q^4 = \frac{9}{P^4} + 1. \tag{3.4}$$

Setting $q = e^{-\pi\sqrt{n/3}}$ in (3.4) and employing the definitions of $h_{k,n}$ and $g_{k,n}$ with $k = 3$, we obtain

$$\frac{h_{3,n}^4}{g_{3,n}^4} + 3h_{3,n}^4 = \frac{9}{3g_{3,n}^4} + 1. \tag{3.5}$$

Setting $k = 3$ in Theorem 3.7 and employing in (3.5) and simplifying, we complete the proof. □

Remark 3.10. *It is obvious from Theorem 3.9 that if we know the explicit values of $h_{3,n}$ and $g_{3,n}$ then the corresponding value of $I_{3,n}$ can be evaluated. Noting $h_{3,9} = \left(1 - \sqrt[3]{2} + \sqrt[3]{4}\right)/\sqrt{3}$ and $g_{3,9} = (1 + \sqrt[3]{2})^2/\sqrt{3}$, we evaluate $I_{3,9}^{12} = 9(1 + 2\sqrt[3]{2} - 2\sqrt[3]{2^2})/(1 + 2^{1/3})^8$.*

Theorem 3.11. *We have*

$$I_{5,n}^6 = \sqrt{5}\left(\frac{1}{g_{5,n}^4} - h_{5,n}^4\right) + 1.$$

Proof. Dividing (2.3) by P^2, setting $q = e^{-\pi\sqrt{n/3}}$ and employing the definitions of $h_{k,n}$ and $g_{k,n}$ with $k = 5$, we obtain

$$\frac{h_{5,n}^2}{g_{5,n}^2} + \sqrt{5}h_{5,n}^2 = \frac{5}{\sqrt{5}g_{5,n}^2} + 1. \tag{3.6}$$

Using Theorem 3.7 with $k = 5$ in (3.6) and simplifying, we complete the proof. □

Remark 3.12. *It is obvious from Theorem 3.11 that if we know the explicit values of $h_{5,n}$ and $g_{5,n}$ then the corresponding value of $I_{5,n}$ can be evaluated. Noting $h_{5,9} = (\sqrt{3} + 1)/(\sqrt{3} + \sqrt{5})$ and $g_{5,9} = \left(3 + \sqrt{3} + \sqrt{5} + \sqrt{15}\right)/2$, we evaluate $I_{5,9}^6 = \left(2 + \sqrt{3}\right)/\left(62 + 31\sqrt{3} + 24\sqrt{5} + 16\sqrt{15}\right)$.*

Theorem 3.13. *We have*

$$I_{9,n}^3 = \sqrt{3}\left(\frac{1}{g_{9,n}} - h_{9,n}\right) + 1.$$

Proof. Dividing (2.4) by P, setting $q = e^{-\pi\sqrt{n/9}}$ and employing the definitions of $h_{k,n}$ and $g_{k,n}$ with $k = 9$, we obtain

$$\frac{h_{9,n}}{g_{9,n}} + \sqrt{3}h_{5,n} = \frac{3}{\sqrt{3}g_{9,n}} + 1. \tag{3.7}$$

Using Theorem 3.7 with $k = 9$ in (3.7) and simplifying, we complete the proof. □

Remark 3.14. *It is obvious from Theorem 3.13 that if we know the explicit values of $h_{9,n}$ and $g_{9,n}$ then the corresponding value of $I_{9,n}$ can be easily evaluated by routine calculations.*

Theorems 3.15–3.23 connect the parameter $I_{k,n}$ with the Ramanujan's remarkable product of theta-functions $a_{k,n}$ for $k = 3, 5, 7, 9, 13$ and 25. If we know the explicit value of $a_{k,n}$ (or $I_{k,n}$), then $I_{k,n}$ (or $a_{k,n}$) can be evaluated. Thus, Theorems 3.15–3.23 may be considered as general theorems for the explicit evaluations of both $a_{k,n}$ and $I_{k,n}$ and new values of $a_{k,n}$ and $I_{k,n}$ can be determined therefrom.

Theorem 3.15. *We have*

$$I_{3,n}^6 - \frac{1}{I_{3,n}^6} = 3\left(a_{3,n} - \frac{1}{a_{3,n}}\right).$$

Proof. Setting $q = e^{-\pi\sqrt{n/3}}$ in Lemma 2.5, employing the definitions of $r_{k,n}$ and $r'_{k,n}$ with $k = 3$ and simplifying, we obtain

$$3\left((r_{3,n}r'_{3,n})^2 - \frac{1}{(r_{3,n}r'_{3,n})^2}\right) = \left(\frac{r_{3,n}}{r'_{3,n}}\right)^6 - \left(\frac{r'_{3,n}}{r_{3,n}}\right)^6. \tag{3.8}$$

Using Lemma 2.1 and Theorem 3.1 with $k = 3$ in (3.8), we complete the proof. □

Remark 3.16. *From Theorem 3.15 we can evaluate $I_{3,n}$ if we know the explicit values of $a_{3,n}$ and vice-versa. Employing the value $a_{3,15} = (2-\sqrt{3})/3$ from [7, p. 337, Entry 8] in Theorem 3.15 with $n = 15$ and solving the resulting equation, we obtain*

$$I_{3,15}^6 = (2-\sqrt{3})/(1+2\sqrt{3}+2\sqrt{5}).$$

Again, noting $I_{3,4} = ((\sqrt{-2+3\sqrt{6}-\sqrt{-6+3\sqrt{6}}})/2)^{1/2}$ from [13, Corollary 24(ii)], employing in Theorem 3.15 with $n = 4$ and solving the resulting equation, we obtain

$$a_{3,4} = \frac{-15+9\sqrt{3}-\sqrt{2(229-132\sqrt{3})}+B}{2\sqrt{-44+27\sqrt{3}-3\sqrt{2(229-132\sqrt{3})}}},$$

where $B = \sqrt{6(125-71\sqrt{3}+3\sqrt{2(229-132\sqrt{3})}-3\sqrt{6(229-132\sqrt{3})})}$.

Theorem 3.17. *We have*

$$I_{5,n}^3 - \frac{1}{I_{5,n}^3} = \sqrt{5}\left(\sqrt{a_{5,n}} - \frac{1}{\sqrt{a_{5,n}}}\right).$$

Proof. Setting $q = e^{-\pi\sqrt{n/5}}$ in Lemma 2.6, employing the definitions of $r_{k,n}$ and $r'_{k,n}$ with $k = 5$ and simplifying, we obtain

$$3\left((r_{5,n}r'_{5,n})^2 - \frac{1}{(r_{5,n}r'_{5,n})^2}\right) = \left(\frac{r_{5,n}}{r'_{5,n}}\right)^3 - \left(\frac{r'_{5,n}}{r_{5,n}}\right)^3. \tag{3.9}$$

Using Lemma 2.1 and Theorem 3.1 with $k = 5$ in (3.9), we complete the proof. □

Remark 3.18. *From Theorem 3.17 we can evaluate $I_{5,n}$ if we know the explicit values of $a_{5,n}$ and vice-versa. Employing the value $a_{5,17} = (\sqrt{17}-4)^2$ from [7, p. 338, Entry 8] in Theorem 3.17 with $n = 17$ and solving the resulting equation, we evaluate $I_{5,17} = (3-\sqrt{5})/2$.*

Again, noting $I_{5,5} = (\sqrt{5}-1)/2$ from Remark 3.2, employing in Theorem 3.17 with $n = 5$ and solving the resulting equation, we evaluate $a_{5,5} = 1/5$.

Theorem 3.19. *We have*

$$I_{9,n}^3 + \frac{1}{I_{9,n}^3} = 3\left(\sqrt{a_{9,n}} + \frac{1}{\sqrt{a_{9,n}}}\right) - 4.$$

Proof. Squaring Lemma 2.7 and dividing by R^3Q^3, we obtain

$$\left(\frac{R}{Q}\right)^3 + \left(\frac{Q}{R}\right)^3 = RQ + \frac{1}{RQ} - 4. \tag{3.10}$$

Setting $q = e^{-\pi\sqrt{n/9}}$ in (3.10), employing the definitions of $r_{k,n}$ and $r'_{k,n}$ with $k = 5$ and simplifying, we obtain

$$3\left(r_{9,n}r'_{9,n} + \frac{1}{r_{9,n}r'_{9,n}}\right) = \left(\frac{r_{9,n}}{r'_{9,n}}\right)^3 + \left(\frac{r'_{9,n}}{r_{9,n}}\right)^3 - 4. \tag{3.11}$$

Using Lemma 2.1 and Theorem 3.1 with $k = 9$ in (3.9), we complete the proof. □

Remark 3.20. *From Theorem 3.19 we can evaluate $I_{9,n}$ if we know the explicit values of $a_{9,n}$ and vice-versa. Employing the value $a_{7,9} = a_{9,7} = \left(\sqrt{\frac{5+\sqrt{21}}{8}} - \sqrt{\frac{\sqrt{21}-3}{8}}\right)^8$ from [7, p. 338, Entry 8] in Theorem 3.19 with $n = 7$ and solving the resulting equation, we evaluate*

$$I_{9,7}^3 = \frac{-77 - 17\sqrt{21} + 19\sqrt{2(3+\sqrt{21})} + 4\sqrt{42(3+\sqrt{21})} + C}{-14 - 2\sqrt{21} + \sqrt{2(3+\sqrt{21})} - \sqrt{42(3+\sqrt{21})}},$$

where $C = \sqrt{6\left(3678 + 802\sqrt{21} - 945\sqrt{2(3+\sqrt{21})} - 205\sqrt{42(3+\sqrt{21})}\right)}$

Again, noting $I_{13,9} = I_{9,13} = (2+\sqrt{3}-\sqrt{3+4\sqrt{3}})/2$ from [13, Corollary 32(ii)], employing in Theorem 3.19 with $n = 13$ and solving the resulting equation, we evaluate

$$a_{9,13} = \frac{-76 - 44\sqrt{3} + 24\sqrt{3+4\sqrt{3}} + 142\sqrt{3(3+4\sqrt{3})} + D}{-10 - 6\sqrt{3} + 3\sqrt{3+4\sqrt{3}} + 2\sqrt{2(3+4\sqrt{3})}},$$

where $D = \sqrt{22725 + 13120\sqrt{3} - 7212\sqrt{3+4\sqrt{3}} - 4164\sqrt{3(3+4\sqrt{3})}}.$

We end this section by presenting three more theorems connecting the parameters $a_{k,n}$ and $I_{k,n}$, from which one parameter can be evaluated if we know the other. The proofs of Theorems 3.21, 3.22 and 3.23 are identical to the proof of Theorem 3.17. So we give only references of the required theta-function identities.

Theorem 3.21. *We have*

$$I_{7,n}^6 - \frac{1}{I_{7,n}^6} + 8\left(I_{7,n}^2 - \frac{1}{I_{7,n}^2}\right) = 7\left(a_{7,n} - \frac{1}{a_{7,n}}\right).$$

Proof. This follows from Lemma 2.8. □

Theorem 3.22. *We have*

$$I_{13,n}^3 - \frac{1}{I_{13,n}^3} + 4\left(I_{13,n} - \frac{1}{I_{13,n}}\right) = \sqrt{13}\left(\sqrt{a_{13,n}} - \frac{1}{\sqrt{a_{13,n}}}\right).$$

Proof. This follows from Lemma 2.9. □

Theorem 3.23. *We have*

$$I_{25,n}^3 + \frac{1}{I_{25,n}^3} + 4\left(I_{25,n}^2 + \frac{1}{I_{25,n}^2}\right) = 5\left(\sqrt{a_{25,n}} + \frac{1}{\sqrt{a_{25,n}}}\right).$$

Proof. This follows from Lemma 2.10. □

4 Formulas for explicit values of $G^3(-e^{-\pi\sqrt{n/3}})$

Theorem 4.1. *We have*

$$(i)\ G^3\left(-e^{-\pi\sqrt{n/3}}\right) = \left(1 + I_{3,n}^{12} - \sqrt{1 + 34I_{3,n}^{12} + I_{3,n}^{24}}\right)/16.$$

$$(ii)\ G^3(-e^{-\pi/\sqrt{3n}}) = \left(1 + I_{3,n}^{12} - \sqrt{1 + 34I_{3,n}^{12} + I_{3,n}^{24}}\right)/16I_{3,n}^{12}.$$

Proof. From Lemmas 2.11 and 2.12, we deduce that

$$\frac{G^3(-q)\left(G^3(-q) - 1\right)}{G^3(-q) + 1} = \frac{\phi^4(q)}{\phi^4(q^3)}\frac{q\psi^4(-q^3)}{\psi^4(-q)}. \tag{4.1}$$

Setting $q = e^{-\pi\sqrt{n/3}}$ in (4.1) and employing the definitions of $h_{k,n}$, $g_{k,n}$ and Theorem 3.7 with $k = 3$, we obtain

$$\frac{G^3(-e^{-\pi\sqrt{n/3}})\left(G^3(-e^{-\pi\sqrt{n/3}})) - 1\right)}{G^3(-e^{-\pi\sqrt{n/3}}) + 1} = I_{3,n}^{12}. \tag{4.2}$$

Solving (4.2) for $G^3(-e^{-\pi\sqrt{n/3}})$ and choosing negative real root as $G^3(-e^{-\pi\sqrt{n/3}}) < 0$, we complete the proof (i).

To prove (ii) we replace n by $1/n$ in (i) and simplify using the result $I_{k,1/n} = 1/I_{k,n}$ from [13, Theorem 16(ii)] with $k = 3$. □

Remark 4.2. *From Theorem 4.1(i) and (ii) it is obvious that if we know the explicit values of $I_{3,n}$ the explicit values of $G^3(-e^{-\pi\sqrt{n/3}})$ and $G^3(-e^{-\pi/\sqrt{3n}})$ can easily be evaluated, respectively. For example, setting $n = 1$ in Theorem 4.1(i) and employing the value $I_{n,1} = 1$ from [13, Theorem 16(i)], we obtain $G^3(-e^{-\pi/\sqrt{3}}) = -1/4$. Again , setting $n = 2$ in Theorem 4.1(ii) and employing the value $I_{3,2} = \left(-44 + 27\sqrt{3} - 3\sqrt{458 - 264\sqrt{3}}\right)^{1/12}$ from [13, Corollary 24(i)], we evaluate*

$$G^3(-e^{-\pi/\sqrt{6}}) = \frac{43 - 27\sqrt{3} + 3\sqrt{458 - 264\sqrt{3}} - 3E}{16\left(-44 + 27\sqrt{3} - 3\sqrt{458 - 264\sqrt{3}}\right)},$$

where $E = \sqrt{6\left(125 - 71\sqrt{3} + 3\sqrt{458 - 264\sqrt{3}} - 3\sqrt{6(458 - 264\sqrt{3})}\right)}$.

Remark 4.3. *In his notebooks Ramanujan recorded some explicit values of $\chi(q)$. For example, Ramanujan recorded explicit value of $\chi(e^{-\pi})$ [7, p. 326, Entry 2] as*

$$\chi(e^{-\pi}) = 2^{1/4}e^{-\pi/24}. \tag{4.3}$$

Here, we offer an explicit formula for $\chi(e^{-\pi n})$ by using the parameter $I_{k,n}$ for any positive real number n. Setting $k = n$ in (1.17) *and simplifying using* (4.3)*, we obtain*

$$\chi(e^{-\pi n}) = \frac{2^{1/4}e^{-\pi n/24}}{I_{n,n}}. \tag{4.4}$$

Thus, to find $\chi(e^{-\pi n})$ it is enough to know the explicit values of $I_{n,n}$. For example, setting $n = 2$ in (4.4) *and employing the value of $I_{2,2}$ from Remark 3.8, we obtain*

$$\chi(e^{-2\pi}) = 2^{1/16}(\sqrt{2} + 1)^{1/4}e^{-\pi/12}.$$

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