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Refinements of the Mitrinović–Adamović inequality, with application

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Abstract: We point out refinement of the Mitrinović–Adamović inequality, and offer an application to the new poof of a recent result.
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1 Introduction

The famous inequality due to D.D. Adamović and D.S. Mitrinović (see [2, p. 238]) states that for any $x \in \left(0, \frac{\pi}{2}\right)$ one has

$$\frac{\sin x}{x} > (\cos x)^{\frac{1}{3}} \tag{1.1}$$

The Seiffert mean P of two positive variables (see [5, 6]) is defined by

$$P(x, y) = \frac{x - y}{2 \arcsin \frac{x-y}{x+y}} \text{ for } x \neq y; P(x, x) = x \tag{1.2}$$

Let $A(x, y) = \frac{x + y}{2,}$ $G(x, y) = \sqrt{xy}$ denote the arithmetic, respectively geometric, means of x and y.

Let $x_0 = \sqrt{xy}, y_0 = \frac{x + y}{2}$ and $x_{n+1} = \frac{x_n + y_n}{2}$, $y_{n+1} = \sqrt{x_{n+1} y_n} (n \geq 0)$ be the Pfaff algorithm (see e.g. [3]).

In 2001, the author [3] has proved that for any $n \geq 0$ one has

$$\sqrt[3]{y_n^2 x_n} < P < \frac{x_n + 2y_n}{3} \tag{1.3}$$

Particularly, for $n = 0$, from (1.3) we get the double inequality

$$\sqrt[3]{A^2G} < P < \frac{G + 2A}{3}, \tag{1.4}$$

while for $n = 1$ we get

$$\sqrt[3]{A\left(\frac{A+G}{2}\right)^2} < P < \frac{1}{3}\left(\frac{A+G}{2} + 2\sqrt{\frac{A+G}{2} \cdot A}\right) \tag{1.5}$$

In what follows, we will use the above mean inequalities, as well as certain algebraic inequalities, in order to obtain refinements of the Mitrinović–Adamović inequality (1.1). An application to a new and simple proof of a result from [1] will be offered, too. For mean inequalities and trigonometric and hyperbolic applications, see also [4].

2 Main results

The main result of this section is contained in the following:

Theorem 2.1. *For any* $x \in \left(0, \frac{\pi}{2}\right)$ *one has*

$$\frac{\sin x}{x} > \left(\frac{\cos x + 1}{2}\right)^{\frac{2}{3}} > \frac{1 + \sqrt{1 + 8\cos x}}{4} > \sqrt[3]{\cos x} \tag{2.1}$$

Proof. First remark that $P(1 + \sin x, 1 - \sin x) = \dfrac{\sin x}{x}$ and $A(1 + \sin x, 1 - \sin x) = 1$, $G(1 + \sin x, 1 - \sin x) = \cos x$.

Applying the left side of (1.5) we get the first inequality of (2.1).

For the second inequality of (2.1) put $u = \cos x$ and $\dfrac{u+1}{2} = v^3$ (here $0 < u, v < 1$). Then $u = 2v^3 - 1$ and the inequality becomes $(4v^2 - 1)^2 > 16v^3 - 7$ or

$$P(v) = 2v^4 - 2v^3 - v^2 + 1 > 0 \tag{2.2}$$

After elementary transformations, $P(v)$ can be written as $P(v) = (v - 1)^2(2v^2 + 2v + 1)$, so (2.2) follows.

For the proof of the last inequality of (2.1) let again $u = \cos x$ and $u = s^3$. Then we have to prove $4s < 1 + \sqrt{1 + 8s^3}$ or $(4s - 1)^2 < 1 + 8s^3$. This becomes $16s^2 - 8s < 8s^3$ or $2s - 1 < s^2$, which is $(s - 1)^2 > 0$, so it is true. □

Remark 1. *Clearly, by* (1.3)*, the first inequality of* (2.1) *can be further improved (e.g. by selecting* $n = 2$*, etc.), and in fact infinitively many improvement are obtainable (as the sequence* $(y_n^2 x_n)$ *is strictly increasing, see [3]).*

3 An application

In what follows, we will apply the following part of inequality (2.1):

$$\frac{\sin x}{x} > \frac{1+\sqrt{1+8\cos x}}{4}. \tag{3.1}$$

In 2015 (see [1]) B. Bhayo, R. Klén and the author have proved the following result:

Theorem 3.1. *The best constants α and β such that*

$$\frac{\cos x+\alpha-1}{\alpha} < \frac{\sin x}{x} < \frac{\cos x+\beta-1}{\beta} \tag{3.2}$$

for any $x \in \left(0, \frac{\pi}{2}\right)$ are $\alpha = \frac{\pi}{\pi-2}$ and $\beta = 3$.

The proof of this result in [1] is based on certain series expansions with Bernoulli numbers, and applications of more auxiliary results.

Our aim is to show that (3.1) will offer an easy proof to this theorem.

The inequality $\frac{\sin x}{x} < \frac{\cos x+\beta-1}{\beta}$ may be written also as $\beta > f(x) = \frac{x - x\cos x}{x-\sin x}$. We will prove that this function $f(x)$ is strictly decreasing. An immediate computation gives $(x-\sin x)^2 f'(x) = -\sin x+\sin x\cos x+x\cos x+x^2\sin x-x = g(x)$. Also, $g'(x) = -2\sin^2 x+x\sin x + x^2\cos x = x^2(-2r^2+r+\cos x)$, where $r = \frac{\sin x}{x}$. Now, the polynomial $P(r) = -2r^2+r+\cos x$ of variable r has roots $\frac{1\pm\sqrt{1+8\cos x}}{4}$. By inequality (3.1) we get $P(r) < 0$ for $x \in \left(0, \frac{\pi}{2}\right)$. Therefore, $g'(x) < 0$, implying $g(x) < g(0) = 0$, so $f(x)$ is indeed strictly decreasing. This implies $f(x) > \lim_{x\to 0} f(x) = 3$, and also $f(x) < f\left(\frac{\pi}{2}\right) = \frac{\pi}{\pi-2}$, which are the best constants in (3.2).

References

[1] Bhayo, B. A., Klén, R. & Sándor, J. New trigonometric and hyperbolic inequalities, to appear, *Miskolc Math. Notes* (see also `https://arxiv.org/pdf/1501.05070v1.pdf`)

[2] Mitrinović, D. S. (1970) *Analytic inequalities*, Springer–Verlag, Berlin.

[3] Sándor, J. (2001) On certain inequalities for means, III., *Arch. Math.* (Besel) 26, 34–40.

[4] Sándor, J. (2012) On Huygens' inequalities and the theory of means, *Intern. J. Math. Math. Sci.* Vol. 2012, Article ID 597490, 9 pages.

[5] Seiffert, H.–J. (1993) Problem 887, *Nieuw Arch. Wiskunde*, 11(4), 176.

[6] Seiffert, H.–J. (1995) Ungleichungen für einen bestimmten Mittelwert, *Nieuw Arch. Wisk.* 13(4), 195–198.